Rauzy fractals, one dimensional Meyer sets, $\beta$-numeration and automata

Paul MERCAT

28/11/2017
Zero Entropy System
Let’s take the following substitution over the alphabet \{a, b, c\}:

\[
\begin{align*}
    s & : \begin{cases} 
        a & \mapsto ab \\
        b & \mapsto ca \\
        c & \mapsto a
    \end{cases}
\end{align*}
\]

Then by iterating the letter a we get an infinite fixed point:

\[
\begin{align*}
    s(a) & = \text{ab} \\
    s^2(a) & = \text{abca} \\
    s^3(a) & = \text{abcaab} \\
    \ldots \\
    s^\infty(a) & = \text{abcaabababcaabcaabcaabcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaababcaabcaabab
If we replace letters of this fixed point by intervals of convenient lengths, we get a self-similar tiling of $\mathbb{R}_+$. 

![Diagram of self-similar tiling](image_url)
To get such a self-similar tiling of $\mathbb{R}_+$, the lengths of each intervalles must satisfy the equality

$$t_{M_s} \cdot \begin{pmatrix} l_a \\ l_b \\ l_c \end{pmatrix} = \beta \begin{pmatrix} l_a \\ l_b \\ l_c \end{pmatrix},$$

where $M_s = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is the incidence matrix of the substitution and $\beta$ is the Perron eigenvalue of $M_s$. Hence we can assume that the lengths $l_i, \ i \in \{a, b, c\}$ live in $\mathbb{Q}(\beta)$. 
Quasicrystal of $\mathbb{R}_+$

If we take for example

$$l_a = 1, \quad l_b = \beta - 1, \quad l_c = \beta^2 - \beta - 1,$$

we get the following subset $Q$ of $\mathbb{Q}(\beta)$.

$$Q = \{0, 1, \beta, \beta^2 - 1, \beta^2, \beta^2 + 1, \beta^2 + 2, \beta^2 + \beta + 1, \beta^2 + \beta + 2, \ldots\}$$

This set have very strong properties since we have:

**Proposition**

$Q$ is a $\beta$-invariant Meyer set of $\mathbb{R}_+$.

But what is a Meyer set?
Meyer sets are a mathematical model for quasicrystals.

**Definition**

A *Meyer set* of $\mathbb{R}_+$ is a set $Q \subset \mathbb{R}_+$ such that
- $Q$ is a Delone set of $\mathbb{R}_+$,
- $Q - Q$ is a Delone set of $\mathbb{R}$.

**Definition**

$Q$ is a *Delone set* of $E$ if
- $Q$ is *uniformly discrete*
  $$\exists \epsilon > 0, \ \forall (x, y) \in Q^2, B(x, \epsilon) \cap B(y, \epsilon) = \emptyset,$$
- $Q$ is *relatively dense in $E$*
  $$\exists R > 0, \ E \subseteq \bigcup_{x \in Q} B(x, R).$$
The quasicrystal $Q$ is a part of $\mathbb{Q}(\beta)$, hence we can look at the action of the Galois group. Here, $\beta$ has two complexes conjugated as conjugates, hence we have an embedding

$$\sigma : \mathbb{Q}(\beta) \hookrightarrow \mathbb{C},$$

by choosing one of the complex conjugates.

**Proposition**

*The set $\sigma(Q) \subseteq \mathbb{C}$ is bounded.*

We call the closure $\overline{\sigma(Q)}$ a **Rauzy fractal**.
The Rauzy fractal $\overline{\sigma(Q)} \subset \mathbb{C}$
Moreover, we can color in red the points of $\sigma(Q)$ that are left bound of an interval of length 1 (i.e. coming from letter $a$), in green the points that are left bound of an interval of length $\beta - 1$ (i.e. coming from letter $b$), and the other ones, for $\beta^2 - \beta - 1$, in blue.

We can also color in the same way by considering the right bound rather than the left one.

### Proposition

Let $u = s^\infty(a)$. Then, the subshift $(\overline{S\mathbb{Z}u}, S)$ is measurably conjugated to a domain exchange on the Rauzy fractal $\sigma(Q)$, for the Haar measure.
Introduction

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Introduction

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Rauzy fractals, one dimensional Meyer sets,
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If $s$ is any substitution over an alphabet $A$, everything generalizes:

- **fixed point**: Up to replace $s$ by a power, $s$ has a fixed point $\omega$.

