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Neighborhoods at infinity and the Plancherel formula for a reductive p-adic symmetric space

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Abstract Yiannis Sakellaridis and Akshay Venkatesh have determined, when the group G is split and the field $\mathbf{F}$ is of characteristic zero, the Plancherel formula for any spherical space X for G modulo the knowledge of the discrete spectrum. The starting point is the determination of good neighborhoods at infinity of X/J, where J is a small compact open subgroup of G. These neighborhoods are related to "boundary degenerations" of X. The proof of their existence is made by using wonderful compactifications. In this article we show the existence of such neighborhoods assuming that $\mathbf{F}$ is of characteristic different from 2 and X is symmetric. In particular, one does not assume that G is split. Our main tools are the Cartan decomposition of Benoist and Oh, our previous definition of the constant term and asymptotic properties of Eisenstein integrals due to Nathalie Lagier. Once the existence of these neighborhoods at infinity of X is established, the analog of the work of Sakellaridis and Venkatesh is straightforward and leads to the Plancherel formula for X.

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1 Introduction

Let G be the group of $\mathbf{F}$-points of a reductive group $\underline{G}$ defined over the non archimedean local field $\mathbf{F}$.

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In a tremendeous work (cf. [14]), Yiannis Sakellaridis and Akshay Venkatesh have determined, when the group G is split and the field $\mathbf{F}$ is of characteristic zero, the Plancherel formula for any spherical space X for G modulo the knowledge of the discrete spectrum.

The starting point is the determination of good neighborhoods at infinity of X/J, where J is a small compact open subgroup of G. Notice that G acts on X on the right. These neighborhoods are related to "boundary degenerations" of X. The proof of their existence is made by using wonderful compactifications.

In this article we will show the existence of such neighborhoods assuming that $\mathbf{F}$ is of characteristic different from 2 and X is symmetric. In particular, one does not assume that G is split. The main tool is the Cartan decomposition (cf. [2]), the definition of the constant term (cf [6]) and asymptotic properties of Eisenstein integrals due to Nathalie Lagier (cf. [12]). The use of Eisenstein integrals to prove results geometric in nature on symmetric spaces goes back to her work (cf. [12] Theorem 7). Notice that our neighborhoods at infinity are quite explicit in terms of the Cartan decomposition.

Once the existence of these neighborhoods at infinity of X is established, the analog of part 3 of [14] is straightforward and leads to the Plancherel formula for X. Notice that our definition of normalized integrals differs slightly from the one in [14] section 15.

Let σ be an involution of $\underline{G}$ defined over F. Let H be the fixed point group of σ in G and let $X = H\backslash G$. We denote by $X(G)$ the group of unramified characters of G and $X(G)_\sigma$ be the connected component of 1 in $\{\chi \in X(G) | \chi \circ \sigma = \chi^{-1}\}$.

Let P be a σ-parabolic subgroup of G i.e. such that P and $\sigma(P)$ are opposed. Let $M := P \cap \sigma(P)$ be the σ-stable Levi subgroup of G. Let U (resp., U^-) be the unipotent radical of P (resp., $P^- := \sigma(P)$) and let δ_P be the modulus function of P. We define:

$$H_P = U^-(M \cap H), X_P = H_P\backslash G.$$

The space X_P is called a "boundary degeneration" of $X = H\backslash G$. It is an important object whose role has been emphasized by Sakellaridis and Venkatesh.

Let $P_\emptyset = M_\emptyset U_\emptyset$ be a minimal σ-parabolic subgroup of G. We will assume in this introduction that $P_\emptyset H$ is the only $(P_\emptyset, H)$-open double coset in G. A split torus is said to be σ-split if all its elements are antiinvariant by σ. Let $A_\emptyset$ be the maximal σ-split torus of the center of $M_\emptyset$. Let $A_\emptyset^+$ be the closed positive chamber in $A_\emptyset$ for $P_\emptyset$. Let $\Sigma(P_\emptyset)$ be the set of simple roots for $A_\emptyset$ in the Lie algebra of the unipotent radical of $P_\emptyset$. The Cartan decomposition asserts (cf. [2]):

$$G = HA_\emptyset^+\Omega,$$

for some compact subset, Ω, of G. Let $P = MU$ be a σ-parabolic sugroup of G, where M is the σ-stable Levi subgroup of P and U is the unipotent radical of P. Let A_M be the maximal σ-split torus of the center of M. Then A_M acts on the left on X_P and this action commutes with the right G-action.

If $C > 0$, let

$$A_\emptyset^+(P, C) := \{a \in A_\emptyset^+ | |\alpha(a)|_F > C, \alpha \; root \; of \; A_\emptyset \; in \; U\}.$$

We denote by $\dot{1}$ (resp., $\dot{1}_P$) the image of the neutral element 1 of G in X (resp., X_P). The following Theorem (cf. Theorem 1) is an easy consequence of [6], Proposition 3.14.

Theorem (Constant term map) *There is a unique* G*-equivariant map* $c_P : C^\infty(X) \to C^\infty(X_P)$ *with the following property. For every compact open subgroup* J *of* G*, there exists* $C > 0$ *such that for all* $f \in C^\infty(X)$ *which is* J*-invariant:*

$$(c_P f)(\dot{1}_P a\omega) = f(\dot{1} a\omega), a \in A_\emptyset^+(P, C), \omega \in \Omega.$$

The following theorem (cf. Theorem 2 for its detailed version) was suggested by the work [14] of Sakellaridis and Venkatesh, who constructed similar maps, in their context, using wonderful compactifications.

Theorem ($exp_{P,J}$-maps) *Let* $P = MU$ *be a standard* σ*-parabolic subgroup of* G *i.e. such that* $P_\emptyset \subset P$*. Let* J *be a compact open subgroup of* G*.*

(i) *There exists* $C > 0$ *such that the correspondence* $\dot{1}xJ \mapsto \dot{1}_P xJ$*, for* $x \in A_\emptyset^+(P, C)\Omega$*, is a well defined bijective map denoted* $exp_{P,J}$ *from the subset* $N_{X,J}(P, C) := \dot{1} A_\emptyset^+(P, C)\Omega J$ *of* X/J*, to the subset* $O' := \dot{1}_P A_\emptyset^+(P, C)\Omega J$ *of* X_P/J*.*
(ii) *For* J *small enough, the map* $exp_{P,J}$ *preserves volumes.*
(iii) *For* f *any right* J*-invariant element of* $C^\infty(H\backslash G)$*, one has:*

$$(c_P f)(exp_{P,J}(x)) = f(x), x \in N_{X,J}(P, C).$$

As said above we need some results of N. Lagier on Eisenstein integrals that we will recall. Let $P = MU$ be a σ-parabolic subgroup of G. Let (δ, E) be a unitary irreducible representation of M. Let $\chi \in X(M)_\sigma$ and let $\delta_\chi = \delta \otimes \chi$. We denote by $i_P^G \delta_\chi$ or π_χ the normalized induced representation and let V_χ denote its space.

Let $\eta \in E'^{M \cap H}$. Let $\chi \in X(M)_\sigma$, sufficiently P-dominant. There is a canonical H-fixed linear form $\xi(P, \delta_\chi, \eta)$ on V_χ, (cf. [4]). One defines the Eisenstein integrals on X, $E(P, \delta_\chi, \eta, v) \in C^\infty(X)$, $v \in V_\chi$. by:

$$E(P, \delta_\chi, \eta, v)(\dot{1}g) = \langle \xi(P, \delta_\chi, \eta), \pi_\chi(g)v \rangle, \quad g \in G.$$

Let A_M be the maximal σ-split torus of the center of M and let μ_δ be the character of A_M by which A_M acts on δ. The following theorem is due to Nathalie Lagier. This is the analog of a lemma of Langlands on asymptotics of smooth coefficients.

One says that the sequence (a_n) *satisfies* $(a_n) \to_P \infty$ *if* $a_n \in A_M$ *and for every root* α *of* A_M *in the Lie algebra of* U, $(|\alpha(a_n)|_F)$ *tends to infinity. Let us assume* $Re(\chi)\delta_P^{-1/2}$ *is* P*-dominant. Then if* $(a_n) \to_P \infty$ *the following limit exists*

$$lim_{n\to\infty}(\chi\delta_P^{-1/2})(a_n^{-1})\mu_\delta(a_n^{-1})E(P,\delta_\chi,\eta,v)(\dot{1}a_n) \tag{1.1}$$

and is equal to

$$< \eta, (A(P^-, P, \delta_\chi)v)(1) >,$$

where $A(P^-, P, \delta_\chi)$ *is the (converging) intertwining integral operator.*

The theorem admits a variation when $(a_n) \to_Q \infty$, with $P \subset Q$. This implies easily (cf. Lemmas 7 and 8):

First Key Lemma

Let us assume that $(a_n) \to_P \infty$ *and that* (g_n) *is a sequence in* G *converging to* g*. If* $\dot{1}a_n g_n = \dot{1}a_n$ *for all* n*, then* $g \in U^-(M \cap H)$*.*

Second Key Lemma

Let J *be a compact open subgroup of* G*. Let* $(a_n) \to_P \infty$, $(a'_n) \to_{P'} \infty$, *for* P, P' σ*-parabolic subgroups of* G *and let* $g, g' \in G$*. Let us assume* $\dot{1}a_n gJ = \dot{1}a'_n g'J$, $n \in \mathbb{N}$*.*

Then $P = P'$ *and a subsequence of* $(a_n{}^{-1}a'_n)$ *is bounded.*

Definition of $exp_{P,J}$ Although we gave a formula for $exp_{P,J}$ it is unclear that it is well defined. This is achieved by using the two previous Lemmas.

Injectivity of $exp_{P,J}$ One wants to prove that, for C large, if $x, x' \in N_{X,J}(P,C)$ and $exp_{P,J}(x) = exp_{P,J}(x')$, then $x = x'$. One introduces the characteristic function f of $x \subset X$ and one will use its constant term $c_P f$. These functions are J-invariant and their values on a J-coset makes sense. From the properties of the constant term, if C is large enough one has:

$$(c_P f)(exp_{P,J}(x)) = f(x) = 1.$$

But, from our hypothesis one deduces:

$$(c_P f)(exp_{P,J}(x)) = (c_P f)(exp_{P,J}(x')).$$

Moreover by the properties of the constant term and because C is large, one has:

$$(c_P f)(exp_{P,J}(x')) = f(x') = 1.$$

This implies that $f(x') = 1$, hence $x = x'$, as wanted.

A compact open subgroup J of G is said to have a strong σ-factorization for $P_\emptyset$ if for all σ-parabolic subgroups $P = MU$ which contains $P_\emptyset$ one has:

(1) $J = J_{U^-} J_M J_U$ for all σ-parabolic subgroups, where $J_M = J \cap M$, ...
(2) For all $a \in A_\emptyset^+$, $a^{-1}J_U a \subset J_U$, $aJ_{U^-}a^{-1} \subset J_{U^-}$.
(3) $J = J_H J_P$, where $J_H = J \cap H$, $J_P = J \cap P$.

(4) J_M satisfies the same properties for $P_\emptyset \cap M$.

There are arbitrary small compact open subgroups with a strong σ-factorization for $P_\emptyset$ (cf. Kato-Takano [10] if the residual characteristic is different from 2, [5] in general and Lemma 6 of this article for the "strong" version).

A choice of a G-invariant measure on X determines a G-invariant measure on X_P.

Third Key Lemma(cf. Lemma 10)

Let J be a compact open subgroup with a strong σ-factorization for $P_\emptyset$. Let $a \in A_\emptyset^+$. Then:

$$\dot{1} aJ = \dot{1} aJ_M J_U, \dot{1}_P aJ = \dot{1}_P aJ_M J_U,$$
$$vol_X(\dot{1} aJ) = vol_{X_P}(\dot{1}_P aJ).$$

The proof is easy. Moreover one can show that the identity of volumes is also true for any small enough compact open subgroup of G. This implies easily the last property of $exp_{P,J}$.

Then, one introduces the restriction e_P of the transpose map of the constant term map to $C_c^\infty(X_P)$. Following an idea given to us by Joseph Bernstein, and using a result of Aizenbud, Avni, Gourevitch [1], one shows that its image is contained in $C_c^\infty(X)$ (cf Theorem 3). This achieves to prove the analog of Theorems 5.1.1 and 5.1.2 of [14].

Then, as was said before, this allows to prove the Plancherel formula, modulo the discrete spectrum of the X_P, by using the same method as [14], Part 3.

More precisely the maps e_P allow to define bounded G-maps, i_P from $L^2(X_P)$ to $L^2(X)$ (cf Theorem 5):

Theorem *For every pair of standard σ-parabolic subgroups of G, P, there exists a canonical G-equivariant map $i_P : L^2(X_P) \to L^2(X)$ characterized by the property that for any $\Psi \in C_c^\infty(X_P)$ and any element a of the set A_P^{++} of strictly P-dominant elements of A_M, we have:*

$$lim_{n\to\infty}(i_P \mathcal{L}_{a^n}\Psi - e_P \mathcal{L}_{a^n}\Psi) = 0$$

where the limit is in $L^2(X)$. Here $\mathcal{L}$ is the normalized unitary representation of A_M deduced from the left action of A_M on X_P.

One introduces the discrete part of $L^2(X_P)$, $L^2(X_P)_{disc}$. Then one has (cf. Proposition 5):

Proposition *Let $L^2(X)_P$ be the image of $L^2(X_P)_{disc}$ under i_P. Then one has:*

$$L^2(X) = \sum_{P\in\mathcal{P}_{st}} L^2(X)_P$$

where $\mathcal{P}_{st}$ is the set of standard σ-parabolic subgroups of G i.e. which contain $P_\emptyset$

Let $P = MU$, $Q = LV$ be two standard σ-parabolic subgroups of G. Let Θ_P (resp., Θ_Q) be the set of elements of $\Sigma(P_\emptyset)$ which are trivial on A_M (resp. A_L). We define

$W(P, Q)$ as the set of elements of $w \in W(A_\emptyset)$ such that $w(\Theta_P) = \Theta_Q$. In particular if $w \in W(P, Q)$, it induces an isomorphism between A_M and A_L. If $W(P, Q)$ is non trivial we say that P and Q are σ-associated. Let $c(P) = \sum_{Q \in \mathcal{P}_{st}} Card\ W(P, Q)$. Then the scattering theory as in [14] leads to (cf. Theorem 6):

Theorem (Scattering Theorem) *Let $P = MU$, $Q = LV$, R be three standard σ-parabolic subgroups of G.*

(i) *If P and Q are not σ-associated, $(i_Q)^t \circ i_P = 0$.*
(ii) *If P and Q are σ-associated, there exist $A_M \times G$-equivariant isometries*

$$S_w : L^2(X_P) \to L^2(X_Q), w \in W(P, Q)$$

where A_M acts on $L^2(X_Q)$ via the isomorphism $A_M \to A_L$ induced by w, with the following properties:

$$\begin{aligned} i_Q \circ S_w &= i_P, \\ S_{w'} \circ S_w &= S_{w'w}, w \in W(P, Q), w' \in W(Q, R), \\ (i_Q)^t \circ i_P &= \sum_{w \in W(P,Q)} S_w. \end{aligned}$$

Let us denote by $(i_P)^t_{disc}$ the composition of $(i_P)^t$ with the orthogonal projection to the discrete spectrum. Finally the map

$$\sum_{P \in \mathcal{P}} \frac{(i_P)^t_{disc}}{c(P)^{1/2}} : L^2(X) \to \oplus_{P \in \mathcal{P}_{st}} L^2(X_P)_{disc}$$

is an isometric isomorphism onto the subspaces of vectors $(f_P)_{P \in \mathcal{P}_{st}} \in \oplus_{P \in \mathcal{P}_{st}} L^2(X_P)_{disc}$ satisfying:

$$S_w f_P = f_Q, w \in W(P, Q).$$

Then, in Theorem 7, we explicate the restriction of the map i_P to $L^2(X_P)_{disc}$ in terms of wave packets of suitable normalized Eisenstein integrals.

2 Notations

If E is a vector space, E' will denote its dual. If $T : E \to F$ is a linear map between two vector spaces, T^t will denote its transpose. If E is real, $E_{\mathbb{C}}$ will denote its complexification. If G is a group, $g \in G$ and X is a subset of G, $g.X$ will denote gXg^{-1}. If J is a subgroup of G, $g \in G$ and (π, V) is a representation of J, V^J will denote the space of invariant elements of V under J and $(g\pi, gV)$ will denote the representation of $g.J$ on $gV := V$ defined by:

$$(g\pi)(g.x) := \pi(x), x \in J.$$

We will denote by (π', V') the contragredient representation of a representation (π, V) of G in the algebraic dual vector space V' of V.

If V is a vector space of vector valued functions on G which is invariant by right (resp., left) translations, we will denote by ρ (resp., λ) the right (resp., left) regular representation of G in V.

If G is locally compact, $d_l g$ or dg will denote a left invariant Haar measure on G and δ_G will denote the modulus function.

Let $\mathbf{F}$ be a non archimedean local field. We assume:

$$\text{The characteristic of } \mathbf{F} \text{ is different from 2.} \tag{2.1}$$

Let $|.|_{\mathbf{F}}$ be the normalized absolute value of $\mathbf{F}$.

One considers various algebraic groups defined over $\mathbf{F}$, and a sentence like:

“let A be a split torus” will mean “let A be the group of $\mathbf{F}$-points of a torus, $\underline{A}$, defined and split over $\mathbf{F}$”. (2.2)

With these conventions, let G be a connected reductive linear algebraic group. Let $\tilde{A}_G$ be the maximal split torus of the center of G. The change with standard notation will become clear later.

Let $\underline{G}$ be the algebraic group defined over $\mathbf{F}$ whose group of $\mathbf{F}$-points is G. Let σ be a rational involution of $\underline{G}$ defined over $\mathbf{F}$. Let H be the group of $\mathbf{F}$-points of an open $\mathbf{F}$-subgroup of the fixed point set of σ. We will also denote by σ the restriction of σ to G.

A split torus A of G is said to be σ-split if A is contained in the set of elements of G which are antiinvariant by σ. We will denote by A_G the maximal σ-split torus of the center of G.

If J is an algebraic subgroup of G stable by σ, one denotes by $\mathrm{Rat}(J)_\sigma$ the group of its rational characters defined over $\mathbf{F}$ which are antiinvariant by σ. Let us define:

$$\mathfrak{a}_G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(G)_\sigma, \mathbb{R}).$$

The restriction of rational characters from G to A_G induces an isomorphism:

$$\mathrm{Rat}(G)_\sigma \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathrm{Rat}(A_G) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{2.3}$$

Notice that $\mathrm{Rat}(A_G)$ appears as a generating lattice in the dual space $\mathfrak{a}'_G$ of $\mathfrak{a}_G$ and:

$$\mathfrak{a}'_G \simeq \mathrm{Rat}(G)_\sigma \otimes_{\mathbb{Z}} \mathbb{R}. \tag{2.4}$$

One has the canonical map $H_G : G \to \mathfrak{a}_G$ which is defined by:

$$e^{\langle H_G(x), \chi\rangle} = |\chi(x)|_{\mathbf{F}},\ x \in G, \chi \in \mathrm{Rat}(G)_\sigma. \tag{2.5}$$

The kernel of H_G, which is denoted by G^1, is the intersection of the kernels of the characters of G, $|\chi|_{\mathbf{F}}, \chi \in \mathrm{Rat}(G)_\sigma$. One defines $X(G)_\sigma = \mathrm{Hom}(G/G^1, \mathbb{C}^*)$. It is a subgroup of the group $X(G)$ of unramified characters of G. It is precisely the connected component of the neutral element of the group of elements of $X(G)$ which are antiinvariant by σ.

One denotes by $\tilde{\mathfrak{a}}_{G,\mathbf{F}}$ (resp., $\mathfrak{a}_{G,\mathbf{F}}$) the image of G (resp., A_G) by H_G. The group G/G^1 is isomorphic to the lattice $\mathfrak{a}_{G,\mathbf{F}}$.