- **self-similar tiling**: We get a self-similar tiling of $\mathbb{R}_+$ or $\mathbb{R}$ by replacing letters by intervals of lengths $l_a$, $a \in A$ given by a Perron left eigenvector of the incidence matrix.

- **quasicrystal**: We get a set $Q_\omega \subset \mathbb{R}$ by taking the bounds of intervals of this self-similar tiling, and up to rescaling we have $Q_\omega \subset \mathbb{Q}(\beta)$ where $\beta$ is the Perron eigenvalue of the incidence matrix $M_s$. If $\beta$ is a Pisot number, $Q_\omega$ is a Meyer set.

- **Rauzy fractal**: $Q_\omega$ is a subset of $\mathbb{Q}(\beta)$, therefore we can embed it into a natural contracting space $E_\beta^c$ where it is a pre-compact subset.

- **Domain exchange**: If the substitution satisfies the strong coincidence condition, then we can color the Rauzy fractal $\sigma_c(Q_\omega)$ in order to define a domain exchange conjugated to the shift.
General definitions of contracting space and Rauzy fractal

There are natural contracting and expanding spaces for the multiplication by $\beta$ on a number field $k = \mathbb{Q}(\beta)$. Call $P$ the set of places of $k$ (i.e. equivalence classes of absolute values), and let

\[ P_e := \{ v \in P \mid |\beta|_v > 1 \} \quad \text{and} \quad P_c := \{ v \in P \mid |\beta|_v < 1 \}. \]

The **contracting space** is $E^c_\beta := \prod_{v \in P_c} k_v$ and the expanding one is $E^e_\beta := \prod_{v \in P_e} k_v$, where $k_v$ denotes the completion of $k$ for the absolute value $v$. We denote by $\sigma_c = \prod_{v \in P_c} \sigma_v : \mathbb{Q}(\beta) \hookrightarrow E^c_\beta$ where $\sigma_v : \mathbb{Q}(\beta) \hookrightarrow k_v$ is a choice of one natural embedding.

**Definition**

We call **Rauzy fractal** the adherence of $\sigma_c(\mathbb{Q}_\omega)$ in $E^c_\beta$.

For the previous example, where $\beta$ is root of $x^3 - x^2 - x - 1$, we have $E^e_\beta = \mathbb{R}$ (there is one real place) and $E^c_\beta = \mathbb{C}$ (there is one complex place).
Main results

Rauzy fractals can approximate any shape

Theorem

For any Pisot number $\beta$ and for any $P \subset E^c_\beta$, bounded and containing 0, there exists substitutions whose Rauzy fractals approximate arbitrarily $P$, for the Hausdorff distance, and whose Perron numbers are powers of $\beta$. Moreover, the proof is constructive.

The Hausdorff distance between two subsets $A \subseteq E$ and $B \subseteq E$ of a metric space $E$ is

$$d(A, B) = \max \left( \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right).$$
Main results

Rauzy fractals approximating various shapes

![Image of fractals approximating various shapes]
Main results
g-β-sets: a nice description of quasicrystals by automata

Definition (Main tool)

A set $Q \subseteq \mathbb{Q}(\beta)$ is a **g-β-set** if we have

$$Q = Q_{L,\beta} = \left\{ \sum_{k=0}^{n} a_k \beta^k \mid n \in \mathbb{N}, \ a_0...a_n \in L \right\},$$

where $\Sigma \subset \mathbb{Q}(\beta)$ is a finite alphabet and $L \subseteq \Sigma^*$ is a regular language.

Proposition

If $\omega$ is a fixed point of a substitution, then $Q_\omega$ is a g-β-set.