There is a surjective map:

$$(\mathfrak{a}'_G)_{\mathbb{C}} \to X(G)_\sigma \to 1 \tag{2.6}$$

denoted by $\nu \mapsto \chi_\nu$ which associates to $\chi \otimes s$, with $\chi \in \mathrm{Rat}(G)_\sigma$, $s \in \mathbb{C}$, the character $g \mapsto |\chi(g)|^s_{\mathbf{F}}$ (cf. [16], I.1.(1)). In other words:

$$\chi_\nu(g) = e^{\langle \nu, H_G(g)\rangle}, \, g \in G, \, \nu \in (\mathfrak{a}'_G)_{\mathbb{C}}. \tag{2.7}$$

The kernel is a lattice in $i\mathfrak{a}'_G$ and it defines a structure of a complex algebraic variety on $X(G)_\sigma$ of dimension $dim_{\mathbb{R}}\mathfrak{a}_G$. Moreover $X(G)_\sigma$ is an abelian complex Lie group whose Lie algebra is equal to $(\mathfrak{a}'_G)_{\mathbb{C}}$.

If χ is an element of $X(G)_\sigma$, let ν be an element of $\mathfrak{a}'_{G,\mathbb{C}}$ such that $\chi_\nu = \chi$. The real part $\mathrm{Re}\,\nu \in \mathfrak{a}'_G$ is independent from the choice of ν. We will denote it by $\mathrm{Re}\,\chi$. If $\chi \in \mathrm{Hom}(G, \mathbb{C}^*)$ is continuous and antiinvariant by σ, the character of G, $|\chi|$, is an element of $X(G)_\sigma$. One sets $\mathrm{Re}\,\chi = \mathrm{Re}\,|\chi|$. Similarly, if $\chi \in \mathrm{Hom}(A_G, \mathbb{C}^*)$ is continuous, the character $|\chi|$ of A_G extends uniquely to an element of $X(G)_\sigma$ with values in $\mathbb{R}^{*+}$, that we will denote again by $|\chi|$ and one sets $\mathrm{Re}\,\chi = \mathrm{Re}\,|\chi|$.

A parabolic subgroup P of G is called a σ-parabolic subgroup if P and $\sigma(P)$ are opposite parabolic subgroups. Then $M := P \cap \sigma(P)$ is the σ-stable Levi subgroup of P. If P is such a parabolic subgroup, P^- will denote $\sigma(P)$.

$$\text{If } P \text{ is a } \sigma\text{-parabolic subgroup of } G,\ PH \text{ is open in } G. \tag{2.8}$$

The sentence: "Let $P = MU$ be a parabolic subgroup of G" will mean that U is the unipotent radical of P and M is a Levi subgroup of G. If moreover P is a σ-parabolic subgroup of G, M will denote its σ-stable Levi subgroup.

If $P = MU$ is a σ-parabolic subgroup of G, we keep the same notations with M instead of G.

The inclusions $A_G \subset A_M \subset M \subset G$ determine a surjective morphism $\mathfrak{a}_{M,\mathbf{F}} \to \mathfrak{a}_{G,\mathbf{F}}$ (resp., an injective morphism, $\tilde{\mathfrak{a}}_{G,\mathbf{F}} \to \tilde{\mathfrak{a}}_{M,\mathbf{F}}$) which extends uniquely to a surjective linear map between $\mathfrak{a}_M$ and $\mathfrak{a}_G$ (resp., injective map, between $\mathfrak{a}_G$ and $\mathfrak{a}_M$). The second map allows us to identify $\mathfrak{a}_G$ with a subspace of $\mathfrak{a}_M$ and the kernel of the first one, $\mathfrak{a}^G_M$, satisfies:

$$\mathfrak{a}_M = \mathfrak{a}^G_M \oplus \mathfrak{a}_G. \tag{2.9}$$

If an unramified character of G is trivial on M, it is trivial on any maximal compact subgroup of G and on the unipotent radical of P, hence on G. This allows to identify $X(G)_\sigma$ to a subgroup of $X(M)_\sigma$. Then $X(G)_\sigma$ is the analytic subgroup of $X(M)_\sigma$ with Lie algebra $(\mathfrak{a}'_G)_{\mathbb{C}} \subset (\mathfrak{a}'_M)_{\mathbb{C}}$. This follows easily from (2.7) to (2.9).

Let $P = MU$ be a σ-parabolic subgroup of G. Recall that A_M is the maximal σ-split torus of the center of M.

Let A_P^+, (resp., A_P^{++}) be the set of P-dominant (resp., strictly dominant) elements in A_M. More precisely, if $\Sigma(P)$ is the set of roots of A_M in the Lie algebra of P, and $\Delta(P)$ is the set of simple roots, one has:

$$A_P^+ \ (resp., A_P^{++}) = \{a \in A_M | |\alpha(a)|_{\mathbf{F}} \geq 1, \quad (resp., > 1) \quad \alpha \in \Delta(P)\}.$$

Let $A_\emptyset$ be a maximal σ-split torus contained in M. Let $\Sigma(U, A_\emptyset)$ be the set of roots of $A_\emptyset$ in the Lie algebra of U, and let $\Delta(P, A_\emptyset)$ be the set of simple roots. One defines for $C > 0$:

$$A_\emptyset^+(P, C) = \{a \in A_\emptyset | |\alpha(a)|_{\mathbf{F}} \geq C, \quad \alpha \in \Delta(U, A_\emptyset)\}. \tag{2.10}$$

Let A be a σ-split torus and $g \in G$. We will say that g is A-good if and only if $g^{-1}.A$ is a σ-split torus. Let us prove:

$$\text{If } g \text{ is } A\text{-good } \sigma(g)g^{-1} \text{ commutes to } A. \tag{2.11}$$

It is enough to prove that if $a \in A$, $(\sigma(g)g^{-1}).a = a$. One has $(\sigma(g)g^{-1}).a = \sigma(g.\sigma(g^{-1}.a)) = \sigma(g.(g^{-1}.a^{-1}) = a$.

For the rest of the article, we fix $P_\emptyset = M_\emptyset U_\emptyset$ a minimal σ-parabolic subgroup of G and let $A_\emptyset$ be the maximal σ-split torus of the center of $M_\emptyset$. It is a maximal σ-split torus of G. One denotes by $A_\emptyset^+$ the set $A_{P_\emptyset}^+$. A σ-parabolic subgroup of G will be said standard (resp., semistandard) if it contains $P_\emptyset$ (resp., $M_\emptyset$). We choose a maximal split torus A_0 which contains $A_\emptyset$. From [7], Lemma 1.9, it is σ-stable. Let K_0 be the stabilizer of a special point of the apartment of the extended building of G associated to A_0.

From [4], Lemma 2.4, there exists a finite set $\mathcal{W}_{M_\emptyset}^G$ of $A_\emptyset$-good elements of G, such that if P is any semi-standard minimal σ-parabolic subgroup of G, $\mathcal{W}_{M_\emptyset}^G$ is a set of representatives of the (P, H)-double open cosets. We will assume that $1 \in \mathcal{W}_{M_\emptyset}^G$. (2.12)

For sake of completeness we will recall the definition of $\mathcal{W}_{M_\emptyset}^G$. Let $(A_i)_{i \in I}$ be a set of representatives of the H-conjugacy classes of maximal σ-split torus of G. Let us assume that $A_\emptyset$ belongs to this set. The groups A_i are conjugate under G (cf. [7], Proposition 1.16). Let us choose for each i in I, an element x_i of G, such that $x_i.A_\emptyset = A_i$ with $x_\emptyset = 1$. Let M_i be the centralizer of A_i in G. If L is a subgroup of G, one denotes by $W_L(A_i)$ the quotient of the normalizer in L of A_i by its centralizer. We will denote $W_G(A_i)$ simply by $W(A_i)$.

Let $\mathcal{W}_i$ be a set of representatives in $N_G(A_\emptyset)$ of $W(A_\emptyset)/W_{H_i}(A_\emptyset)$ where $H_i = x_i^{-1}.H$. Then ([7], Theorem 3.1) one can take $\mathcal{W}_{M_\emptyset}^G = \cup_{i \in I} \mathcal{W}_i x_i^{-1}$.

For $g \in G$ we define $\dot{g}$ to be the left coset Hg and we define:

$$\mathcal{X}^G_{M_\emptyset} := \left\{ \dot{x} | x^{-1} \in \mathcal{W}^G_{M_\emptyset} \right\}.$$

3 The G-spaces X_P, the constant terms and the maps $c_{P,Q}$

3.1 The G-spaces X_P

One has (cf. [5], Lemma 9.4):

Let $P = MU$ be a σ-parabolic subgroup of G. The union of the (P, H) open double cosets in G is equal to $G' := \underline{P}\ \underline{H} \cap G$. The set G' is also equal to the set of $g \in G$ such that $g^{-1}.P$ is a σ-parabolic subgroup. (3.1)

Let us prove:

Let $P = MU$ be a σ-parabolic subgroup of G and $g \in G$ such that $g.A_M$ is σ-split. Then $g.P$ is a σ-parabolic subgroup of G. (3.2)

One has $P = P_\nu$ for some $\nu \in \mathfrak{a}'_M$ in the sense of [5], (2.14). Then $g.P_\nu = P_\mu$ where μ is the conjugate of ν by g. Our hypothesis implies that $\sigma(\mu) = -\mu$. This implies that $g.P_\nu$ is a σ-parabolic subgroup as P_μ and $\sigma(P_\mu) = P_{-\mu}$ are opposite parabolic subgroups.

One easily extends [5], Eq. (7.1), by replacing $A_\emptyset$ by A_M, the proof being identical:

Let $P = MU$ be a σ-parabolic subgroup of G. Let y, y' be A_M-good elements of G such that $PyH = Py'H$. Then there exist $m \in M, h \in H$ such that $y' = myh$. (3.3)

We define an equivalence relation $\approx_M$ on $\mathcal{X}^G_{M_\emptyset}$ by $x \approx_M x'$ if and only if $Px^{-1}H = Px'^{-1}H$, which by the above equation is equivalent to $xM = x'M$, as x^{-1}, x'^{-1} are $A_\emptyset$-good. Let $\mathcal{X}^G_M$ be a set of representatives of the equivalences classes of this relation. Let us define

$$\mathcal{W}^G_M := \{ y \in \mathcal{W}^G_{M_\emptyset} | (y^{-1})^{\cdot} \in \mathcal{X}^G_M \}.$$

From the above and from (2.12) one has:

The set $\mathcal{X}^G_M$ is a set of representatives of the open (H, P)-double cosets in G. (3.4)

Lemma 1 *Let $P = MU$ be a semistandard σ-parabolic subgroup of G.*

(i) *The set of elements g of G such that $g.A_M$ is σ-split is denoted $X_M^{Lev} \subset G$. It is left invariant by H. Its quotient by H on the left is denoted by $X_M \subset H\backslash G$. One has $\mathcal{X}_M^G \subset X_M$ and:*

$$X_M = \cup_{x \in \mathcal{X}_M^G} xM \subset H\backslash G,$$

the union being disjoint.

(ii) *For each $x \in \mathcal{X}_M^G$, xM is closed in X.*

(iii) *We endow X_M with the topology induced by the topology of X. Then for each $x \in \mathcal{X}_M^G, xM$ is open and closed in X_M. Moreover the canonical map $(M \cap x^{-1}.H\backslash M) \to xM$, $(M \cap x^{-1}.H)m \mapsto xm$, is an homeomorphism.*

(iv) *For all $x \in X_M$, xP is open in X and $X_M P = X_M U$ is the union of the open orbits of P in X.*

Proof (i) If $g \in X_M^{Lev}$, $g.P$ is a σ-parabolic subgroup [cf. (3.2)]. From (3.1) one has $g^{-1} \in G'$. One deduces from (2.12) and the definition of the relation $\approx_M$ that $g^{-1} \in PyH$ for some $y \in \mathcal{W}_M^G$. From (3.3), one deduces that there exists $m \in M, h \in H$ such that $g^{-1} = myh$. The equality of (i) follows immediately. From (3.3), if x, x' are distinct elements of $\mathcal{X}_M^G$, the sets HxP and $Hx'P$ are disjoint. The disjointness follows.

(ii) Changing H into $x^{-1}.H$, one is reduced to prove (ii) when x is equal to $\dot{1}$. If (m_n) is a sequence in M such that $(\dot{m}_n)$ converges in X to l, then $(\sigma(m_n)^{-1}m_n)$ converges. The Cartan decomposition for $M \cap H\backslash M$ (cf. [2] Theorem 1.1) allows to extract a subsequence of (m_n) denoted again by (m_n) such that $m_n = h_n x a_n \omega_n$, where (ω_n) converges, $a_n \in A_\emptyset$, $x^{-1} \in M$ is $A_\emptyset$-good and $h_n \in M \cap H$. Then using (2.11) one has:

$$\sigma(m_n)^{-1}m_n = \sigma(\omega_n^{-1})\,\sigma(x^{-1})xa_n^2\omega_n, n \in \mathbb{N}.$$

Hence (a_n^2) is convergent and (a_n) is bounded. Extracting again a subsequence we can assume that (a_n) is convergent. This implies that $(M \cap H)m_n$ is convergent in $M \cap H\backslash M$ and l is element of $\dot{1}M$. This proves (ii).

(iii) The fact that xM is closed follows from (ii). As $\mathcal{X}_M^G$ is finite, (i) implies that xM is open in X_M. The last assertion follows from [4], Lemma 3.1(iii).

(iv) From (3.1) to (3.2) and the definition of X_M, one sees that HxP is open in G. This achieves to prove the first assertion of (iv). The second follows from this and from (3.4). □

Definition 1 Let $P = MU$ be a σ-parabolic subgroup of G. Then X_M is a P^--space with the given action of M and with the trivial action of U^-. We define:

$$X_P = X_M \times_{P^-} G.$$

Then X_M identifies to a subset of X_P. If $x \in X_M$, its image in X_P will be denoted by x_P.

If $x, x' \in X_M$ the notation $x \approx_M x'$ will mean that x, x' are in the same M-orbit in X_M. The following assertion follows from the definition of X_P.

Let $x, x' \in X_M$. The following conditions are equivalent:

(i) $x_P G = x'_P G$.
(ii) $xM = x'M$ in other words $x \approx_M x'$. (3.5)

We define $H_P := U^-(M \cap H)$. If $y \in G$, let us denote by σ_y the rational involution of G defined by:

$$\sigma_y(g) = y^{-1}\sigma(ygy^{-1})y,$$

whose fixed point set is equal to $y^{-1}.H$ Moreover σ_y depends only on $\dot{y}$.

Let $x \in X_M \subset H\backslash G$. The stabilizer of x_P in G is equal to $(x^{-1}.H)_P := U^-(M \cap x^{-1}.H)$. (3.6)

Definition 2 Let $a \in A_M$. From (3.5) any element $y \in X_P$ is of the form $y = x_P g$ for a unique element $x \in \mathcal{X}_M^G$ and some element $g \in G$, which is defined up to the left action of $U^-(M \cap x^{-1}.H)$. We see easily from (3.6) that $ay := x_P ag$ is well defined. It defines an action of A_M on X_P which commutes to the right G-action.

From the equality in Lemma 1, one deduces the following equality:

Lemma 2 (i) *One has:*

$$X_P = \cup_{x \in \mathcal{X}_M^G} x_P G, \tag{3.7}$$

the union being disjoint and for $x \in \mathcal{X}_M^G$, the map $g \mapsto x_P g$ goes through a bijection from $U^-(M \cap x^{-1}.H)\backslash G$ to $x_P G$.

(ii) *For $x \in \mathcal{X}_M^G$, $x_P P$ is the unique open orbit in $x_P G$.*

(iii) Let $(X_M)_P$ be the image of X_M in X_P or equivalently the set $\{x_P | x \in X_M\}$. The union of the open P-orbits in X_P is equal to $(X_M)_P P = (X_M)_P U$ and the map from $X_M P = X_M U$ to $(X_M)_P P$ defined by $xu \mapsto x_P u$ is a bijective P-equivariant map.

Proof One deduces (i) from the equality in Lemma 1.

(ii) It follows from (3.6) that $x_P G$ is isomorphic to $U^-(M \cap x^{-1}.H)\backslash G$. Then (ii) follows from the fact that there is a unique open (U^-, P)-double coset in G.

The first part of (iii) is clear. It follows from (3.6) that the map $X_M \times U \to (X_M)_P P$, $(x, u) \mapsto (x_P u)$ is bijective. One checks easily that it is P-equivariant. □

Let P be a standard σ-parabolic subgroup of G. Let us prove:

$$\{a \in A_\emptyset \| \alpha(a)|_{\mathbf{F}} \geq C, \alpha \in \Delta(P_\emptyset \cap M)\} = A_\emptyset^+(P_\emptyset, C)A_M. \tag{3.8}$$

The right hand side is clearly included in the left hand side of the equality to prove. Let a be an element of the left hand side. Let $b \in A_M$ be strictly P-dominant. Then for large $n \in \mathbb{N}$, one has $ab^n \in A_\emptyset^+(P_\emptyset, C)$. Our claim follows.

Proposition 1 *There exists a compact subset Ω of G such that for all σ-parabolic subgroups P of G containing $M_\emptyset$, one has:*

$$X_P = \cup_{x \in \mathcal{X}^G_{M_\emptyset}} x_P A^+_\emptyset A_M \Omega.$$

Proof The claim is true for $P = G$ from the Cartan decomposition for symmetric space (cf. [2] Theorem 1.1). In general one has $G = P^- K_0$ hence

$$X_P = X_M P^- K_0 = X_M K_0 = \cup_{x \in \mathcal{X}^G_M} x_P M K_0.$$

The M-space $xM \subset H\backslash G$ is a symmetric space for M for the involution σ_x restricted to M. As x is $A_\emptyset$ good, $P_\emptyset \cap M$ is a σ_x-parabolic subgroup of M (cf. [5] Lemma 2.2). From the Cartan decomposition for this symmetric space, it is enough to prove the following lemma.

Lemma 3 *The open orbits of $P_\emptyset \cap M$ in xM are the orbits $y(P_\emptyset \cap M)$, where y describes the set of elements in $\mathcal{X}^G_{M_\emptyset}$ such that $y \approx_M x$.*

By conjugating on the left by x^{-1} and changing H into $x^{-1}.H$ one is reduced to prove the lemma for $x = \dot{1}$. Any open $(P_\emptyset \cap M)$-orbit in $(M \cap H)\backslash M$ is of the form $(M \cap H)z(P_\emptyset \cap M)$ where z^{-1} is $A_\emptyset$-good and element of M (cf. (2.12)). As HP is open, the product map $H \times P \to HP$ is open (cf. [4], Lemma 3.1(iii)). Hence, as $HzP_\emptyset = H((H \cap M)z(P_\emptyset \cap M))U$, one sees that $HzP_\emptyset$ is open. Then (2.12) implies the existence of an element y of $\mathcal{X}^G_{M_\emptyset}$ such that:

$$HzP_\emptyset = HyP_\emptyset. \tag{3.9}$$

As $z \in M$, z is A_M-good. As y is also A_M-good, it follows from (3.3) that $z = hym'$ for some $m' \in M$, $h \in H$ and one has $y \approx_M z$. Let us prove:

$$(HzP_\emptyset) \cap HM = Hz(P_\emptyset \cap M).$$

Let $p \in P_\emptyset$ and let us write $p = p'u$ with $p' \in P_\emptyset \cap M$ and $u \in U$. Let us show that $zp'u \in HM$ if and only if $u = 1$. Let $m' := zp' \in M$. If $zp'u \in HM$, there exist $h \in H, m \in M$ such that $m'u = hm$. Then one has

$$h = m'm^{-1}(m.u).$$

Hence both sides of the equality are elements of $H \cap P = H \cap M$. It follows that $u = 1$. Our claim follows.

As $z \in M$, $\dot{z} \approx_M \dot{1}$. Taking into account $y \approx_M z$, one has $y \approx_M \dot{1}$ and one shows similarly that:

$$(HyP_\emptyset) \cap HM = Hy(P_\emptyset \cap M). \tag{3.10}$$

From this and (3.9) one sees that:

$$Hy(P_\emptyset \cap M) = Hz(P_\emptyset \cap M).$$

This shows that any open $P_\emptyset \cap M$-orbit in $\dot{1}M$ has the required form.

Reciprocally from (3.10) one sees that for all $y \in \mathcal{X}^G_{M_\emptyset}$ such that $y \approx_M \dot{1}$, $y(P_\emptyset \cap M)$ is open in X_M as it is equal to the intersection of an open set of X with the open subset $\dot{1}M$ of X_M (cf. Lemma 1(iii)). This proves the Lemma. □

Remark 1 (1) There is a minor change with [14]. Here we are interested to $X = H\backslash G$ but Sakellaridis and Venkatesh study the bigger space $(\underline{H}\backslash\underline{G})(\mathbf{F})$. The space X appears as one of the finitely many G-orbits in $\underline{X}(\mathbf{F})$ and every G-orbit in $\underline{X}(\mathbf{F})$ is of the same type than X.

(2) If $P = MU$ is a standard σ-parabolic subgroup of G, we define Θ_P as the set of simple $A_\emptyset$-roots in the Lie algebra of M which are simple for $P_\emptyset$. Notice that $\Theta_{P_\emptyset} = \emptyset$. We could define also $A_{\Theta_P} = A_P$. Then A_{Θ_P} plays here the role of A_{X,Θ_P} in [14].

3.2 Constant term

Let J be a totally discontinuous group acting continuously on a totally disconnected topological space Y. We will say that the action is smooth if the stabilizer of any element of Y is open and we will denote by $C^\infty(Y)$ the space of functions which are fixed by the right action of some compact open subgroup of G.

Let us recall (cf. [6], Proposition 3.14) the following result.

Let $P = MU$ be a σ-parabolic subgroup of G. Let (π, V) be a smooth G-submodule of $C^\infty(H\backslash G)$. The map $f \to f_P$ is the unique morphism of P-modules from V to the space $C^\infty((M \cap H)\backslash M)$ endowed with the right action of M tensored by $\delta_P^{1/2}$ and the trivial action of U, such that: For all compact open subgroup, J, of G there exists $C > 0$, such that for all $f \in V^J$:

$$f(a) = \delta_P^{1/2}(a) f_P(a), a \in A_M(P^-, C) = A_M \cap A^+_\emptyset(P^-, C), \tag{3.11}$$

where $A^+_\emptyset(P^-, C)$ has been defined in (2.10).

We have a similar statement by replacing the preceding equality by

$$f(a) = \delta_P^{1/2}(a) f_P(a), a \in A^+_\emptyset(P^-, C)$$

We have slightly modified the statement of l.c. by replacing A_0 by A_M and $A_\emptyset$ but unicity still holds due to [6] Equation (3.8). It is useful to introduce:

$$\tilde{f}_P = \delta_P^{1/2} f_P. \tag{3.12}$$

Let us assume that V is of finite length. Let (δ, E) be the unormalized Jacquet module of V. Then there exists a finite family of complex characters $\chi_1, \dots, \chi_r$ of A_M such that

$$(\delta(a) - \chi_1(a)) \dots (\delta(a) - \chi_r(a)) = 0, a \in A_M.$$

From the intertwining properties of the P-module map $f \mapsto \tilde{f}_P$, one deduces that it factors through the module of M. Hence (3.13)

$$(\rho(a) - \chi_1(a)) \dots (\rho(a) - \chi_r(a)) \tilde{f}_P = 0, a \in A_M,$$

where ρ denotes the right regular representation of M on $C^\infty((M \cap H)\backslash M)$.

Theorem 1 *Let $P = MU \subset Q = LV$ be two standard σ-parabolic subgroups of G. If $C \geq 0$, let $A_\emptyset^+(P, Q, C)$ be the set of $a \in A_\emptyset$ such that $|\alpha(a)|_{\mathbf{F}} \geq C$ for all roots α of $A_\emptyset$ in the Lie algebra of $U \cap L$ and $|\alpha(a)|_{\mathbf{F}} \geq 0$ for all roots of $A_\emptyset$ in the Lie algebra of $U_\emptyset \cap L$.*

(i) *There exists a unique G-equivariant map $c_{P,Q}$ from $C^\infty(X_Q)$ to $C^\infty(X_P)$ satisfying the following property:*
For all compact open subgroups J of G, there exists $C > 0$ such that for all $f \in C^\infty(X_Q)$ which is right J-invariant, one has:

$$(c_{P,Q} f)(x_P a) = f(x_Q a), a \in A_\emptyset^+(P, Q, C),\ x \in \mathcal{X}_{M_\emptyset}^G. \tag{3.14}$$

The map does not depend on the choice of $\mathcal{X}_{M_\emptyset}^G$.

(ii) *Let R be an other standard σ-parabolic subgroup of G such that $Q \subset R$. Then one has:*

$$c_{P,R} = c_{P,Q} \circ c_{Q,R}.$$

(iii) *Let $\mathcal{V}$ be a smooth G-submodule of finite length of $C^\infty(X_Q)$. Then there exists a finite family of complex characters $\chi_1, \dots, \chi_r$ such that for all $f \in \mathcal{V}$:*

$$((\lambda(a) - \chi_1(a)) \dots (\lambda(a) - \chi_r(a)) c_{P,Q} f)(x_P g) = 0, x \in \mathcal{X}_M^G, g \in G, a \in A_M,$$

where λ denotes the representation of A_M on $C^\infty(X_P)$ given by (cf. Defintion 2 for the significance of ax):

$$\lambda(a) f(y) = f(ax), a \in A_M, f \in C^\infty(X_P).$$

For the proof we will need two lemmas.

Lemma 4 *Let $x \in X_M$.*

(i) *If $f \in C^\infty(X_Q)$ and $g \in G$, let $f_{x_Q,g}$ be the map $l \mapsto f(x_Q lg)$ viewed as a map on $(x^{-1}.H) \cap L \backslash L$. We define a function $f_{x_Q,P^- \cap L}$ on G by $g \mapsto (f_{x_Q,g})\tilde{}_{P^- \cap L}(1)$, where we use the notation (3.12). It is left invariant by $(x^{-1}.H)_P$ and it is right J-invariant if f is right J-invariant.*

(ii) *The point (i) allows to define a map $c_{P,Q,x} : C^\infty(X_Q) \to C^\infty(x_P G)$ by*

$$(c_{P,Q,x} f)(x_P g) = f_{x_Q,P^- \cap L}(g).$$

It intertwines the right regular representations of G on $C^\infty(X_Q)$ and $C^\infty(x_P G)$.

(iii) *One has*

$$(c_{P,Q,x} f)(x_P mg) = (f_{x_Q,g})\tilde{}_{P^- \cap L}(m), m \in M.$$

(iv) *For all compact open subgroup J of G, there exists $C > 0$ such that for all $x \in \mathcal{X}^G_{M_\emptyset}$, for all $f \in C^\infty(X_Q)$ which is right J-invariant, one has:*

$$(c_{P,Q,x} f)(x_P a) = f(x_Q a), a \in A^+_\emptyset(P, Q, C),\ x \in \mathcal{X}^G_{M_\emptyset}.$$

(v) *We have unicity of the G-maps satisfying the condition above on the sets $A_M \cap A^+_\emptyset(P, Q, C)$.*

Proof (i) Due to the intertwining properties of the constant term map (cf. (3.11)) the map $\varphi \mapsto \tilde{\varphi}_{P^- \cap L}$ intertwines the right regular representations of $P^- \cap L$ on $C^\infty((x^{-1}.H) \cap L) \backslash L)$ and on $C^\infty((x^{-1}.H) \cap M) \backslash M)$, where $U^- \cap L$ acts trivially on the latter space. Also one remarks that $f_{x_Q,vg} = f_{x_Q,g}$ for $g \in G, v \in V$. Altogether this shows (i) and that the map $c_{P,Q,x}$ is well defined. The map $c_{P,Q,x}$ intertwines the right regular representations of G as the equality $(c_{P,Q,x}(\rho(g) f))(x_P g') = (c_{P,Q,x} f)(x_P g' g)$ follows from the definitions. This achieves to prove (ii).

(iii) By (ii), it is enough to prove this for $g = 1$. The intertwining properties of the map $\varphi \mapsto \tilde{\varphi}_{P^- \cap L}$ described above allows to prove (iii).

(iv) By (iii) and from the second equality of (3.11) for $P^- \cap L$ and $J \cap L$, one deduces (iv).

(v) Let us prove unicity for $x = 1$. For general x, one has simply to change H into $x^{-1}.H$. Let c be a G-map satisfying the hypothesis of (v). This is a map from $\mathcal{V} = C^\infty(X_Q)$ to $C^\infty(H_P \backslash G)$. As c is a G-map, it is entirely determined by the linear form ξ on $\mathcal{V}$ defined by:

$$\xi(f) = c(f)(1), f \in \mathcal{V}.$$

This linear form goes through the quotient to the unormalized Jacquet module $(\delta, \mathcal{E})$ of $\mathcal{V}$ for P^-. Let ξ' be the corresponding linear form on $\mathcal{E}$. We denote by j the natural projection from $\mathcal{V}$ onto $\mathcal{E}$. We fix J and let C given by condition (iv). Let a be a strictly P-dominant element contained in $A_M \cap A^+_\emptyset(P, Q, C)$, which obviously exists. Then one has:

$$\xi(\rho(a)f) = \xi'(\delta(a)j(f)) = c(f)(a),\ f \in \mathcal{V}^J.$$

Let us assume that J has an Iwahori factorization for (P, P^-). Then by the generalized Jacquet Lemma of Bernstein (cf. [13] theorem VI.9.1) the map from $\mathcal{V}^J$ to $\mathcal{E}^{J\cap M}$ induced by j is surjective. As $\delta(a)$ induces a bijective endomorphism of $\mathcal{E}^{J\cap M}$, one sees that ξ' is uniquely determined on this space by the second equality above. By varying J one sees that ξ' itself is uniquely determined. The same is true for ξ hence for c. This achieves to prove (v). □

Lemma 5 *Let $x, y \in X_M$. If $x \approx_M y$, one has $c_{P,Q,x} = c_{P,Q,y}$.*

From Lemma 4(v), it is enough to prove the following assertion.

Let J be a compact open subgroup of G. There exists $C > 0$ such that for all $f \in C^\infty(X_Q)$ which is J-invariant

$$(c_{P,Q,x}f)(y_P a) = f(y_Q a),\ a \in A_M \cap A_\emptyset^+(P, Q, C). \tag{3.15}$$

Let $m \in M$ such that $y = xm$. Then $y_P = x_P m$, $y_Q = x_Q m$. By the interwinining properties of $c_{P,Q,x}$ and the commutation of $a \in A_M$ with m, one has

$$(c_{P,Q,x}f)(y_P a) = c_{P,Q,x}(\rho(m)f)(x_P a),\ a \in A_M. \tag{3.16}$$

One remarks that $\rho(m)f$ is fixed by $m.J$. Hence as x satisfies Lemma 4(iv) , there exists $C > 0$ such that for all $f \in C^\infty(X_Q)$ right invariant by J:

$$c_{P,Q,x}(\rho(m)f)(x_P a) = (\rho(m)f)(x_Q a),\ a \in A_M \cap A_\emptyset^+(P, Q, C).$$

As $(\rho(m)f)(x_Q a) = f(y_Q a)$, together with (3.16) this proves (3.15) and the lemma. □

Proof of Theorem 1 (i) We define $c_{P,Q}(f)$ for $f \in C^\infty(X_Q)$ by:

$$(c_{P,Q}f)(x_P g) := (c_{P,Q,x}f)(x_P g),\ x \in \mathcal{X}_M^G.$$

From Lemma 4(iv) and (v), one sees that this is well defined and that it has the required properties including unicity. Also from Lemma 5, $c_{P,Q}$ does not depend on the choice of $\mathcal{X}_M^G$ in $\mathcal{X}_{M_\emptyset}^G$. Also, as changing our choice of $\mathcal{X}_{M_\emptyset}^G$ involves only right multiplication by elements of $M_\emptyset$, one sees that $c_{P,Q}$ even does not depend of the choice of $\mathcal{X}_{M_\emptyset}^G$.

(ii) follows easily from the unicity statement in (i).

(iii) As the right action of G on X_P commutes with the left action of A_M, it is enough to prove the equality for $g = 1$. We use the notation of Lemma 4(i). The map $f \mapsto (\tilde{f}_{x_Q,1})_{P^-\cap L}$ is a P^--map from $\mathcal{V}$ to a P^--submodule of $C^\infty((x^{-1}.H)\cap M\backslash M)$ endowed with the right action of M and the trivial action of U^-. This submodule is a quotient of the unormalized Jacquet module of $\mathcal{V}$ for P^-. Hence it is an M-module of finite length. Then (iii) follows from the definition of $c_{P,Q}$ above and from (3.13) applied to L instead of G. □

4 Neighborhoods at infinity of X_Q and mappings exp_{X_P,X_Q}

4.1 Choice of measures

We fix on G (resp., H, resp., the unipotent radical of a semistandard σ-parabolic $P = MU$ of G) the Haar measure such that its intersections with K_0 is of volume 1. From this we deduce a measure on $H\backslash G$. We choose the Haar measure on M such that:

$$\int_G f(g)dg = \int_{U\times M\times U^-} f(umu^-)\delta_P(m)^{-1}dudmdu^-, \, f \in C_c^\infty(G). \tag{4.1}$$

Also there exists a constant $\gamma(P)$ such that:

$$\int_G f(g)dg = \gamma(P)\int_{M\times U^-\times K_0} f(mu^-k)dmdu^-dk. \tag{4.2}$$

The set $X_M U$ is an open subset of $H\backslash G$ (cf. Lemma 1(iv)) which is right invariant by P. Hence the measure on $H\backslash G$ induces a right P-invariant measure on $X_M U$. But the map $X_M \times U \to X_M U$, $(x, u) \mapsto xu$ is a homeomorphism. As the Haar measure on U has been fixed, there is a canonical measure m_{X_M} on X_M such that:

$$\int_{X_M U} f(y)dy = \int_{X_M\times U} f(xu)dm_{X_M}(x)du, \, f \in C_c(X_P). \tag{4.3}$$

One checks easily that this measure satisfies:

$$\int_{X_M} f(xm)dm_{X_M}(x) = \delta_P(m)^{-1}\int_{X_M} f(x)dm_{X_M}(x), m \in M. \tag{4.4}$$

Let $x \in X_M$. As U^-P is open in G, $x_P P$ is an open set in X_P which depends only on xM. By looking to the stabilizer of x and x_P one sees that the map $xp \mapsto x_P p$ is a well defined continuous bijection between xP and $x_P P$ which depends only on xM hence on $x_P P$. Thus, our choice of P-invariant measure on xP induces "by transport de structure" a P-invariant measure on $x_P P$. We fix on $x_P G$ the G-invariant measure which agrees with this measure on $x_P P$. Hence we have a right invariant measure by G on X_P. We want to deduce from m_{X_M} an M-invariant measure on X_M. This will depend on our choice of $\mathcal{X}_M^G$. If $x \in \mathcal{X}_M^G$, the map $(M \cap x^{-1}.H)\backslash M \to xM$, $(M \cap x^{-1}.H)m \mapsto xm$ is a homeomorphism (cf. e.g. [4] Lemma 3.1(iii)). The measure on X_M determines a measure on $(M \cap x^{-1}.H)\backslash M$. Let us show:

$$\text{The function } \delta_P \text{ is trivial on } M \cap x^{-1}.H. \tag{4.5}$$

The group P is a σ_x-parabolic subgroup of G (cf. [5], Lemma 2.2(iii) where one has to change x in x^{-1}). This implies that δ_P is antiinvariant by σ_x and hence trivial on the fixed points of σ_x. This proves our claim. This determines "par transport de structure" a function denoted $\delta_{P,x}$ on xM. Multiplying the restriction to xM of the canonical quasiinvariant measure m_{X_M} by $\delta_{P,x}$ one gets an M-invariant measure on xM and on $(M \cap x^{-1}.H)\backslash M$. Hence one has:

$$\text{Our choice of } \mathcal{X}_M^G \text{ determines an } M\text{-invariant measure on } X_M. \tag{4.6}$$

It allows to identify $C^\infty(X_M)$ to a subspace of the dual of $C_c^\infty(X_M)$ (we will see later that this subspace of the dual is the full smooth dual, cf. after (8.1)).

One deduces also a measure on $x^{-1}.H$ by conjugacy. Together with our choice of measure on M and on $(M \cap x^{-1}.H)\backslash M$, this determines a measure on $(M \cap x^{-1}.H)\backslash x^{-1}.H$.

We introduce a unitary action $\mathcal{L}$ of A_M (cf. (4.4) for unitarity) on the space $L^2(X_P)$ called normalized action:

$$\mathcal{L}_a f(x) = \delta_P^{1/2}(a) f(ax), x \in X_P, \tag{4.7}$$

where ax is the left action of $a \in A_M$ on $x \in X_P$ of Definition 2.

4.2 Compact open subgroups with a σ-factorization

First we give a definition.

A compact open subgroup J of G is said to have a σ-factorization (resp. strong σ-factorization) for standard σ-parabolic subgroups of G if it satisfies the following conditions:
(i) For every standard σ-parabolic subgroup $P = MU$ of G the product map $J_{U^-} \times J_M \times J_U \to J$ is bijective, where $J_{U^-} = J \cap U^-$, $J_M = J \cap M$, $J_U = J \cap U$.
(ii) Let $A \subset A_\emptyset$ be the maximal σ-split torus of the center of M and let A^- (resp. $A_\emptyset^-$) be the set of its P-(resp. $P_\emptyset$)-antidominant elements. For all a belonging to A^- (resp. $A_\emptyset^-$ for the strong σ-factorization) one has (4.8)

$$aJ_Ua^{-1} \subset J_U, a^{-1}J_{U^-}a \subset J_{U^-}.$$

(iii) One has $J = J_H J_P$, where $J_H = J \cap H$, $J_P = J \cap P$.
(iv) For every σ-parabolic subgroup $P = MU$ of G which contains $P_\emptyset$, $J \cap M$ enjoys the same properties as J for M and $P_\emptyset \cap M$.

From [5] Prop 2.3, there exist arbitrary small compact open subgroups of G with a σ-factorization. We will need the following lemma later.

Lemma 6 *There exists a basis of neighborhoods of the identity in G, $(J'_n)_{n\in\mathbb{N}}$, made of a decreasing sequence of compact open subgroups of G with a strong σ-factorization and such that for all $n \in \mathbb{N}$, J'_n is a normal subgroup of J'_0.*

Proof We keep the notation of [5] Prop. 2.3, Then, as the basis of $\underline{u}_\emptyset$ and $\underline{u}_\emptyset^-$ is made of weight vectors $\underline{a}_\emptyset$, one has:

$$\Lambda \underline{g} = \Lambda \underline{u} \oplus \Lambda \underline{m} \oplus \Lambda \underline{u}^-,$$

where $\Lambda \underline{u} = \Lambda \underline{g} \cap \underline{u}$, $\Lambda \underline{m} = \Lambda \underline{g} \cap \underline{m}$, $\Lambda \underline{u}^- = \Lambda \underline{g} \cap \underline{u}^-$ and $\Lambda \underline{u}$ (resp., $\Lambda \underline{u}^-$) is stable by the adjoint action of $A_\emptyset^-$ (resp., $A_\emptyset^+$). Then one shows as in the proof of [5] proposition 2.3, where only (ii) has to be modified, that there exists a basis of neighborhoods $(J_n)_{n\in\mathbb{N}}$ of the identity in G made of a decreasing sequence of compact open subgroups of G with a strong σ-factorization.