The aim of the following will be to give a reciprocal to this proposition.
**g-β-set coming from a substitution**

For the example

\[ s : \begin{cases} 
    a &\mapsto ab \\
    b &\mapsto ca \\
    c &\mapsto a 
\end{cases} \]

the mirror of the language \( L \), recognized by the following automaton, define a g-β-set which is a quasicrystal coming from the substitution \( s \), for \( β \) the Tribonacci number.
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Rauzy fractals, one dimensional Meyer sets, $\beta$-numeration and automata

Main results

$\times \beta$

$+0 \quad +1$

$\times \beta$

$+0 \quad +1 \quad +0 \quad +\beta^2 - \beta - 1$

$\times \beta$

$+0 \quad +1 \quad +0 \quad +\beta^2 - \beta - 1 \quad +0 \quad +0 \quad +1$

$\times \beta$

$+0 \quad +1 \quad +0 \quad +\beta^2 - \beta - 1 \quad +0 \quad +0 \quad +1$
Stability of the set of $g\beta$-sets

Properties (Properties of $g\beta$-sets)

If $\beta$ is an algebraic number without conjugate of modulus one, and if $Q_1$ and $Q_2$ are two $g\beta$-sets, then

- $Q_1 \cup Q_2$, $Q_1 \cap Q_2$ and $Q_1 \setminus Q_2$ are $g\beta$-sets,
- $Q_1 + Q_2$ is a $g\beta$-set,
- $\forall t \in \mathbb{Q}(\beta)$, $Q_1 + t$ is a $g\beta$-set,
- $\forall c \in \mathbb{Q}(\beta)$, $cQ_1$ is a $g\beta$-set,
- $\forall k \geq 1, n \geq 1$, a $g\beta^k$-set is a $g\beta^n$-set.

Moreover, everything is computable, and emptyness and inclusion are decidable.

Hence, it is easy to approximate any shape by $g\beta$-sets.
Main result:
Characterization of Meyer sets coming from substitutions

It is easy to prove that Rauzy fractals can approximate any shape with the previous properties of g-$\beta$-sets and with the following theorem.

**Theorem**

Let $\beta$ be a Pisot number, and let $Q \subseteq \mathbb{Q}(\beta)$ a $\beta$-invariant Meyer set. Then, the Meyer set $Q$ comes from a substitution if and only if it is a g-$\beta$-set that contains 0.

We have already seen that these conditions are necessary. Let us show that these are sufficient, and how to construct such substitution.
\( \beta \)-expansion algorithm in a \( \beta \)-invariant Meyer set

Let \( Q \) be a Meyer set and \( \beta \) be a Pisot number with \( \beta Q \subset Q \) and \( 0 \in Q \). Then we can define the following algorithm that gives an unique finite \( \beta \)-expansion of any element of \( Q \).

**Data:** \( x \in Q \)

**Result:** coefficients \( t_0 \) of a \( \beta \)-expansion of \( x \)

**while** \( x \neq 0 \) **do**

\[
\begin{align*}
    x &\leftarrow x - t_0 \text{ for } t_0 = \inf\{t \geq 0 \ | \ x - t \in \beta Q\}; \\
    x &\leftarrow x/\beta; \\
    \text{print } t_0; \\
\end{align*}
\]

**end**

The expansion of \( x \) is given by the successive elements \( t_0 \).
Proof

With the previous algorithm, we define the language

\[ L_Q := \{ a_0...a_n \in \Sigma_Q^* \mid a_0...a_n \text{ expansion of } x \text{ given by the algorithm } \} 0^* \]

over the finite alphabet

\[ \Sigma_Q := \{ \inf \{ t \geq 0 \mid x - t \in \beta Q \} \mid x \in Q \} . \]

In others word, \( L_Q \) is the unique subset of \( \Sigma_Q^* \) containing the empty word \( \epsilon \), such that \( Q = Q_{L_Q} \) and such that

\[ a_0...a_n \in L_Q \iff \left\{ \begin{array}{l}
    a_0 = \min \{ t \in \Sigma_Q \mid \sum_{k=0}^n a_k \beta^k \in \beta Q + t \} \\
    a_1...a_n \in L_Q
\end{array} \right. \]

Proposition

The following two sentences are equivalent.

- \( Q \) comes from a substitution.
- \( L_Q \) is a regular language.
Hence, to prove the main theorem, it is enough to prove the following lemma:

**Lemma**

We have the equivalence between:

- $L_Q$ is a regular language.
- $Q$ is a $g-\beta$-set.