As $\Lambda \underline{g}$ is compact and open in $\underline{g}$, there exists $n_0 \in \mathbb{N}$ such that the adjoint action of J_{n_0} preserves $\Lambda \underline{g}$. Hence by l.c. Lemma 10.1(iii), there exists $N \in \mathbb{N}$ such that for all n greater than N, J_{n_0} normalizes J_n. The sequence (J'_n) defined by $J'_n = J_{N+n}$ has the required properties. □

4.3 Statement of Theorem 2

Theorem 2 *Let $P = MU \subset Q = LV$ two standard σ-parabolic subgroups of G. Let K be a compact open subgroup of G having a strong σ-factorization. Let Ω be as in Proposition 1. We may and will assume that $K \subset \Omega$ and that Ω is biinvariant by K. Let J be a compact open subgroup of G such that for all ω in Ω, $x^{-1} \in \mathcal{W}_{M_\emptyset}^G$, $(x\omega).J \subset K$.*

We define for $C > 0$ and $x \in \mathcal{X}_M^G$:

$$N_{X_Q}(x, P, C) := \cup_{y\in\mathcal{X}_{M_\emptyset}^G, y\approx_M x}\, y_Q A_\emptyset^+(P, Q, C)\Omega.$$

Then there exists $C > 0$ such that:

(i) *The union*

$$N_{X_Q}(P, C) := \cup_{x\in\mathcal{X}_M^G} N_{X_Q}(x, P, C)$$

is disjoint.

(ii) *The subset $N_{X_Q}(P, C)$ of X_Q is right J-invariant. We view $N_{X_Q,J}(P, C) := N_{X_Q}(P, C)/J$ as a subset of X_Q/J. The map $N_{X_Q,J}(P, C) \to X_P/J$ which associates $x_P a\omega J$ to $x_Q a\omega J$ with $x \in \mathcal{X}_{M_\emptyset}^G$, $a \in A_\emptyset^+(P, Q, C)$, $\omega \in \Omega$ is well defined on $N_{X_Q,J}(P, C)$. It is denoted $exp_{X_P,X_Q,J}$. Let R be a σ-parabolic subgroup of G such that $P \subset Q \subset R$. Then the image by $exp_{X_Q,X_R,J}$ of $N_{X_R,J}(P, C)$ is contained in $N_{X_Q,J}(P, C)$.*

(iii) *The map $exp_{X_P,X_Q,J}$ is injective on $N_{X_Q,J}(P, C)$.*

(iv) *As a map from a set of J-invariant subsets of X_Q to a set of J-invariant subsets of X_P, $exp_{X_P,X_Q,J}$ preserves volumes.*

From the definitions, one sees:

Corollary 1 *If $a \in A_L$ and $z \in N_{X_Q,J}(P,C)$, one sees from the definitions that $az \in N_{X_Q,J}(P,C)$ and that:*

$$exp_{X_P,X_Q,J}(az) = aexp_{X_P,X_Q,J}(z), z \in N_{X_Q,J}(P,C).$$

First reduction for the proof of Theorem 2

We will reduce the proof of the theorem to the case where $Q = G$. The proof when $Q = G$ will be done in Sect. 6. Let us assume that the theorem has been proved for $Q = G$. Let us prove it for arbitrary Q.

We will define $exp_{X_P,X_Q,J}$ and prove part (ii) of Theorem 2. We define $N'_{X_Q,J}(P,C) = exp_{X_Q,X,J}(N_{X,J}(P,C))$ which is well defined for C large. Then, from (3.8), the definition of the left A_L-action (cf. Definition 2) and the definition of $exp_{X_Q,X,J}$ one has:

$$N_{X_Q,J}(P,C) = A_L N'_{X_Q,J}(P,C). \tag{4.9}$$

Let $y \in N_{X_Q,J}(P,C)$. By the above equality and the definition of $N'_{X_Q,J}(P,C)$, there exist $a \in A_L$ and $z \in N_{X,J}(P,C)$ such that $y = aexp_{X_Q,X,J}(z)$. Let us prove that $aexp_{X_P,X,J}(z)$ does not depend on the choice of a and z as above.

Let us assume that there exists $a' \in A_L$ and $z' \in N_{X,J}(P,C)$ with $y = a'exp_{X_Q,X,J}(z')$. By choosing $b \in A_L$ sufficiently Q-dominant we can assume that ba, ba' are Q-dominant. As $z \in N_{X,J}(P,C)$ one may write $z = xa_z\omega J$ for some $x \in \mathcal{X}^G_{M_\emptyset}, a_z \in A^+_\emptyset(P,C), \omega \in \Omega$. By abuse of notation, as it may depends on this writing, one defines $baz := xbaa_z\omega J$. One defines similarly $b'a'z'$. Then $baz, ba'z' \in N_{X,J}(P,C)$. From our hypothesis one has:

$$baexp_{X_Q,X,J}(z) = ba'exp_{X_Q,X,J}(z').$$

From Theorem 2(ii) for $Q = G$ (i.e. the definition of $exp_{X_Q,X,J}$), one has:

$$baexp_{X_Q,X,J}(z) = exp_{X_Q,X,J}(baz), ba'exp_{X_Q,X,J}(z') = exp_{X_Q,X,J}(ba'z').$$

From the injectivity in (iii) for $Q = G$, one deduces:

$$baz = ba'z'.$$

One sees from the definition of $exp_{X_P,X,J}$ in (ii) that:

$$exp_{X_P,X,J}(baz) = baexp_{X_P,X,J}(z), exp_{X_P,X,J}(ba'z') = ba'exp_{X_P,X,J}(z').$$

As $baz = ba'z'$, one deduces from this the equality:

$$aexp_{X_P,X,J}(z) = a'exp_{X_P,X,J}(z').$$

This proves our claim and it allows to define

$$exp_{X_P,X_Q,J}(y) := aexp_{X_P,X,J}(z).$$

Let $y = x_Q a\omega J \in N_{X_Q,J}(P,C)$ with $x \in \mathcal{X}^G_{M_\emptyset}$, $a \in A^+_\emptyset(P,Q,C)$. By choosing $b' \in A_L$ sufficiently Q-dominant, one has $a' := b'a \in A^+_\emptyset(P,C)$. Let $b = b'^{-1}$ and $y' = x_Q a'\omega J$ One has $y = by'$ and $y' = exp_{X_Q,X,J}(xa'\omega J)$. Our definition of $exp_{X_P,X_Q,J}$ shows that:

$$exp_{X_P,X_Q,J}(x_Q a\omega J) = bx_P a'\omega J = x_P a\omega J.$$

This achieves to prove that $exp_{X_P,X_Q,J}$ is well defined by the formula given in the theorem. This implies that the image of $N_{X,J}(R,C)$ is contained in $N_{X_Q,J}(P,C)$. This achieves the proof of Theorem 2(ii) and Corollary 1 follows.

(iii) Let $y, y' \in N_{X_Q,J}(P,C)$ with $exp_{X_P,X_Q,J}(y) = exp_{X_P,X_Q,J}(y')$. One wants to prove that $y = y'$. By multiplying y and y' by a sufficiently Q-dominant element of A_L, one may assume that $y, y' \in N'_{X_Q,J}(P,C)$. Then $y = exp_{X_Q,X,J}(z)$, $y' = exp_{X_Q,X,J}(z')$ with $z, z' \in N_{X,J}(P,C)$. From our definition of $exp_{X_P,X_Q,J}$, one deduces the equality:

$$exp_{X_P,X,J}(z) = exp_{X_P,X,J}(z').$$

From the injectivity of $exp_{X_P,X,J}$ one sees that $z = z'$, hence $y = y'$. This achieves to prove (iii).

(iv) One has the equality

$$vol_{X_P}(axJ) = \delta_P(a) vol_{X_P}(xJ), x \in X_P, a \in A_P.$$

Using this equality for P and Q, using Theorem 2 for $Q = G$ and P successively equal to P and Q, and our definition of $exp_{X_P,X_Q,J}$ one deduces (iv) for all Q.

It remains to prove (i). One has $y_P G = x_P G$ if and only if $x \in \mathcal{X}^G_M$ and $y \in X_M$ is such that $x \approx_M y$ (cf. (3.5)). From the "if part" and the definition above of $exp_{X_P,X_Q,J}$, the image of $N_{X_Q,}(x, P, C)$ by $exp_{X_Q,X_P,J}$ is contained in $x_P G$. Then the "only if part" implies (i). □

The following proposition is an easy consequence from the definition in part (ii) of the Theorem above.

Proposition 2 *With the notation of Theorem 2, one has*

$$exp_{X_P,X_Q,J}(exp_{X_Q,X,J}(xJ)) = exp_{X_P,X,J}(xJ), x \in N_{X,J}(P,C).$$

The following assertion is an immediate corollary of the Cartan decomposition for X_Q.

Let $C > 0$. The complementary set in X_Q of the union of $N_{X_Q}(P,C)$ when P describes the maximal standard σ-parabolics is a compact set modulo the action of A_L. (4.10)

5 Eisenstein integrals and some results of Nathalie Lagier

5.1 Eisenstein integrals

Let $P = MU$ be a semi standard σ-parabolic subgroup of G. Let (δ, E) be a unitary irreducible smooth representation of M. Let $\chi \in X(M)_\sigma$ and let $\delta_\chi = \delta \otimes \chi$ and let us denote by E_χ the space of δ_χ. Let $(i_P^G \delta_\chi, i_P^G E_\chi)$ be the normalized parabolically induced representation.

The intertwining linear map from $i_P^G \check{\delta}$ to $(i_P^G \delta)\check{}$ which associates to $\check{v} \in i_P^G \check{\delta}$ the linear form on $i_P^G \delta$ given by the absolutely converging integrals:

$$v \mapsto \int_{U^-} \langle \check{v}(u^-), v(u^-) \rangle du^-, \quad v \in i_P^G \delta \tag{5.1}$$

is an isomorphism.

The restriction of functions to K_0 determines a bijection between $i_P^G E_\chi$ and $i_{K_0 \cap P}^{K_0} E$. If v is an element of $i_{K_0 \cap P}^{K_0} E$, v_χ will denote its unique extension to an element of $i_P^G E_\chi$.

$$\text{Let } \mathcal{V}(\delta, H) = \oplus_{x \in \mathcal{X}_M^G} \mathcal{V}(\delta, x, H) \text{ where } \mathcal{V}(\delta, x, H) = (E')^{M \cap x^{-1}.H}. \tag{5.2}$$

Let $\eta = (\eta_x)_{x \in \mathcal{X}_M^G} \in \mathcal{V}(\delta, H)$. Let J_χ be the subspace of elements of $i_P^G E_\chi$ whose support is contained in $P\mathcal{W}_M^G H$ which is the union of the open (P, H) double cosets in G. One defines a linear form on J_χ by

$$\langle \tilde{\xi}(P, \delta_\chi, \eta), v \rangle = \sum_{x \in \mathcal{X}_M^G} \int_{M \cap x^{-1}.H \backslash x^{-1}.H} \langle \eta_x, v(yx^{-1}) \rangle dy, \quad v \in J_\chi.$$

From [4], Theorem 2.7, one sees that

There exists a non zero product q of functions on $X(M)_\sigma$ of the form $\chi \mapsto \chi(m) - c$, for some $m \in M$ and $c \in \mathbb{C}^*$, such that if $q(\chi) \neq 0$, $\tilde{\xi}(P, \delta_\chi, \eta)$ extends to a unique H-invariant linear form on $i_P^G E_\chi$, denoted by $\xi(P, \delta_\chi, \eta)$. Moreover for every element v of $i_{K_0 \cap P}^{K_0} E$, the map $\chi \mapsto q(\chi) \langle \xi(P, \delta_\chi, \eta), v_\chi \rangle$ extends to a polynomial function on $X(M)_\sigma$. (5.3)

When $\xi(P, \delta_\chi, \eta)$ is defined, one defines for $v \in i_P^G E_\chi$:

$$E(P, \delta_\chi, \eta, v)(\dot{g}) = \langle \xi(P, \delta_\chi, \eta), (i_P^G \delta_\chi)(g) v \rangle, g \in G.$$

Now, one uses (9.3) which extends results of [4] and [12] when the characteristic of **F** is equal to zero to the case where this characteristic is different from 2. From (9.3), [12], Theorem 4(ii), [4], Theorem 2.14 and Equation (2.33), one sees that if $\chi \in X(M)_\sigma$ is such that $Re(\chi\delta_P^{-1/2})$ is strictly P-dominant, $\xi(P, \delta_\chi, \eta)$ is defined and one has:

$$E(P, \delta_\chi, \eta, v)(\dot{g}) = \sum_{x \in \mathcal{X}_M^G} \int_{M \cap x^{-1}.H \backslash x^{-1}.H} \langle \eta_x, v(yx^{-1}g) \rangle dy, \quad g \in G, \quad v \in i_P^G E_\chi \tag{5.4}$$

the integrals being convergent.

5.2 Some results of Nathalie Lagier

One has the following assertion which follows from [16], Theorem IV.1.1. Let $P = MU$, $P' = MU'$ be two σ-parabolic subgroups of G with Levi subgroup M.

There exists $R > 0$ such that if $\chi \in X(M)_\sigma$ satisfies

$$\langle \mathrm{Re}(\chi), \alpha \rangle > R, \alpha \in \Delta(P) \cap \Delta(P'^{-}),$$

the following integrals are convergent: (5.5)

$$(A(P', P, \delta_\chi)v)(g) := \int_{U \cap U' \backslash U'} v(u'g)du', v \in i_P^G E_\chi.$$

Then $A(P', P, \delta_\chi)$ is an intertertwining operator between $i_P^G\delta_\chi$ and $i_{P'}^G\delta_\chi$.

The following results are due to Nathalie Lagier (cf. [12], Theorem 5). We use the notation and hypothesis of the preceding subsection.

Let P be a standard σ-parabolic subgroup of G. Let (a_n) be a sequence in A_M such that $(a_n) \to_P \infty$ i.e. such that for every root α of A_M in the Lie algebra of U, $(|\alpha(a_n)|_F)$ tends to infinity.

Let (δ, E) be a smooth unitary irreducible representation of M and let μ_δ be its central character. Let $\chi \in X(M)_\sigma$. Let us assume that the real part of $\tilde{\chi} := \chi\delta_P^{-1/2}$ is strictly P-dominant and satisfies (5.5) for $P' = P^-$. Let $v \in i_P^G E_\chi$ and $g \in G$. Recall that we have choosen $\mathcal{X}_M^G \subset \mathcal{X}_{M_\emptyset}^G$ such that $\dot{1} \in \mathcal{X}_M^G$. Then one has:

If $\eta \in \mathcal{V}(\delta, x, H)$ with $x \in \mathcal{X}_M^G$ different from $\dot{1}$, one has: (5.6)

$$lim_{n\to\infty}\tilde{\chi}(a_n^{-1})\mu_\delta(a_n^{-1})E(P, \delta_\chi, \eta, v)(\dot{1}a_n g) = 0.$$

and

If $\eta \in \mathcal{V}(\delta, 1, H)$, i.e. $\eta \in E'^{M\cap H}$, one has the equality of

$$lim_{n\to\infty}\tilde{\chi}(a_n^{-1})\mu_\delta(a_n^{-1})E(P, \delta_\chi, \eta, v)(\dot{1}a_n g) \tag{5.7}$$

with

$$\langle \eta, (A(P^-, P, \delta_\chi)v)(g)\rangle,$$

Let ε be the trivial representation of $M_\emptyset$. Let $\chi \in X(M_\emptyset)_\sigma$ be such that the real part of $\tilde{\chi} := \chi\delta_P^{-1/2}$ is strictly $P_\emptyset$-dominant. Let η be the linear form on $\mathbb{C}$ corresponding to 1 and let $x \in \mathcal{X}_M^G$. We consider the Eisenstein integrals for $x^{-1}.H\backslash G$. Then x^{-1} might be viewed as an element of a set $\mathcal{X}_M^G$ for $x^{-1}.H$. We view η as an element of $E'^{M\cap H} = \mathcal{V}(\varepsilon, 1, H)$ and of $E'^{M\cap xx^{-1}.H} = \mathcal{V}(\varepsilon, x^{-1}, x^{-1}.H)$. Let $v \in i_{P_\emptyset}^G\chi$. We denote by $E_x(P_\emptyset, \chi, \eta, v)$ the Eisenstein integral for $x^{-1}.H\backslash G$. Then one has:

$$E(P_\emptyset, \chi, \eta, v)(xg) = E_x(P_\emptyset, \chi, \eta, v)((x^{-1}.H)g), g \in G,$$

as it follows easily from (5.4). Using this, it follows from [12], Theorem 5:

(5.8)

Let ε be the trivial representation of $M_\emptyset$. Let $\chi \in X(M_\emptyset)_\sigma$ be such that the real part of $\tilde{\chi} := \chi\delta_P^{-1/2}$ is strictly $P_\emptyset$-dominant. Let $P = MU$ be a standard σ-parabolic subgroup of G. Let (a_n) be a sequence in A_M such that $(a_n) \to_P \infty$.
Let η be the linear form on $\mathbb{C}$ corresponding to 1. Let $x \in \mathcal{X}_M^G$. With the notation as above, for $v \in V := i_{P_\emptyset}^G\mathbb{C}_\chi$, let $E_v := E(P_\emptyset, \chi, \eta, v)$. Then the sequence $(\tilde{\chi}(a_n^{-1})E_v(xa_n))$ has a limit. If $x \notin \dot{1}P$ this limit is equal to zero. Moreover if $x = \dot{1}$, one has:

$$\lim_{n\to\infty}(\tilde{\chi}(a_n^{-1})E_v(\dot{1}a_n)) = l(v), v \in V$$

where l is a non zero linear form on V.

Actually l is explicit but what is important for us here is that it is non zero.

5.3 Applications of the results of N. Lagier

Lemma 7 *Let $P = MU$ be a standard σ-parabolic subgroup of G. Let (a_n) be a sequence in A_M such that $(a_n) \to_P \infty$. If (g_n) is a sequence in G converging to $g \in G$ and such that for all $n \in \mathbb{N}$, $\dot{1}a_n g_n = \dot{1}a_n$, then g is an element of $H_P = U^-(M \cap H)$.*

Proof One applies (5.7). We use the notation of this result. If J is a compact open subgroup of G, for n large enough $g_n J = gJ$. Hence, if $v \in i_P^G E_\chi$,

$$E(P, \delta_\chi, \eta, v)(\dot{1} a_n g_n) = E(P, \delta_\chi, \eta, v)(\dot{1} a_n g),$$

for n large enough.

First, let δ be the trivial representation of M. One applies (5.7) to v and $(i_P^G \chi(g))v$ in order to deduce from the preceding equality

$$(A(P^-, P, \chi)v)(g) = (A(P^-, P, \chi)v)(1), v \in i_P^G \mathbb{C}_\chi$$

for χ sufficiently P-dominant. If χ is such that $A(P^-, P, \chi)$ is bijective, one deduces the following equality:

$$v(g) = v(1), v \in i_{P^-}^G \mathbb{C}_\chi. \tag{5.9}$$

Let us show that this implies $g \in U^- M$. Let us write $g = p^- k$ with $k \in K_0$ and $p^- \in P^-$. If $k \notin K_0 \cap P^-$, there exists $v \in i_{P^-}^G \mathbb{C}_\chi$ such that $v(k) = 0$ and $v(1) = 1$, as the space of restrictions to K_0 of the elements of $i_{P^-}^G \mathbb{C}_\chi$ is equal to $i_{K_0 \cap P^-}^{K_0} \mathbb{C}$. This is a contradiction to (5.9). Hence $g = u^- m$ with $u^- \in U^-$ and $m \in M$.

Then applying (5.7) to any (δ, E, η), we get similarly the equality:

$$< \delta'(m)\eta, e >=< \eta, e >, e \in E.$$

The abstract Plancherel formula (cf. [3], Sect. 0.2) for $H \cap M \backslash M$ implies $m \in M \cap H$. $\square$

Lemma 8 *Let $P = MU$, $P' = M'U'$ be two standard σ-parabolic subgroups of G. Let (a_n) (resp., (a'_n)) be a sequence in A_M (resp. $A_{M'}$) such that $(a_n) \to_P \infty$ (resp. $(a'_n) \to_{P'} \infty$). Let J be a compact open subgroup of G. Let us assume that there exists $g, g' \in G$ such that for all $n \in \mathbb{N}$, $\dot{1} a_n g J = \dot{1} a'_n g' J$. Then, taking possibly subsequences, one has:*

(i) *for all χ such that the real part of $\tilde{\chi} := \chi \delta_P^{-1/2}$ is strictly $P_\emptyset$-dominant $\tilde{\chi}(a_n^{-1} a'_n)$ has a non zero limit.*
(ii) *The sequence $(a_n^{-1} a'_n)$ is bounded.*
(iii) *One has $P = P'$.*
(iv) *If Q is a σ-parabolic subgroup of G such that $P \subset Q$, one has $\dot{1}_Q a_n g J = \dot{1}_Q a'_n g' J$ for n large.*

Proof (i) For all $n \in \mathbb{N}$, there exists $j_n \in J$ such that

$$\dot{1} a_n g = \dot{1} a'_n g' j_n. \tag{5.10}$$

As J is compact, one may take a subsequence and we may assume that (j_n) converges to $j \in J$. Let $g'' = g' j g^{-1}$. One will apply the result (5.8). With its notations, let $v \in i_{P_\emptyset}^G \mathbb{C}_\chi$ and let us denote by E_v the function $E(P_\emptyset, \chi, \eta, v)$. As E_v is right invariant by an open compact subgroup of G, one has $E_v(\dot{1}a'_n g' j_n g^{-1}) = E_v(\dot{1}a'_n g'')$ for n large. From (5.8), one has:

$$lim_n \tilde{\chi}(a_n^{-1})E_v(\dot{1}a_n) = l(v), lim_n \tilde{\chi}(a_n'^{-1})E_v(\dot{1}a'_n g' j_n g^{-1}) = l'(v) \tag{5.11}$$

where l, l' are non zero linear forms on $i_{P_\emptyset}^G E_\chi$. Also from (5.10) one has:

$$\dot{1}a_n = \dot{1}a'_n g' j_n g^{-1}. \tag{5.12}$$

Let us show that there exists $v_1 \in V = i_{P_\emptyset}^G \mathbb{C}_\chi$ such that $l(v_1)$ and $l'(v_1)$ are non zero. Let $v \in V$ such that $l(v) \neq 0$. Then l does not vanish on $v + \mathrm{Ker}(l)$. If l' vanished identically on $v + \mathrm{Ker}(l)$ it would vanish on V, a contradiction which shows that l' does not vanish identically on $v + \mathrm{Ker}(l)$. This proves our claim.

For such a v_1, one sees from (5.11) to (5.12) that:

$$\text{The sequence } (\tilde{\chi}(a_n a_n'^{-1})) \text{ tends to a non zero limit.} \tag{5.13}$$

This proves (i).

(ii) By varying χ such that $\chi = Re\chi$ and such that $Re\chi$ describes a basis of $\mathfrak{a}_\emptyset^*$ one gets (ii).

(iii) If P is different from P', by exchanging possibly the role of P and P', there exists a simple root α of $A_\emptyset$ in the Lie algebra of U which is not a root in the Lie algebra of U', hence which is a root in the Lie algebra of M'. Then $|\alpha(a'_n)|_{\mathbf{F}} = 1$ and $|\alpha(a_n a_n'^{-1})|_{\mathbf{F}}$ is unbounded. This would contradict (ii). Hence $P = P'$ and (iii) is proved.

(iv) From (ii), one writes $a'_n = a_n b_n$ where the sequence (b_n) in A_M is bounded. Taking a subsequence we can assume that (b_n) converges to $b \in A_M$.

Taking into account (5.10), one has $\dot{1}a_n = \dot{1}a_n c_n$ where $c_n = b_n g' j_n g^{-1}$. One deduces from Lemma 7 that the limit c of (c_n) is in H_P. One has $a'_n g' J = a_n c_n g J$. Hence for n large one has:

$$\dot{1}_P a'_n g' J = \dot{1}_P a_n c_n g J = \dot{1}_P a_n c g J.$$

As $c \in H_P$ and as $a_n \in A_M$ normalize H_P, one deduces that, for n large:

$$\dot{1}_P a'_n g' J = \dot{1}_P a_n g J.$$

This proves (iv) for $Q = P$.

Let $g, g' \in G$. In view of Theorem 1, applied to the right translates of f by g, g', there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ greater than N and for all $f \in C^\infty(H_Q\backslash G)$ which is J-invariant $(c_{P,Q} f)(\dot{1}_P a_n g) = f(\dot{1}_Q a_n g)$ and $(c_{P,Q} f)(\dot{1}_P a'_n g') = f(\dot{1}_Q a'_n g')$. Let f be the characteristic function of $\dot{1}_Q a_n g J \subset X_Q$. Let n be an integer greater than N and let $x = a_n g$, $x' = a'_n g'$. From the above remark, one has:

$$(c_{P,Q}f)(\dot{1}_P x) = f(\dot{1}_Q x) = 1,$$
$$(c_{P,Q}f)(\dot{1}_P x') = f(\dot{1}_Q x').$$

From (iv) for $Q = P$, one has $\dot{1}_P x' J = \dot{1}_P x J$. By J-invariance, this implies:

$$(c_{P,Q}f)(\dot{1}_P x') = (c_{P,Q}f)(\dot{1}_P x).$$

Hence, one has

$$f(\dot{1}_Q x') = 1$$

and $x' \in \dot{1}_Q x J$. This implies $\dot{1}_Q x J = \dot{1}_Q x' J$. □

Lemma 9 *Let $P = MU$, $P' = M'U'$ be two standard σ-parabolic subgroups of G. Let (a_n) (resp., (a'_n)) be a sequence in A_M (resp., $A_{M'}$) such that $(a_n) \to_P \infty$ (resp., $(a'_n) \to_{P'} \infty$). Let $g, g' \in G$ and $x, y \in \mathcal{X}^G_{M_\emptyset}$. Let us assume that the sequences $(x a_n g J)$ and $(y a'_n g' J)$ are equal. Then one has $P = P'$, $xP = yP$ and $y = xm$ for some $m \in M$.*

Proof Let $\chi \in X(M_\emptyset)_\sigma$ such that $\chi = |\chi|$ and such that $\mathrm{Re}(\tilde{\chi})$ is strictly $P_\emptyset$-dominant. By exchanging possibly the role of x and y, and by taking a subsequence, one may assume that $\tilde{\chi}(a_n) \geq \tilde{\chi}(a'_n)$. Changing H into $x^{-1}.H$, one is reduced to the case where $x = \dot{1}$. Using the notation and the result of (5.8), one sees that there exists a non zero linear form l on $V_\chi := i^G_{P_\emptyset}\mathbb{C}_\chi$ such that for all $v \in V_\chi$, one has:

$$lim_n \tilde{\chi}(a_n)^{-1} E_v(\dot{1} a_n g) = l(v). \tag{5.14}$$

Let $v \in V_\chi$ such that $l(v) \neq 0$. One chooses $j_n \in J$ such that $\dot{1} a_n g = y a'_n g' j_n$. By extracting a subsequence, one may assume that (j_n) converges to $j \in J$. One has:

$$E_v(\dot{1} a_n g) = E_v(y a'_n g' j) \text{ for } n \text{ large} \tag{5.15}$$

and $\tilde{\chi}(a_n) \geq \tilde{\chi}(a'_n)$. Let us assume $y \notin \dot{1} P'$. Then from (5.8)

$$lim_n \tilde{\chi}(a'_n)^{-1} E_v(y a'_n g' j) = 0.$$

Together with (5.15) this contradicts (5.14). Hence $y \in \dot{1} P'$ which implies (cf. (3.3)) $y = \dot{1} m'$ for some $m' \in M'$. This implies the equality $y a'_n g' = \dot{1} a'_n m' g'$ as $a'_n \in A_{M'}$. Hence one has $\dot{1} a_n g J = \dot{1} a'_n m' g' J$. Using Lemma 8, one sees that $P = P'$. Hence $M = M'$ and the lemma follows. □

6 End of Proof of Theorem 2

6.1 Definition of $exp_{X_P,X,J}$

We have to deal only with the case $Q = G$ i.e. $X_Q = X$. If it does not exist a constant $C > 0$ satisfying (i) of Theorem 2 for $Q = G$, there would exist $x, y \in \mathcal{X}^G_{M_\emptyset}$, and sequences (a_n), $(a'_n) \in A^+_\emptyset$, (ω_n), $(\omega'_n) \in \Omega$ such that $xP \neq yP$, $(|\alpha(a_n)|_{\mathbf{F}})$ tends to infinity for all roots α of $A_\emptyset$ in the Lie algebra of U and such that:

$$x a_n \omega_n J = y a'_n \omega'_n J, n \in \mathbb{N}.$$

By extracting subsequences, one may assume that ω_n (resp., ω'_n) converges to ω (resp. ω'). Let $Q = LV$ be the standard σ-parabolic subgroup of G such that for $\alpha \in \Delta(P_\emptyset)$, the sequence $(|\alpha(a_n)|_{\mathbf{F}})$ is unbounded if and only if $\alpha \in \Delta(Q, A_\emptyset)$. Clearly one has $Q \subset P$.

By extracting subsequences, one will show that one can write $a_n = b_n c_n$ where the sequence (b_n) in A_L satisfies $(b_n) \to_Q \infty$ and where the sequence (c_n) converges in G. Let $(\delta_1, \dots, \delta_p)$ be the union of $\Delta(Q, A_\emptyset)$ viewed as subset of $\mathfrak{a}'_\emptyset$ and of a basis of $\mathfrak{a}'_G$ viewed as a subset of $\mathfrak{a}'_\emptyset$ (cf. (2.9)). Let us look to the map $\phi : A_\emptyset \to \mathbb{R}^p$ given by $a \mapsto (\delta_1(H_\emptyset(a)), \dots, \delta_p(H_\emptyset(a))$. Its image is a lattice of dimension p as the image $\mathfrak{a}_{\emptyset,\mathbf{F}}$ of $A_\emptyset$ by $H_\emptyset$ is a lattice of dimension equal to the dimension of $\mathfrak{a}_\emptyset$. Its restriction to A_L has the same property as it factors through H_L and $(\delta_1, \dots, \delta_p)$ might be viewed as a basis of $\mathfrak{a}'_L$. Hence $\phi(A_L)$ is of finite index in $\phi(A_\emptyset)$. Hence one can find $x_1, \dots, x_q \in A_\emptyset$ such that for all $a \in A_\emptyset$ there exists $b \in A_L$ and $i \in \{1, \dots, q\}$ such that $\phi(a) = \phi(bx_i)$. This allows to define b_n and $c_n = a_n(b_n)^{-1}$. One has $c_n = x_{i_n}$ for some $i_n \in \{1, \dots, q\}$. Then extracting a subsequence one may even assume that (c_n) is constant hence it converges. Moreover as $\phi(b_n) = \phi(a_n) - \phi(x_{i_n})$ one has $(b_n) \to_Q \infty$.

Hence, for n large, $x a_n \omega_n J = x b_n c \omega J$ where c is the limit of (c_n). We introduce similarly Q', b'_n and c'_n. From Lemma 9 applied to G one deduces $Q' = Q$ and $xQ = yQ$. Hence, as $Q \subset P$, one has $xP = yP$. A contradiction which shows that there exists $C > 0$ which satisfies (i). It is clear that any constant greater than such a constant enjoys the same property.

Let us assume that there is no constant satisfying (i) which satisfies also (ii). Proceeding as above, there would exist two standard σ-parabolic subgroups $Q = LV$, $Q' = L'V' \subset P$ of G , two sequences (b_n) in A_L, (b'_n) in $A_{L'}$, $c, c' \in G$ and $x, y \in \mathcal{X}^G_{M_\emptyset}$ such that, $(b_n) \to_Q \infty$, $(b'_n) \to_{Q'} \infty$, and

$$\begin{aligned} x b_n c J &= y b'_n c J. \\ x_P b_n c J &\neq y_P b'_n c J. \end{aligned} \tag{6.1}$$

From Lemma 9, one sees that $Q = Q'$ and $x \approx_L y$. In particular $y = xl$ for some $l \in L$ and, as l commutes to the elements (b'_n) of $A_{L'}$, one has:

$$xb_n cJ = xb'_n lc' J.$$

Conjugating by x^{-1}, one gets an equality of left $x^{-1}.H$ cosets. From Lemma 8(i), (ii) and (iii), applied to $x^{-1}.H$ instead of H, one deduces that $(b'_n {b_n}^{-1})$ is bounded. Hence, by taking a subsequence one can assume that it has a limit. Then from Lemma 8(iv) one gets for n large:

$$(x^{-1}.H)_P b_n cJ = (x^{-1}.H)_P b'_n lc' J.$$

Hence there exists a sequence in J, (j_n) such that

$$(x^{-1}.H)_P b_n c j_n = (x^{-1}.H)_P b'_n lc'.$$

Hence $b_n c j_n c'^{-1} l^{-1} b'^{-1}_n \in (x^{-1}.H)_P$. As the stabilizer of x_P is equal to $(x^{-1}.H)_P$, one deduces from this the equality:

$$x_P b_n c j_n = x_P b'_n lc'.$$

As $y = xl$ and $l \in L \subset M$, one has $y_P = x_P l$. As $l \in L$ commutes to $b'_n \in A_L$, one deduces from this the equality

$$x_P b_n cJ = y_P b'_n c' J,$$

for n large. This contradicts our hypothesis (6.1). Hence there exists $C > 0$ which satisfies (i) and (ii).

6.2 Injectivity of $exp_{X_P, X, J}$

Let us prove that one can choose $C > 0$ such that $exp_{X_P, X}$ is injective on $N_{X,J}(P, C)$. Let us assume that every constant $C > 0$ satisfying conditions (i), (ii) of Theorem 2 does not satisfy condition (iii). From the finiteness of $\mathcal{X}^G_{M_\emptyset}$ and proceeding as in Sect. 6.1, one sees that there would exist $x, x' \in \mathcal{X}^G_{M_\emptyset}$, two σ-parabolic subgroups $Q = LV$, $Q' = L'V' \subset P$ of G, a sequence (a_n) in A_L, a sequence (a'_n) in $A_{L'}$ such that $(a_n) \to_Q \infty$, $(a'_n) \to_{Q'} \infty$ and two elements d and d' of $A_\emptyset \Omega$ such that:

$$xa_n dJ \neq x' a'_n d' J$$

and

$$x_P a_n dJ = x'_P a'_n d' J.$$

Let f_n be the characteristic function of $xa_n dJ$. For n_0 large enough one can use Theorem 1(i) for the right translates of f_{n_0} by d and d' and one has, by setting $a = a_{n_0}$, $f = f_{n_0}$, etc.:

$$f(xad) = (c_{P,G} f)(x_P ad), \; f(x'a'd') = (c_{P,G} f)(x'_P a'd').$$

But, by our assumptions $f(xad) = 1$ and $x_P adJ = x'_P a'd'J$. Hence, by J invariance, one has:

$$f(x'a'd') = 1,$$

which implies

$$xadJ = x'a'd'J.$$

This is a contradiction to our hypothesis. This achieves to prove that there exists a constant $C > 0$ such that the properties (i), (ii) and (iii) of Theorem 2 are satisfied.

6.3 Volumes

The following lemma will allow to finish the proof of Theorem 2.

Lemma 10 *Let K be a compact open subgroup of G with a strong σ-factorization for standard σ-parabolic subgroups (cf. (4.8)). Let $P = MU$ be a standard σ-parabolic subgroup of G. Let $a \in A_\emptyset$ which is $P_\emptyset$-dominant. Then*

(i)

$$HaK = HaK_M K_U.$$

where $K_M = K \cap M$, $K_U = K \cap U$.

(ii)

$$vol_X \dot{1}aK = vol_{X_P} \dot{1}_P aK.$$

Proof (i) As $K_{M_\emptyset} K_{U_\emptyset} = K \cap P_\emptyset$ and $K_M K_U = K \cap P$, it is enough to prove (i) when $P = P_\emptyset$. Let us assume this in the sequel. If $u^- \in K_{U^-}$, as a is $P_\emptyset$-dominant, one has $a.u^- = au^-a^{-1} \in K_{U^-} \subset K = K_H K_M K_U$ (cf. (4.8)(ii) and (iii)). Hence one has:

$$Hau^- = H(a.u^-)a \in HK_M K_U a.$$

But $K_M K_U a = a(a^{-1}.K_M)(a^{-1}.K_U)$. As $M = M_\emptyset$ and $a \in A_\emptyset$, $a^{-1}.K_M = K_M$. As a is $P_\emptyset$-dominant $a^{-1}.K_U \subset K_U$ (cf. (4.8)(ii)). Altogether, this shows:

$$HaK_{U^-} \subset HaK_M K_U.$$

One deduces (i) from the equality $K = K_{U^-} K_M K_U$.

Let us prove (ii). Let P be a standard σ-parabolic subgroup of G. As $U^- \subset H_P$ and $K = K_{U^-} K_M K_U$, and $a.K_{U^-} \subset U^-$, (cf. (4.8)(ii)) one has:

$$\dot{1}_P aK = \dot{1}_P aK_M K_U.$$

Then (ii) follows from (i), from the fact that $aK_M K_U \subset P$ and from our choice of measure on X_P (cf. Sect. 4.1). $\square$

End of proof of Theorem 2 Let K and J be as in the theorem. Using (4.9) and Proposition 2, the proof of (iv) reduces to prove the statement for subsets of $N_X(x, P, C)/J$ for $x \in \mathcal{X}_M^G$. Using our choices of volumes and translating sets on the left by x^{-1} and changing H in $x^{-1}.H$, one is reduced to the case $x = 1$. For K and J as in the theorem, we have:

$$\omega.J \subset K, \omega \in \Omega.$$

Let $\omega \in \Omega$ and one sets $J' := \omega.J \subset K$. As Ω is compact and is left K-invariant, Ω/J is finite and J' varies in a finite set. Let us assume that $C > 0$ satisfies Theorem 2(i), (ii) and (iii) for all groups J' . One has to prove that for $a \in A_\emptyset^+(P, C)$:

$$vol_X(\dot{1}a\omega J) = vol_{X_P}(\dot{1}_P a\omega J).$$

As the measures on X and X_P are right invariant by G, in order to prove this equality, it is enough to prove the equality:

$$vol_X(\dot{1}aJ') = vol_{X_P}(\dot{1}_P aJ').$$

Let K_a (resp., K'_a) be the stabilizer in K of $\dot{1}a$ (resp., $\dot{1}_P a$). We need the following fact. Let K_1 be a closed subgroup of K. Let us assume that a Haar measure is given on K and let $K_1\backslash K$ be endowed with the image of this measure. Let $X \subset K$ and Y its image in $K_1\backslash K$. Then $vol_{K_1\backslash K}(Y) = vol_K(K_1 X)$. From this applied to $K_1 = K_a$ and $K_1 = K'_a$ and from Lemma 10(ii), it is enough to prove the equality:

$$K_a J' = K'_a J'.$$

The image of the set $\dot{1}aK'_a J'$ by the map $exp_{X_P,X,J'}$ is equal to $\dot{1}_P aJ'$, as it follows from the definition in Theorem 2 and the equality $\dot{1}_P aK'_a J' = \dot{1}_P aJ'$. From the definition of $exp_{X_P,X,J'}$, this image is also equal to the image of $\dot{1}aJ'$. Hence from the part (iii) of Theorem 2, one deduces the equality:

$$\dot{1}aJ' = \dot{1}aK'_a J'.$$

Looking to the orbit of $\dot{1}a$ under K one deduces from this the inclusion:

$$K'_a J' \subset K_a J'.$$

We recall that $K \subset \Omega$. To prove the reverse inclusion let us remark that $\dot{1}aK_a J'$ is equal to $\dot{1}aJ'$. From the definition of $exp_{X_P,X,J'}$ one deduces the equality:

$$\dot{1}_P aK_a J' = \dot{1}_P aJ'$$

which implies as above:

$$K_a J' \subset K'_a J'.$$

This implies the required equality. This finishes the proof of the theorem. □

6.4 Eventual equivariance

It is good in Theorem 2 to emphasize on the dependence of $N_{X_Q,J}(P,C)) \subset X_Q$ on Ω by denoting it $N_{X_Q,J}(P,C,\Omega)$. If it satisfies the properties (ii), (iii), (iv) of Theorem 2 and Proposition 3 it will be said J-good. We want to prove the analogue of the eventual equivariance of Proposition 4.3.3.

Lemma 11 *Let h be an element of $C_c^\infty(G)$ biinvariant by J and let S be its support. Let $N = N_{X_Q,J}(P,C,\Omega)$ be good and let Ω' be a compact subset of G which contains $\Omega S S^{-1} \cup \Omega S \cup \Omega S^{-1} \cup \Omega$ and satisfies the properties of Ω in Theorem 2.. Let $C' \leq C$ such that $N' = N_{X_Q,J}(P,C',\Omega')$ is good and let $N'' = N_{X_Q,J}(P,C',\Omega) \subset N$.*

Let ϕ be the map $exp_{X_P,X_Q,J}$ from N' to X_P/J. The image of N' (resp. N'') is denoted N'_P (resp. N''_P). If f is in $C_c^\infty(N'_P)$ let $\phi^* f = f \circ \phi \in C_c^\infty(N')$. We view f (resp. $\phi^* f$) as an element of $C_c^\infty(X_P)^J$ (resp. $C_c^\infty(X_Q)^J$).

(i) Let $f \in C_c^\infty(N''_P) \subset C_c^\infty(N'_P)$. Then $f \star h$ has compact support in N'_P and hence is an element $C_c^\infty(N'_P)$. Moreover one has:

$$\phi^*(f \star h) = (\phi^* f) \star h.$$

(ii) Let $f \in C_c^\infty(N'_P)$. Then

$$(f \star h) \circ \phi_{|N''} = [(f \circ \phi) \star h]_{|N''}$$

Proof (i) One has $\Omega S \subset \Omega'$ and the definition of ϕ implies $N''_P S \subset N'_P$. The assertion on the support in (i) follows. Also one has for $z \in N''_P S$ and $x \in S$, $\phi(zx^{-1}) = \phi(z)x^{-1}$. Then a direct computation leads to the equality in (i).

(ii) is proved in a similar way. □

7 Bernstein maps and scattering theorem

7.1 Constant term and exp-mappings

The following proposition is an immediate corollary of Theorems 1 and 2.

Proposition 3 *Let $P \subset Q$ be two standard σ-parabolic subgroups of G. Let J be a compact open subgroup of G small enough to satisfy the conditions of Theorem 2.*

There exists $C > 0$ *such that* $exp_{X_P, X_Q, J}$ *is well defined on* $N_{X_Q,J}(P, C)$ *and satisfies for all* J*-invariant function* f *on* X_Q*:*

$$(c_{P,Q} f)(exp_{P,Q,J}(xJ)) = f(xJ), xJ \in N_{X_Q,J}(P, C).$$

Remark 2 In [14], for G split and X spherical, the exp-mappings are introduced before the maps $c_{P,Q}$, by means of wonderful compactifications, and the maps $c_{P,Q}$ are defined by the relation above.

7.2 Bernstein maps $e_{Q,P}$

We thank Joseph Bernstein for having suggested to us the proof of the following Theorem.

Theorem 3 *Let* $P = MU \subset Q = LV$ *two standard* σ*-parabolic subgroups of* G*. The right* G*-invariant measure on* X_P *allows to identify* $C_c^\infty(X_P)$ *to a subspace of the dual of* $C^\infty(X_P)$*. Let* $e_{Q,P}$ *be the restriction of the transpose map of* $c_{P,Q}$ *to* $C_c^\infty(X_P)$*.*
Let J *and let* $C > 0$ *be as in Theorem 2.*

(i) *Let* $xJ \in N_{X_Q,J}(P, C)$ *and* $y = exp_{X_P,X_Q,J}(xJ)$*. Then the image by* $e_{Q,P}$ *of the characteristic function of* $yJ \subset X_P$ *is the characteristic function of* $xJ \subset X_Q$*.*
(ii) *For* $f \in C_c^\infty(X_P)$ *supported in* $exp_{X_P,X_Q,J}(N_{X_Q,J}(P, C))$*,* $e_{Q,P} f$ *has its support in* $N_{X_Q,J}(P, C)$ *and*

$$(e_{Q,P} f)(xJ) = f(exp_{X_P,X_Q}(xJ)), xJ \in N_{X_Q,J}(P, C).$$

(iii) *The map* $e_{Q,P}$ *has its image in* $C_c^\infty(X_Q)$*.*

Proof (i) We fix a compact open subgroup J and C as in the preceding proposition from which we use the notations. Let $xJ \in N_{X_Q,J}(P, C) \subset X_Q/J$. Let f be the characteristic function of $exp_{X_P,X_Q,J}(xJ)$ which is a J-invariant function on X_P. Let $g \in C^\infty(X_Q)^J$. One has

$$\langle e_{Q,P} f, g\rangle = \langle f, c_{P,Q} g\rangle$$

and by the preceding proposition one sees:

$$\langle e_{Q,P} f, g\rangle = g(xJ).$$

This implies that $e_{Q,P} f$ is the characteristic function of xJ. This proves (i).
(ii) follows by linear combinations.
(iii) Let $a \in A_M$ be strictly P-dominant. Let $y \in X_P$. From the Cartan decomposition for X_P one sees that for n large, $a^n y$ is of the form $a^n y = exp_{X_P,X_Q,J}(x_n J) \in X_P/J$ for some $x_n J \in N_{X_Q,J}(P, C) \subset X_Q/J$.

For $n \in \mathbb{Z}$, let f_n be the characteristic function of $a^n yJ \subset X_P$. One has just seen that for n large in $\mathbb{N}$, $e_{Q,P}(f_n)$ is in $C_c^\infty(X_Q)^J$. Let us assume that it is not true for all

$n \in \mathbb{N}$. Then there would exist $N \in \mathbb{N}$ such that $e_{Q,P}(f_n) \in C_c^\infty(X_Q)^J$ for $n > N$ and such that $e_{Q,P}(f_N) \notin C_c^\infty(X_Q)^J$.

We want to apply Theorem A of [1] in order to prove that the $C_c^\infty(G)^J$-module $C_c^\infty(X_P)^J$ is finitely generated. For this it is necessary to see that one may apply it to each homogeneous space $x_P G$ which is isomorphic to $U^-(M \cap x^{-1}.H)\backslash G$. The first thing to prove is that for each parabolic subgroup R of G, the number of $(U^-(M \cap x^{-1}.H), R)$-double cosets is finite. By using conjugacy, one can assume that R contains A_0. By the Bruhat decomposition, one has $G = \cup_i P x_i R$, where (x_i) is a finite family of elements of G normalizing A_0. It is enough, to prove our claim, to show that for each i, $R_i := (x_i.R) \cap M$ has a finite number of orbits in the symmetric space $(M \cap x^{-1}.H)\backslash M$. But R_i is a parabolic subgroup of L and our claim follows from [8], Corollary 6.16.

The second thing to prove, in order to apply Theorem A of [1] is that :

For each finite length smooth G-module V, the dimension of the space $V'^{U^-(M\cap x^{-1}.H)}$ is finite. (7.1)

But this dimension is precisely the dimension of $j(V)'^{M\cap x^{-1}.H}$ where $j(V)$ is the Jacquet module of V with respect to P^-. This space is finite dimensional (cf. [6], Theorem 4.4.)

Now, one can apply Theorem A of [1] to conclude that the $C_c^\infty(G)^J$-module $C_c^\infty(X_P)^J$ is finitely generated. Moreover the algebra $C_c^\infty(G)^J$ is Noetherian (cf. [13] Corollary of Theorem VI.10.4).

Hence, it follows that an ascending chain of $C_c^\infty(G)^J$-submodules of $C_c^\infty(X_P)^J$ is stationnary.

We apply this to the $C_c^\infty(G)^J$-submodules of $C_c^\infty(X_P)^J$, M_n, generated by $f_0, \dots f_{-n}$. Hence there exists $n \in \mathbb{N}$ and $\phi_0, \dots \phi_n \in C_c^\infty(G)^J$ such that:

$$f_{-n-1} = f_0 * \phi_0 + \cdots + f_{-n} * \phi_n.$$

Using that the right G-action and the left A_M-action commute (cf. Definition 2) and applying the left action of a^{n+1+N} to the above identity, one gets:

$$f_N = f_{n+1+N} * \phi_0 + \cdots + f_{1+N} * \phi_n.$$

From Theorem 1, $c_{P,Q}$ is a morphism of G-modules. Hence it is also the case for $e_{Q,P}$. Hence $e_{Q,P}(f_N)$ is in $C_c^\infty(X_Q)^J$. From the definition of N, we get a contradiction. Hence in particular, $e_{Q,P} f$ is in $C_c^\infty(X_Q)^J$. The theorem follows by linearity.

Let (π, V) be a smooth representation of a parabolic subgroup $P = MU$ of G. One denotes by (π_P, V_P) the tensor product of the quotient of V by the M-submodule generated by the $\pi(u)v - v$, $u \in U$, $v \in V$, with the representation of M on $\mathbb{C}$ given by $\delta_P^{-1/2}$. We call it the normalized Jacquet module of V along P. We denote the natural projection map from V to V_P by j_P and sometimes π_P will be denoted $j_P(\pi)$.

Lemma 12 *Let P be a semistandard σ-parabolic subgroup of G.*

(i) If $f \in C_c^\infty(X)$ has its support in $X_M P$ we define, using (4.5), $f^P \in C^\infty(X_M)$ by

$$f^P(xm) = \delta_P^{1/2}(m) \int_U f(\dot{x}mu)du, x \in \mathcal{X}_M^G, m \in M.$$

Then $f^P \in C_c^\infty(X_M)$.

(ii) The map $f \mapsto f^P$ goes through the quotient to an intertwining map between the normalized Jacquet module $C_c^\infty(X_M P)_P$ of the P-module $C_c^\infty(X_M P)$ and $C_c^\infty(X_M)$.

(iii) This intertwining map is bijective and its inverse defines an intertwining injective map $m_P^X : C_c^\infty(X_M) \to C_c^\infty(X)_P$.

(iv) One can replace X by X_P in (i), (ii) and (iii) and one gets an injective intertwining map $m_P : C_c^\infty(X_M) \to C_c^\infty(X_P)_P$.

Proof (i) follows easily from the definition.

(ii) It is clear that our map goes through the quotient to a map between the normalized Jacquet module $C_c^\infty(X_M P)_P$ of the P-module $C_c^\infty(X_M P)$. On the other hand, for $f \in C_c^\infty(X_M P)$ one has:

$$(\rho(m_0)f)^P(xm) = \delta_P^{1/2}(m) \int_U f(xmm_0 m_0^{-1}um_0)du.$$

One makes the change of variable $u' = m_0^{-1}um_0$ to achieve to prove the intertwining property of (ii).

As an U-space, $X_M P$ is isomorphic to $X_M \times U$ where U acts trivially on the first factor. This implies easily (iii).

(iv) is proved similarly. □

Proposition 4 *We denote by $j_P(e_P)$ the map between the normalized Jacquet modules $C_c^\infty(X_P)_P$ and $C_c^\infty(X)_P$ determined by $e_P := e_{G,P}$. Then*

$$j_P(e_P) \circ m_P = m_P^X.$$

Proof One has to prove;

$$j_P(e_P)(m_P(f)) = m_P^X(f) \tag{7.2}$$

for all $f \in C_c^\infty(xM)$ and $x \in \mathcal{X}_M^G$. Changing H to $x^{-1}.H$, one is reduced to prove (7.2) for $x = 1$. One writes the Cartan decomposition for $M \cap H\backslash M$:

$$M \cap H\backslash M = \cup_{x \in \mathcal{X}_{M_\emptyset}^M} x A_\emptyset^+(P_\emptyset, P, 0)\Omega_M,$$

where Ω_M is a compact set of M and $\mathcal{X}_{M_\emptyset}^M$ is the analog of $\mathcal{X}_{M_\emptyset}^G$. The M-module of compactly supported smooth functions on $\dot{1}M$ is the linear span of the characteristic functions of $\dot{1}xa\omega J$ where J describes a basis of neighborhoods of 1 in M made of compact open subgroup of M, $x \in \mathcal{X}^M_{M_\emptyset}$, $\omega \in \Omega_M$, $a \in A^+_\emptyset(P_\emptyset, P, 0)$. As m_P, m^X_P, e_P are M-equivariant, one has to prove (7.2) for every f among a set of generators of this M-module. Again we reduce to $x = 1$. Taking into account (3.8), one can write $a = a'b$ with $a' \in A^+_\emptyset$ and $b \in A_M$. As b commutes to J, one is reduced to prove (7.2) for the characteristic functions of $\dot{1}a\omega J$, with $a \in A^+_\emptyset$ and $\omega \in \Omega_M$.

As $\omega J = \omega J \omega^{-1}\omega$, the characteristic functions of $\dot{1}aJ'$ where J' describes the set of $\omega.J$ for J as above, $a \in A^+_\emptyset$, $\omega \in \Omega_M$ is a set of generators of $C^\infty_c(\dot{1}M)$.

Let (J'_n) be as in Lemma 6. By continuity and compacity, there exists a neighborhood $\mathcal{V}$ of 1 in M such that:

$$\omega.\mathcal{V} \subset (J'_0)_M, \omega \in \Omega_M.$$

One can assume that all the groups J above are contained in $\mathcal{V}$. Hence all the groups J' are contained in $(J'_0)_M$. For such a group, let $n \in \mathbb{N}$ such that $(J'_n)_M \subset J'$. Then as J' is the disjoint union of the left $(J'_n)_M$-cosets, the characteristic function of $\dot{1}aJ'$ is a linear combination of the characteristic functions of $\dot{1}aj'(J'_n)_M$ where j' describes J'. But as J'_n is normal in J'_0 (cf. Lemma 6) and $J' \subset (J'_0)_M$, $(J'_n)_M$ is normal in J'. Hence $\dot{1}aj'J'_n = \dot{1}aJ'_nj'$. Hence, again by M-equivariance, one has to prove (7.2) for f equal to the characteristic function of $\dot{1}a(J'_n)_M$, $n \in \mathbb{N}$, $a \in A^+_\emptyset$.

For simplicity we write J instead of J'_n and let $g = vol(J_U)\delta_P(a)^{1/2}1_{\dot{1}aJ_M}$ and let $f = 1_{\dot{1}_PaJ_MJ_U} \in C^\infty_c(X_P)$. Then $f^P = g$. Then, by definition of m_P, one has:

$$m_P(g) = j_P(f)$$

where $j_P(f)$ is the image of f in the normalized Jacquet module of $C^\infty_c(X_P)$. Similarly the characteristic function h of $\dot{1}aJ_MJ_U$ satisfies $h^P = g$. Hence one has:

$$m^X_P(g) = j_P(h)$$

and

$$(j_P(e_P))(m_P(g)) = j_P(e_P(f)).$$

It remains to prove:

$$(j_P(e_P))(m_P(g)) = m^X_P(g)$$

i.e.

$$(j_P(e_P))(j_P(f)) = j_P(h).$$

For this, it is enough to prove:

$$e_P(f) = h.$$

One has

$$\dot{1}aJ_M J_U = \dot{1}aJ$$

from Lemma 10. As J_{U^-} is normalized by $a \in A_\emptyset^+$ (cf. Lemma 6), one has

$$\dot{1}_P a J_M J_U = \dot{1}_P a J.$$

Then the required equality follows from Theorem 3(i). □

7.3 Discrete spectrum

An irreducible subrepresentation of $C^\infty(X)$, (π, V), is said discrete if the action of A_G is unitary and the elements of V are square integrable mod A_G. Obviously if ψ is an element of the group $X(G)_{\sigma,u}$ of unitary elements of $X(G)_\sigma$, the representation π_ψ of G in the space $V_\psi := \{\psi v | v \in V\}$ is also a discrete series. Moreover π_ψ is isomorphic to $\pi \otimes \psi$. Let χ be a unitary character of A_G and let $L^2(X, \chi)_{disc}$ be the sum of all X-discrete series on which A_G acts by χ.

Theorem 4 *Let J be a compact open subgroup of G and χ a unitary character of A_G. Then the space $L^2(X, \chi)^J_{disc}$ of J-invariants of $L^2(X, \chi)_{disc}$ is finite dimensional.*

Proof One will see that the proof of Theorem 9.2.1 of [14] adapts by changing $Z(G)^0$ to A_G, and, for a standard σ-parabolic subgroup $P = MU$ of G by changing $Z(X_P)$ to A_M acting on the left. As in [14], the proof is essentially reduced to the case $A_G = \{1\}$.

Let A_P^+ be the set of P-dominant elements of A_M. Let N'_P be equal to $N_{X,J}(P, C)$ for $C > 0$ large enough in such a way that the exp-maps are defined and such that the identity of Proposition 3 holds. Let $N_P = N'_P \setminus_{Q \subset P, Q \in \mathcal{P}, Q \neq P} N'_Q$. Then the N_P covers X. We remark that $exp_{X_P,X,J}(N'_P)$ is stable by the left action of A_P^+ as well as $N''_P := exp_{X_P,X,J}(N_P)$. One sees from the definitions that there is a finite subset Ω_P of X_P/J, such that $N''_P = A_P^+ \Omega_P$. Let $(\hat{A})^{J_M}_{\mathbb{C}}$ be the set of complex characters of A_M which are trivial on $A_M \cap J$. Let $\mathcal{P}$ be the set of standard σ-parabolic subgroups of G. We choose a map $R : \mathcal{P} \to \mathbb{N}$, $P \mapsto r_P$ and we define $\mathfrak{S}_R := \prod_{P \in \mathcal{P}} ((\hat{A})^{J_M}_{\mathbb{C}})^{r_P}$. An element of $x \in \mathfrak{S}_R$ is denoted $[(\chi_i)_{i=1,\dots,r_P}]_{P \in \mathcal{P}}$. We consider for $a \in A_M$

$$\prod_{i=1,\dots,r_P} (\mathcal{L}_a - \chi_i(a)), \tag{7.3}$$

where $\mathcal{L}$ has be defined in (4.7). Let $x \in \mathfrak{S}_R$. We consider the subspace $V_x \subset C^\infty(X)^J$ of J-invariant functions on X, f, such that for all standard σ-parabolic subgroup P of G and $a \in A_M$, $c_{P,G} f$ is annihilated by (7.3). Then V_x is invariant by the Hecke algebra of C_c^∞ functions on G which are right and left invariant by J: this is due to the fact that $c_{P,G}$ is a G-morphism and that the right action of G on $C^\infty(X_P)$ commutes with the left action of A_M.

Recall that from our hypothesis on C that:

$$(c_{P,G} f)(exp_{X_P,X,J}(x)) = f(x), x \in N_P.$$

Then V_x is finite dimensional, as it is shown in the proof of Theorem 9.2.1 of [14]. The rest of the proof is entirely analogous to the proof of this Theorem. □

Corollary 1 *Let J be a compact open subgroup of G. There exists finitely many discrete series for X, $(\pi_i, V_i), i = 1, \ldots, n$ such that any discrete series, (π, V) for X which has a non zero vector fixed by J is of the form $(\pi_i)_\chi$ where χ is element of the group $X(G)_{\sigma,u}$ of unitary elements of $X(G)_\sigma$ and $i \in \{1, \ldots, n\}$.*

Proof Looking to Lie algebras one sees that the restriction map from the group $X(G)_{\sigma,u}$ of unitary elements of $X(G)_\sigma$ to the group $X(A_G)_u$ of unitary elements of $X(A_G)$ is surjective. On the other hand the action by multiplication of $X(A_G)_u$ on $(\hat{A}_G)_u^J$ has finitely many orbits (cf. 2.6). Hence one is reduced to the case where the restriction of the central character of π is one of the representatives of these orbits. Then the corollary follows immediately from the Theorem. □

The proof of the following Lemma is immediate.

Lemma 13 *Let $\delta_{P,\mathcal{X}_M^G}$ be the function on X_M such that, for all $x \in \mathcal{X}_M^G$, its restriction to xM is equal to the function $\delta_{P,x}$ occuring in (4.6). For a function f on X_P we associate the map $T(f)$ on G with values in the space of functions on X_M defined by:*

$$(T(f)(g))(x) = \delta_{P,\mathcal{X}_M^G}^{-1/2}(x) f(xg), x \in X_M, g \in G.$$

(i) One has

$$T(f)(mg) = (\rho \otimes \delta_P^{1/2})(m)T(f)(g), m \in M, g \in G.$$

(ii) The map T induces a bijective G-intertwining map between $C_c^\infty(X_P)$ and $i_{P^-}^G C_c^\infty(X_M)$ (resp., $C^\infty(X_P)$ and $i_{P^-}^G C^\infty(X_M)$).

(iii) Let χ be a unitary character of A_M. The map T induces a bijective isometric G-intertwining map between $L^2(X_P)$ and the unitarily induced representation from P^- to G of $L^2(X_M)$ (resp., $L^2(X_P, \chi)_{disc}$ and the unitarily induced representation from P^- to G of $L^2(X_M, \chi)_{disc}$).

Proof (i) is immediate.

(ii) From (i), it remains only to prove the bijectivity. The inverse map to T is easily described using the fact that $X_P = X_M \times_{P^-} G$.

(iii) follows easily from the definition of the scalar product on unitary induced representations from P to G (cf. (5.1)) and from the definition of the M-invariant measure on X_M (cf. (4.6) and (4.3)). □

We define $L^2(X_M)_{disc}$ etc as in [14] section 9.

Lemma 14 *$L^2(X_P)_{disc}$ is unitarily equivalent to the unitary induced representation from P^- to G of $(L^2(X_M)_{disc})$*

Proof The Lemma follows from the analog of Corollary 9.3.4 in [14] and of Lemma 13(iii). Notice that this Corollary follows from l.c. Equation (9.1). To establish its analog, one remarks that A_M acts freely on the left on X_P. □

Lemma 15 *The G-space X_P satisfies the discrete series conjecture 9.4.6 of [14] for the parabolic subgroup P^- and the torus of unitary unramified characters of P^-, $D^* := X(M)_{\sigma,u}$.*

Proof From Corollary 1 of Theorem 4, there is a denumerable family of $X(M)_{\sigma,u}$-orbits of discrete series. Then the Lemma follows from Lemma 14. □

7.4 Bernstein maps

The proof of the following theorem is entirely analogous to the proof of Theorem 11.1.2 in [14].

Theorem 5 *For every pair of standard σ-parabolic subgroups of G, $P \subset Q$, there exists a canonical G-equivariant map $i_{Q,P} : L^2(X_P) \to L^2(X_Q)$ characterized by the property that for any $\Psi \in C_c^\infty(X_P)$ and any element a of the set A_P^{++} of strictly P-dominant elements of A_M, we have:*

$$lim_{n\to\infty}(i_{Q,P}\mathcal{L}_{a^n}\Psi - e_{Q,P}\mathcal{L}_{a^n}\Psi) = 0$$

where the limit is in $L^2(X_Q)$.

Then as a corollary of Theorem 5 and of the analog of Proposition 11.6.1 of [14], one has the following analog of l.c Corollary 11.6.2. The proof requires the criteria for discrete series of symmetric spaces due to Kato and Takano [11]:

Proposition 5 *Let $L^2(X)_P$ be the image of $L^2(X_P)_{disc}$ under $i_P := i_{G,P}$. Then one has:*

$$L^2(X) = \sum_{P\in\mathcal{P}_{st}} L^2(X)_P$$

where $\mathcal{P}_{st}$ is the set of standard σ-parabolic subgroups of G.

7.5 Scattering theory

From Lemma 15, one proves the analog of Proposition 13.2.1 in [14] in which we use A_M and A_L instead of $A_{X,\Theta}$ and $A_{X,\Omega}$ and where $P = MU$, $Q = LV$ are σ-parabolic subgroups of G. This is a step for the analog of Proposition 13.3.1 in l.c.. We will only recall part (2) of it.

Proposition 6 *Let $P = MU$, $Q = LV$ be two standard σ-parabolic subgroups of G. If the dimensions of A_M and A_L are distinct, $L^2(X)_P$ is orthogonal to $L^2(X)_Q$.*

Let Θ_P (resp., Θ_Q) be the set of elements of $\Sigma(P_\emptyset)$ which are trivial on A_M (resp. A_L). We define $W(P, Q)$ as the set of elements of $w \in W(A_\emptyset)$ such that $w(\Theta_P) = \Theta_Q$. In particular if $w \in W(P, Q)$, it induces an isomorphism between A_M and A_L. If $W(P, Q)$ is non trivial we say that P and Q are σ-associated. Let $c(P) = \sum_{Q \in \mathcal{P}_{st}} Card\ W(P, Q)$.

The proof of the analog of l.c. Theorem 14.3.1 (Tiling property of scattering morphisms) is entirely similar. Then one proves the following theorem like Theorem 7.3.1 of l.c. is proved in Section 14 of l.c.. Notice that one needs for this proof to establish part of this Theorem for spaces X_P, but this works like for X. We recall that i_P is the map $i_{G,P}$.

Theorem 6 (Scattering Theorem) *Let $P = MU$, $Q = LV$, R be three standard σ-parabolic subgroups of G.*

(i) *If P and Q are not σ-associated, $(i_Q)^t \circ i_P = 0$.*
(ii) *If P and Q are σ-associated, there exist $A_M \times G$-equivariant isometries*

$$S_w : L^2(X_P) \to L^2(X_Q), w \in W(P, Q)$$

where A_M acts on $L^2(X_Q)$ via the isomorphism $A_M \to A_L$ induced by w, with the following properties:

$$\begin{aligned} i_Q \circ S_w &= i_P, \\ S_{w'} \circ S_w &= S_{w'w}, w \in W(P, Q), w' \in W(Q, R), \\ (i_Q)^t \circ i_P &= \sum_{w \in W(P,Q)} S_w. \end{aligned}$$

Let us denote by $(i_P)^t_{disc}$ the composition of $(i_P)^t$ with the orthogonal projection to the discrete spectrum. Finally the map

$$\sum_{P \in \mathcal{P}} \frac{(i_P)^t_{disc}}{c(P)^{1/2}} : L^2(X) \to \oplus_{P \in \mathcal{P}_{st}} L^2(X_P)_{disc}$$

is an isometric isomorphism onto the subspaces of vectors $(f_P)_{P \in \mathcal{P}_{st}} \in \oplus_{P \in \mathcal{P}_{st}} L^2(X_P)_{disc}$ satisfying:

$$S_w f_P = f_Q, w \in W(P, Q).$$

In the next section we will explicate the maps i_P.

8 Explicit Plancherel formula

8.1 Injectivity of the map $\mathfrak{a}'/W(A) \to \tilde{\mathfrak{a}}'/W(\tilde{A})$

Lemma 16 *(i) Let A be a maximal σ-split torus and let $\tilde{A}$ be a maximal split torus containing A. It is σ-stable (cf.* [7], *Lemma 1.9).*

(ii) The set of non zero weights of A (resp., $\tilde{A}$) in the Lie algebra of G is a root system $\Delta(A)$ (resp., $\Delta(\tilde{A})$) which appears as a subset of $\mathfrak{a}'$(resp., $\tilde{\mathfrak{a}}'$).
The set $\Delta(A)$ is equal to the set of non zero restrictions of elements of $\Delta(\tilde{A})$.

(iii) Let $W(A)$ (resp. $W(\tilde{A})$) be the quotient of the normalizer of A (resp., $\tilde{A}$), $N_G(A)$ (resp. $N_G(\tilde{A})$), by its centralizer, $C_G(A)$ (resp., $C_G(\tilde{A})$).
Then $W(A)$ (resp., $W(\tilde{A})$) identifies with the Weyl group of $\Delta(A)$ (resp., $\Delta(\tilde{A})$) and is the set of restrictions to $\mathfrak{a}$ of the elements of $W(\tilde{A})$ which normalize $\mathfrak{a}$.

(iv) Let $\mu, \nu \in \mathfrak{a}'$ which are conjugate by an element of $W(\tilde{A})$, then they are conjugate by an element of $W(A)$.

Proof (i) follows from [7], Lemma 2.4.

(ii) and (iii) follows from [8], Propositions 5.3 and 5.9.

(iv) It is clear that one may replace μ and ν by a conjugate element by $W(A)$. Hence one may assume that μ and ν are dominant for some choice of a set positive roots of $\Delta(A)$, $\Delta^+(A)$. Then we choose a set of positive roots for $\Delta^+(\tilde{A})$ whose non zero restrictions are precisely the elements of $\Delta^+(A)$. Hence μ and ν are dominant for $\Delta^+(\tilde{A})$ and conjugate by an element of $W(\tilde{A})$. Hence they are equal, which proves (iv). □

Remark 3 It follows from (iv) of the previous lemma that the map $\mathfrak{a}'/W(A) \to \tilde{\mathfrak{a}}'/W(\tilde{A})$ is injective. This allows to apply the analog of Lemma 14.2.3 of [14].

8.2 Coinvariants

Let $P = MU$ be a semistandard σ-parabolic subgroup of G. Let us prove:

Using our G-invariant measure on X_P, the smooth dual of $C_c^\infty(X_P)$ is isomorphic to $C^\infty(X_P)$. (8.1)

An element of the smooth dual of $C_c^\infty(X_P)$ is fixed by some compact open subgroup J of G and is the composition of the J-average with a linear form on the space of J-fixed elements of $C_c^\infty(X_P)$. A basis of this later space is given by the characteristic functions of J-cosets. Hence a linear form on this space is given by integration against a J-fixed element of $C^\infty(X_P)$. This proves (8.1).

Similarly one has:

Using our choice of an M-invariant measure on X_M (cf. (4.6)), we will identify the smooth dual of $C_c^\infty(X_M)$ with $C^\infty(X_M)$. This identification depends on our choice of $\mathcal{X}_M^G$. (8.2)

Let (π, V) be a smooth representation of G of finite length. Let us define the space of coinvariants as in [14](6.1) :

$$C_c^\infty(X_P)_\pi := Hom_{\mathbb{C}}(Hom_G(C_c^\infty(X_P), \pi), \pi). \tag{8.3}$$

As $Hom_G(C_c^\infty(X_P), \pi)$ is finite dimensional (cf. (7.1)), one has:

$$Hom_G(C_c^\infty(X_P)_\pi, \pi) = Hom_G(C_c^\infty(X_P), \pi).$$

Definition 3 If π is a smooth admissible representation of G, there is a canonical projection

$$C_c^\infty(X_P) \to C_c^\infty(X_P)_\pi \to 0.$$

If $\pi = i_{P^-}^G \delta$, we denote this map $i_{P,\delta}^t$

The canonical map from $C_c^\infty(X_P)$ to $C_c^\infty(X_P)_\pi$ is defined as follows. If $f \in C_c^\infty(X_P)$, one defines $\phi \in C_c^\infty(X_P)_\pi$ by associating to each $T \in Hom(C_c^\infty(X_P), \pi)$, the element $\phi(T) := T(f)$ of the space of π. It is easy to see that this map is surjective.

Let (δ, E) be a unitary irreducible smooth representation of M. Let $T \in Hom_M(C_c^\infty(X_M), \delta)$. Due to (8.2), the transpose map T^t might be viewed as an element $\tilde{T}^t$ of $Hom(\check{\delta}, C^\infty(X_M))$. Let us define $\eta_T = (\eta_{T,x})_{x \in \mathcal{X}_M^G} \in \mathcal{V}(\check{\delta}, H)$ [cf. (5.2) for the notation] by:

$$\eta_{T,x}(\check{e}) := \tilde{T}^t(\check{e})(x), \check{e} \in \check{E}. \tag{8.4}$$

One defines $Hom_M(C_c^\infty(X_M), \delta)^{disc}$ as the space of $T \in Hom_M(C_c^\infty(X_M), \delta)$ such that the image of $\tilde{T}^t$ is a discrete series for X_M. Let us define:

$$C_c^\infty(X_P)_\delta := (Hom_M(C_c^\infty(X_M), \delta)^{disc})' \otimes i_{P^-}^G \delta. \tag{8.5}$$

$$C_c^\infty(X_P)_\delta[\delta] = (Hom_M(C_c^\infty(X_M), \delta)^{disc})' \otimes \delta. \tag{8.6}$$

Hence we have:

$$C_c^\infty(X_P)_\delta = i_{P^-}^G C_c^\infty(X_P)_\delta[\delta]. \tag{8.7}$$

It can be viewed as a quotient of $C_c^\infty(X_P)$ as follows (cf. [14] before Equation (15.12)). From the Lemma 13, one has an injective map defined by induction:

$$0 \to Hom_M(C_c^\infty(X_M), \delta)_{disc} \to Hom_G(C_c^\infty(X_P), i_{P^-}^G \delta).$$

Hence, using the transpose map and taking into account the notation (8.3) one has a surjective map:

$$C_c^\infty(X_P)_{i_{P^-}^G \delta} = Hom_G(C_c^\infty(X_P), i_{P^-}^G \delta)' \otimes i_{P^-}^G \delta \to C_c^\infty(X_P)_\delta \to 0.$$

Together with Definition 3, this shows that

$$C_c^\infty(X_P)_\delta \text{ is a quotient of } C_c^\infty(X_P). \tag{8.8}$$

The smooth dual of $C_c^\infty(X_P)_\delta$ is denoted $C^\infty(X_P)^{\check\delta}$ and one has

$$C^\infty(X_P)^{\check\delta} = Hom_M(C_c^\infty(X_M), \delta)^{disc} \otimes i_{P^-}^G \check\delta.$$

From (8.8) it can be viewed as a subspace of $C^\infty(X_P)$.

8.3 Eisenstein integral maps and their transpose

Definition 4 We use the fact that the Eisenstein integral associated to δ_χ are well defined for χ in the complementary set of the zero set of a non zero polynomial function on $X(M)_\sigma$. For such a χ, we define a map called Eisenstein integral map in [14]:

$$E_{P,\delta_\chi} \in Hom_G(Hom_M(C_c^\infty(X_M), \check{\delta_\chi})^{disc} \otimes i_P^G \delta_\chi, C^\infty(X))$$

by

$$E_{P,\delta_\chi}(T \otimes v) = E(P, \delta_\chi, \eta_T, v), T \in Hom_M(C_c^\infty(X_M), \check{\delta_\chi})^{disc}, v \in i_P^G \delta_\chi.$$

We keep the notation of the preceding subsection. Let us denote by ev_1 the map

$$ev_1 : (Hom_M(C_c^\infty(X_M), \check{\delta_\chi})^{disc})' \otimes i_P^G \check{\delta_\chi} \to (Hom_M(C_c^\infty(X_M), \check{\delta_\chi})^{disc})' \otimes \check E$$

defined by:

$$ev_1(\theta \otimes v) = \theta \otimes v(1), \theta \in (Hom_M(C_c^\infty(X_M), \check{\delta}_\chi)^{disc})', v \in i_P^G \check{\delta_\chi}.$$

If $\phi \in C_c^\infty(X_M)$, let $q_\delta(\phi)$ be the canonical element of $(Hom_M(C_c^\infty(X_M), \check\delta)^{disc})' \otimes \check E$ defined as follows. The latter space appears as the smooth dual of $Hom_M(C_c^\infty(X_M), \check\delta)^{disc}) \otimes E$ and we define

$$\langle q_\delta(\phi), T \otimes e\rangle := \langle e, T(\phi)\rangle, e \in E, T \in Hom_M(C_c^\infty(X_M), \check\delta)^{disc}.$$

Identifying the smooth dual of $C_c^\infty(X_M)$ to $C^\infty(X_M)$ (cf. (8.2)), one has also:

$$\langle q_\delta(\phi), T \otimes e\rangle = \langle \tilde T^t(e), \phi\rangle. \tag{8.9}$$

Let us denote, by abuse of notation, the restriction of the transpose map of E_{P,δ_χ} to $C_c^\infty(X)$ by E^t_{P,δ_χ} .

Lemma 17 *One has*

$$E^t_{P,\delta_\chi} \in Hom_G(C_c^\infty(X), (Hom_M(C_c^\infty(X_M), \check{\delta_\chi})^{disc})' \otimes i_P^G \check{\delta_\chi})$$

and

$$ev_1((E_{P,\delta_\chi})^t(f)) = q_\delta(f^P),\ f \in C_c^\infty(X),$$

where f^P has been defined in Lemma 12.

Proof Let $e \in E$, $T \in Hom_M(C_c^\infty(X_M), \check{\delta}_\chi)^{disc}$. Let J be a compact open subgroup of G with a σ-factorization for P and such that J_M fixes e and f and let $v_\chi := v^{P,J}_{e,\delta_\chi}$ the element of $i_P^G \delta_\chi$ which is invariant by J, whose support is equal to PJ and whose value at 1 is equal to e (for the existence see e.g. [5] Equation (3.2)). Notice that, from (4.8), one has:

$$v_\chi \text{ has its support equal to } PJ_{U^-} = PJ_H \subset PH\ . \tag{8.10}$$

We will compute in two ways:

$$I := \langle E^t_{P,\delta_\chi}(f), T \otimes v_\chi \rangle.$$

We take into account the expression of the duality of $i_P^G \delta$ and $i_P^G \check{\delta}$ (cf. (5.1) and (8.10)). This leads to our first expression of I:

$$I = vol(J_{U^-}) \langle ev_1(E^t_{P,\delta_\chi}(f)), T \otimes e \rangle. \tag{8.11}$$

In order to compute I in an other way we use a transposition:

$$I = \int_{H\backslash G} f(\dot{g}) E_{P,\delta_\chi}(T \otimes v_\chi)(\dot{g}) d\dot{g}.$$

For $Re\chi$ sufficiently P-dominant, one has from (5.4) and the definition of η_T (cf. (8.4)):

$$I = \int_{H\backslash G} f(\dot{g}) \sum_{x \in \mathcal{X}_M^G} \int_{M\cap x^{-1}.H\backslash x^{-1}.H} \tilde{T}^t(v_\chi(yx^{-1}g))(x) dy d\dot{g}.$$

One makes the change of variable $g' = x^{-1}.g$ and then the Fubini theorem that one can use because f is compactly supported. One gets:

$$I = \sum_{x \in \mathcal{X}_M^G} \int_{(M\cap x^{-1}.H)\backslash G} f(x.g) \tilde{T}^t(v_\chi(gx^{-1}))(x) d\dot{g}.$$

We make the change of variable $g'' = gx^{-1}$. We use the integration formula (4.1) and our choice of measure on $M \cap x^{-1}.H\backslash M$. As v_χ has its support in PJ_{U^-} and f and v_χ are J-invariant, one gets:

$$I = vol(J_{U^-}) \sum_{x \in \mathcal{X}_M^G} \int_{M\cap x^{-1}.H\backslash M} \delta_P(m^{-1}) \int_U f(xum) du \tilde{T}^t(v_\chi(m))(x) dm.$$

But the change variable $u' = m^{-1}um$ shows that:

$$I = vol(J_{U^-}) \sum_{x \in \mathcal{X}_M^G} \int_{M\cap x^{-1}.H\backslash M} \int_U f(xmu) du \tilde{T}^t(v_\chi(m))(x) dm.$$

From the intertwining property of T one has:

$$\tilde{T}^t(v_\chi(m))(x) = \delta_P^{1/2}(m)\tilde{T}^t(e)(xm).$$

With our choices of measures one deduces:

$$I = vol(J_{U^-}) \sum_{x \in \mathcal{X}_M^G} \int_{M\cap x^{-1}.H\backslash M} f^P(\dot{x}m)\tilde{T}^t(e)(\dot{x}m) dm.$$

In other words

$$I = vol(J_{U^-})\langle f^P, \tilde{T}^t(e)\rangle,$$

and (8.9) implies:

$$I = vol(J_{U^-})\langle q_\delta(f^P), T \otimes e\rangle.$$

From (8.11) and Lemma 17 one deduces the equality:

$$ev_1((E_{P,\delta_\chi})^t(f)) = q_\delta(f^P).$$

□

8.4 Canonical quotient and the small Mackey restriction

We follow the terminology of [14], section 15. Let τ be a finite length smooth representation of M. If the intertwining integral:

$$A(P, P^-, \tau) : i_{P^-}^G \tau \to i_P^G \tau$$

is well defined, the *canonical quotient* is the composition:

$$(i_{P^-}^G \tau)_P \xrightarrow{j_P(A(P,P^-,\tau))} (i_P^G \tau)_P \to \tau$$

where the right map is the evaluation at 1 (cf. [14] Equation (15.8)). If $\tau = C_c^\infty(X_P)_\delta[\delta]$, the canonical quotient in this case is denoted c_δ and taking into account (8.7) one has:

$$c_\delta : (C_c^\infty(X_P)_\delta)_P \to C_c^\infty(X_P)_\delta[\delta].$$

Let (π, V) be a smooth representation of G. The Mackey restriction (cf. [14] section 15.5.3) is the map

$$Mack : Hom_G(C_c^\infty(X), \pi) \to Hom_M(C_c^\infty(X_M), \pi_P)$$

obtained by taking the Jacquet functor to any element T of $Hom_G(C_c^\infty(X), \pi)$, and restricting it to $C_c^\infty(X_M)$ which is identified by m_P^X (cf. Lemma 12) with a subspace of the normalized Jacquet module of $C_c^\infty(X)$.

If $\pi = i_{P^-}^G \tau$, and the intertwining integral $A(P, P^-, \tau) : i_{P^-}^G \tau \to i_P^G \tau$ is bijective the small Mackey restriction is the composition of the canonical quotient with the Mackey restriction $Mack$:

$$sMack : Hom_G(C_c^\infty(X), \pi) \to Hom_M(C_c^\infty(X_M), \tau).$$

If $\pi = C_c^\infty(X_P)_\delta$, and $T \in Hom_G(C_c^\infty(X), \pi)$ one has

$$sMack(T) \in Hom_M(C_c^\infty(X_M), Hom_M(C_c^\infty(X_M), \delta)^{disc})' \otimes \delta).$$

8.5 Normalized Eisenstein integrals

Definition 5 Let P be a semistandard σ-parabolic subgroup of G. We define the normalized integral

$$E^0_{P,\delta_\chi} \in Hom_G(Hom_M(C_c^\infty(X_M), \check{\delta_\chi})^{disc} \otimes i_{P^-}^G \delta_\chi, C^\infty(X))$$

by:

$$E^0_{P,\delta_\chi} := E_{P,\delta_\chi} \circ (Id \otimes A(P^-, P, \delta_\chi)^{-1})$$

which is rational in $\chi \in X(M)_\sigma$.

By the formula of the transpose of intertwining integrals (cf. [16] IV.1(11) and denoting by $(E^0_{P,\delta_\chi})^t$ the restriction of the transpose of E^0_{P,δ_χ} to $C_c^\infty(X)$, one has

$$(E^0_{P,\delta_\chi})^t = (Id \otimes A(P, P^-, \check{\delta_\chi})^{-1}) \circ (E_{P,\delta_\chi})^t.$$

From this it follows

$$sMack((E^0_{P,\delta_\chi})^t) \in Hom_M(C_c^\infty(X_M), Hom_M(C_c^\infty(X_M), \check{\delta_\chi})^{disc})' \otimes \check{\delta_\chi})$$

is equal to

$$ev_1 j_P(A(P, P^-, \check{\delta_\chi}) \circ (E^0_{P,\delta_\chi})^t).$$

From Lemma 17, one deduces:

The map $sMack((E^0_{P,\delta_\chi})^t)$ is equal to the map q_{δ_χ}. (8.12)

Our definition of normalized Eisenstein integrals differs from the one in [14], Equation (15.30) for G split and X spherical. Here we do not use the Radon transform, but we use that the opposite of a σ-parabolic subgroup is a σ-parabolic subgroup. From (8.12), our Eisenstein integrals maps have the same small Mackey restrictions than the ones defined in l.c. (cf. (15.36)). (8.13)

8.6 Explicit Plancherel formula

Let

$$L^2(X_M)_{disc} = \int_{\hat{M}}^{\oplus} \check{I}_\delta d\nu_{disc}(\delta)$$

where $\check{I}_\delta$ is a unitary representation of M isomorphic to a direct sum of copies of δ. From Lemma 14 , one has

$$L^2(X_P)_{disc} = \int_{\hat{M}}^{\oplus} \check{H}_\delta d\nu_{disc}(\delta)$$

where $\check{H}_\delta$ is the unitarily induced representation from P^- to G of $\check{I}_\delta$. Let $\check{H}_\delta^\infty$ be its space of smooth vectors. With the notation of (8.5), its space of smooth vectors is equal to $C^\infty(X_P)^\delta$.

Let $f \in C_c^\infty(X_P)$ and let us write its discrete component

$$f_{disc} = \int_{\hat{M}} f^\delta d\nu_{disc}(\delta),$$

where $f^\delta \in C^\infty(X_P)^\delta$. Its image by the Bernstein morphism $i_P(f_{disc})$ satisfies:

$$i_P(f_{disc}) = \int_{\hat{M}} i_{P,\delta}(f^\delta) d\nu_{disc}(\delta).$$

for some maps $i_{P,\delta} : \check{H}_\delta^\infty \to C^\infty(X)$ defined for almost all δ (cf. [14] Equation (15.6)).

One has the analog of Lemma 15.5.4 of [14]. As the analog of the beginning of section 15.6 of [14] is identical, together with (8.12), this leads to the analog of Th 15.6.3 in [14]:

Proposition 7 *The small Mackey restrictions* $sMack((E^0_{P,\delta_\chi})^t)$ *and* $sMack(i^t_{P,\delta_\chi})$ *are equal for almost all* $\chi \in X(M)_{\sigma,u}$.

Also by the uniqueness result of [4] recalled in (5.3), for almost all $\chi \in X(M)_{\sigma,u}$, every element F of $Hom_G(C_c^\infty(X), i^G_{P^-}\check{\delta_\chi})$ has its transpose given in term of the normalized Eisenstein integral i.e. is of the form

$$(E(P, \delta_\chi, \eta_T, v) \circ A(P^-, P, \delta_\chi)^{-1})$$

for a unique $T \in Hom_M(C_c^\infty(X_M), \check{\delta_\chi})$. Using (8.12) or rather its immediate generalization by replacing $Hom_M(C_c^\infty(X_M), \check{\delta_\chi})^{disc}$ by $Hom_M(C_c^\infty(X_M), \check{\delta_\chi})$ one sees that the small Mackey restriction of F is equal to T. Hence one has:

Proposition 8 *The small Mackey restriction*

$$sMack : Hom_G(C_c^\infty(X), i^G_{P^-}\check{\delta_\chi}) \to Hom_M(C_c^\infty(X_M), \check{\delta_\chi})$$

is injective for almost all $\chi \in X(M)_{\sigma,u}$.

Corollary 2 *For almost all* $\chi \in X(M)_{\sigma,u}$, *one has:*

$$i_{P,\delta_\chi} = E^0_{P,\delta_\chi}.$$

Theorem 7 *Let* $f \in C_c^\infty(X_P)$ *and let us write its discrete component*

$$f_{disc} = \int_{\hat{M}}^{\oplus} f^\delta d\nu_{disc}(\delta),$$

where $f^\delta \in C^\infty(X_P)^\delta$.

Its image by the Bernstein morphism $i_P(f)$ *satisfies:*

$$i_P(f)(x) = \int_{\hat{M}} E^0_{P,\delta}(f^\delta)(x) d\nu(\delta), x \in X.$$

In combination with the scattering theorem (cf. Theorem 6), one deduces:

Theorem 8 *The norm on* $L^2(X)_P$, $\|.\|_P$, *admits the decomposition:*

$$\|\Phi\|_P^2 = \frac{4}{Card(W(P, P))} \int_{\hat{M}} \|E^{0t}_{P,\delta}(\Phi)\|_\delta^2 d\nu(\delta),$$

where the measure and norms on the right hand side of the equality are the discrete part of the Plancherel decomposition of $L^2(X_P)$.

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9 Appendix: Rational representations

In this section we establish some results on rational representations of G which are needed to extend the results of [12] and [4], which are established when $\mathbf{F}$ is of characteristic zero, to the case where $\mathbf{F}$ is simply of characteristic different from 2.

9.1 Rational representations and parabolic subgroups

Let $\underline{G}$ be a reductive algebraic group defined over a non archimedean local field $\mathbf{F}$, whose group of $\mathbf{F}$-points is equal to G. We will use similar notations for the subgroups of G.

Let A_0 be a maximal split torus of G and let $P_0 = M_0U_0$ be a minimal parabolic subgroup of G with $A_0 \subset M_0$. Let T be a maximal $\mathbf{F}$-torus of $\underline{G}$ which contains $\underline{A}_0$. Let B be a Borel subgroup of $\underline{G}$, which contains T and is contained in $\underline{P}_0$. One denotes by $\Sigma(T)$ the set of roots of T in the Lie algebra of $\underline{G}$. One denotes by $\Lambda(T)$ (resp., $\Lambda(T)_{rac}$) the weight lattice (resp., the root lattice) of T with respect to $\underline{G}$. We adopt similar notations for A_0. Let Γ be the absolute Galois group of $\mathbf{F}$ which acts on these lattices. Let $\Lambda^+(T)$ be the set of dominant weights for T relative to B. Let $\Lambda^+(A_0)$ (resp., $\Lambda^+(A_0)_{rac}$) the set of dominant elements for P_0 of $\Lambda(A_0)$ (resp., $\Lambda(A_0)_{rac}$).

Definition 6 Let $P = MU$ be a parabolic subgroup of G which contains P_0, where M is its Levi subgroup which contains A_0. One denotes by $\Lambda^+_M(T)$ the set of elements λ of $\Lambda^+(T)$such that G has a rational finite dimensional irreducible representation, defined over $\mathbf{F}$, with highest weight λ relative to B, (π_λ, V_λ), with the following property:

Any non zero vector of weight λ under T, v_λ, transforms under M by a rational character of M, denoted Λ. (9.1)

The goal of this subsection is to produce sufficiently many elements of $\Lambda^+_M(T)$.

Proposition 9 (i) *Let T_{an} be the anisotropic component of T. There exists $n \in \mathbb{N}^*$ such that any element λ of $n\Lambda^+(A_0)$ extends uniquely to an element μ of $\Lambda(T)_{rac}$ trivial on T_{an}.*

(ii) *If λ is orthogonal to the simple roots of A_0 in the Lie algebra of $U_0 \cap M$ then μ is element of $\Lambda^+_M(T)$.*

For the proof we need several lemmas.

Let β be an element of the set, $\Sigma(A_0)$, of roots of A_0 in the Lie algebra of G. One defines:

$$\underline{\beta} := \sum_{\alpha \in \Sigma(T), \alpha_{|A_0} = \beta} \alpha.$$

One sees easily that:

$$\text{There exists } n' \in \mathbb{N}^* \text{ such that, for all } \beta \in \Sigma(A_0), \text{ there exists } n'_\beta \in \mathbb{N}^* \text{ such that } n'_\beta \underline{\beta}_{|A_0} = n'\beta. \tag{9.2}$$

We fix, once for all, such integers n' and n'_β

Lemma 18 *Every element λ of $n'\Lambda_{rac}(A_0)$ extends uniquely to an element μ of $\Lambda_{rac}(T)$ trivial on the anisotropic component T_{an} of T, invariant by Γ and by $W(\underline{M}_0, T)$.*

Let us denote by $(n'\Lambda_{rac}(A_0))\tilde{}$ the lattice generated by the $n'_\beta \underline{\beta}$, $\beta \in \Sigma(A_0)$. From their definition, one sees that the elements of $(n'\Lambda_{rac}(A_0))\tilde{}$ are invariant under Γ and are elements of $\Lambda_{rac}(T)$. One remarks that every element μ of $(n'\Lambda_{rac}(A_0))\tilde{}$ is invariant by the Weyl group of $\underline{M}_0$ relative to T, $W(\underline{M}_0, T)$.

Let us show any element μ is trivial on T_{an}. One can choose T such that it contains a maximal torus defined over $\mathbf{F}$, T_1, of the derived group of $\underline{M}_0$. Actually, by conjugacy, one sees that any T has this property. Moreover T contains the maximal anisotropic torus C_{an} of the center of M_0. The product $T_1 C_{an} \underline{A}_0$ is a torus. For reasons of dimension it is a maximal torus of G. Hence $T = T_1 C_{an} \underline{A}_0$. Notice that $T_1 C_{an}$ is the anisotropic component T_{an} of T. As μ is $W(\underline{M}_0, T)$-invariant, the restriction of μ to T_1 is trivial. As C_{an} is anisotropic, the invariance by Γ of μ shows that its restriction to C_{an} is trivial. This proves the existence part of the Lemma. As $T = T_{an}\underline{A}_0$ the unicity follows. □

Lemma 19 (i) *There exists $n \in \mathbb{N}$ such that $n\Lambda(A_0) \subset n'\Lambda_{rac}(A_0)$.*

(ii) *If $\lambda \in n\Lambda^+(A_0)$, its extension μ to T given by the preceding lemma is the highest weight of a rational representation of G, defined over* $\mathbf{F}$, *denoted (π_μ, V_μ).*

Proof (i) The lattice $n'\Lambda_{rac}(A_0)$ is contained in the lattice $\Lambda(A_0)$. As these lattices are of the same rank, there exists $n \in \mathbb{N}^*$ such that $n\Lambda(A_0) \subset n'\Lambda_{rac}(A_0)$.

(ii) From (i) and the preceding lemma, if $\lambda \in n\Lambda^+(A_0)$, μ is in $\Lambda_{rac}(T) \subset \Lambda(T)$. Moreover if α is a root of T in the Lie algebra of $\underline{G}$, $\langle \mu, \alpha \rangle = \langle \lambda, \alpha_{|A_0} \rangle$. Hence μ is a dominant weight. From the preceding Lemma, it is invariant by Γ. Then [15], Theorem 3.3 and Lemma 3.2 implies (ii). □

Lemma 20 *Let $\lambda \in n\Lambda^+(A_0)$ and μ as in Lemma 18. Then, with the notation of the preceding lemma, M_0 acts on a non zero highest weight vector of (π_μ, V_μ) by a rational character of M_0 again denoted by μ.*

Proof As π_μ is defined over $\mathbf{F}$, it is enough to prove that v_μ transforms under a rational character of M_0. In order to prove this, one can work with the algebraic closure. The invariance of μ by $W(\underline{M}_0, T)$ (cf. Lemma 18), the fact that the space of weight μ in V_μ is of dimension one (cf. [9], Proposition 33.2) together with the Bruhat decomposition of $\underline{M}_0$ allow to prove the Lemma. □

Proof of Proposition 9 (i) follows from Lemmas 18 and 19.

Let $\lambda \in n\Lambda^+(A_0)$ be as in the statement of Proposition 9(ii) i.e. λ is orthogonal to the simple roots of A_0 in the Lie algebra of $U_0^- \cap M$. Let μ be as in Lemma 18. Let

(π_μ, V_μ) be as in Lemma 19, and let v_μ be a non zero highest weight vector. One has to prove that v_μ transforms under M by a rational character of M that we will still denote by μ. It is enough to prove that the line $\mathbf{F}\mu$ is stable by the action of M. One shows, using the preceding Lemma, by a proof analogous to the one of [9], Proposition 31.2 and using the density of $U_0^- M_0 U_0$ in G, that the A_0-weight space of V_μ for the weight λ is one dimensional. The Weyl group, $W(M, A_0)$, of M relative to A_0 fixes λ from the hypothesis on λ. One finishes the proof of our assertion on the action M on v_μ by using the Bruhat decomposition of M relative to $P_0 \cap M$. Hence $\mu \in \Lambda_M^+(T)$. □

9.2 *H*-distinguished rational representations of *G*

Proposition 9 allows to extend the results of [4] section 2.7 and especially Propositions 2.9, 2.11 to a non archimedean local field, $\mathbf{F}$, of characteristic different from 2. Let $\Sigma(G, A_0)$ (resp., $\Sigma(P_0, A_0)$ or $\Sigma(P_0)$) the set of roots of A_0 in the Lie algebra of G (resp., P_0). We denote by $\Delta(P_0)$ the set of simple roots of $\Sigma(P_0)$.

Let $P = MU$ be a standard σ-parabolic subgroup of G. We will use the notation of the main body of the article. Let us assume that $A_\emptyset \subset A_0$, which is automatically σ-stable, and $P_0 \subset P_\emptyset$. Let $\{\alpha_1, \ldots, \alpha_{m_0}\}$ be the simple roots of $\Sigma(P_0)$ written in such a way that $\{\alpha_1, \ldots, \alpha_{m_\emptyset}\}$ are the simple roots in the Lie algebra of $U_\emptyset$, $\{\alpha_1, \ldots, \alpha_m\}$ are the simple roots in the Lie algebra of U. One has the fundamental weights of $\Sigma(P_0, A_0)$, $\delta_1, \ldots, \delta_l$.

Let $i = 1, \ldots, m$ and $\lambda_i = n\delta_i$ with n as in Proposition 9. From this proposition, $\lambda_i \in \Lambda_M^+(T)$ and there exists a unique rational character of T, μ, trivial on T_{an} and whose restriction to A_0 is equal to λ_i and μ is the highest weight of an irreducible finite dimensional rational representation of G, denoted by (π_μ, V_μ). Moreover if v_μ is a non zero highest weight vector in V_μ, the space $\mathbf{F}v_\mu$ is P-invariant. We denote again by μ the rational character of M which describes the action of M on v_μ. One denotes by v'_μ the unique element of V'_μ of weight μ^{-1} under M and such that $\langle v'_\mu, v_\mu\rangle = 1$.

Let $\nu := \mu(\mu^{-1} \circ \sigma) \in \Lambda(T)$ and let $(\tilde{\pi}_\nu, \tilde{V}_\nu)$ be the rational representation of G $(\pi_\mu \otimes (\pi'_\mu \circ \sigma), V_\mu \otimes V'_\mu)$. Let $\tilde{v}_\nu := v_\mu \otimes v'_\mu$ which is of weight ν under the representation $\tilde{\pi}_\nu$ restricted to M. Then there exists a non zero H-invariant vector, under $\tilde{\pi}_\nu$ in $\tilde{V}'_\nu = (V_\mu \otimes V'_\mu)' \simeq V'_\mu \otimes V_\mu \simeq End(V_\mu)$, namely the identity that we will denote $e'_{\nu,H}$. It satisfies $\langle e'_{\nu,H}, \tilde{v}_\nu\rangle = 1$.

Let us show that $\nu = 2\mu$. As σ preserves T_{an}, the character $\mu^{-1} \circ \sigma$ of T is trivial on T_{an}. Its restriction to A_0 is equal to λ. From the unicity statement of μ, it is equal to μ. This proves our claim.

From this it follows that

Proposition 2.9 and 2.11 of [4] extend to a non archimedean local field, $\mathbf{F}$, of characteristic different from 2. This shows that the results of [4], section 2.8, 2.9 are valid for such a field. Also, the Lemma 1 (resp., section 3.2) in [12] is true also for such a field $\mathbf{F}$ due to Proposition 2.3 of [5] (resp., the Proposition 9 of the present article). Hence the results of [12] are valid for such a field $\mathbf{F}$. (9.3)

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