The direct part is obvious. To prove the converse, we have to construct the language $L_Q$ from any regular language $L$ such that $Q = Q_L$. 

Proof
Step 1/3 : get a regular language over the alphabet $\Sigma_Q$

Let $L$ be a regular language over an alphabet $\Sigma \subset \mathbb{Q}(\beta)$ such that $Q = Q_L$.

Lemma (Change of the alphabet)

The following language is regular

$$L_{Q,\Sigma_Q} := \{a_0...a_n \in \Sigma_Q^* \mid n \in \mathbb{N}, \sum_{k=0}^{n} a_k \beta^k \in Q\},$$

and we have $Q_{L_{Q,\Sigma_Q}} = Q$.

Proof.

$$L_{Q,\Sigma_Q} = Z(p_1(L_{rel} \cap \Sigma_Q^* \times L0^*))$$

where $Z : L \mapsto \bigcup_{n \in \mathbb{N}} L0^{-n}$,

$$L_{rel} = \{(u, v) \in (\Sigma_Q \times \Sigma)^* \mid \sum_{k=0}^{n} (u_k - v_k) \beta^k = 0\}.$$

This last language is regular thanks to the main result of my paper « Semi-groupes fortement automatiques ».
Lemma (Stabilization by suffix)

*The greatest language* \( L' \subset L_{Q, \Sigma_Q} \) *such that*

\[
u \in L' \implies \text{every suffix of } u \text{ is in } L'
\]

*is a regular language, and we have* \( Q = Q_{L'} \).

**Proof.**

Take a deterministic automaton recognizing the mirror of \( L_{Q, \Sigma_Q} \). Remove every non final state. Then this new automaton recognize the mirror of \( L' \). And we have \( L_Q \subset L' \subset L_{Q, \Sigma_Q} \), hence \( Q = Q_{L'} \).
Lemma (Minimal words in lexicographic order describing $Q$)

We have the equality

$$L_Q = L' \setminus p_1(L' \times L' \cap L^{\text{rel}} \cap L^>)$$,

where

$$L^{\text{rel}} := \{(u, v) \in (\Sigma_Q \times \Sigma_Q)^* \mid \sum_{k=0}^{n} (u_k - v_k) \beta^k = 0\}$$

and

$$L^> := \{(u, v) \in (\Sigma_Q \times \Sigma_Q)^* \mid u > v \text{ for the lexicographic order}\}$$,

where we choose the natural order on $\Sigma_Q$, given by the embedding into the expanding space $E^e_\beta = \mathbb{R}$.

Hence $L_Q$ is regular, and this proves the theorem.
Proof of last lemma.

- $L' \times L' \cap L^{rel} \cap L^>$ is the couple of words of same length, giving the same element of $Q$, and with the left one strictly less than the right one for the lexicographic order.

- Hence $L' \setminus p_1(L' \times L' \cap L^{rel} \cap L^>)$ is the set of elements of $L'$ which are minimal in lexicographic order among the words of $L'$ of same length describing the same point of $Q$.

- We deduce the equality with $L_Q$: the language is still stable by suffix and the first letter is the minimal one, as in the definition of $L_Q$.

- The language $L^>$ is easily seen to be regular: we can recognize it with an automaton having two states.

- The language $L^{rel}$ is regular, thanks to my article « Semi-groupes fortement automatiques ».

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Rauzy fractals, one dimensional Meyer sets,
Let’s take the $g$-$\beta$-set defined by

$$\beta^3 = \beta^2 + \beta + 1.$$ 

The regular language $L$ described by this automaton is

$$L = 0^*1^* \cup 0^*1^+0100\{0, 1\}^*.$$ 

This $g$-$\beta$-set satisfy every hypothesis of the theorem, hence we can compute a substitution from it.
Corresponding substitution whose Perron number is $\beta$:

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Rauzy fractal
Construction of a domain exchange

\[-\beta^2 + 2\beta, \; \beta^2 - \beta - 1, \; \beta - 1, \; 1, \; -\beta^2 + 2\beta + 1, \; \beta^2 - \beta, \; \beta\]

Domain exchange on the model set defined by the unit disk window, and the integer ring \(\mathcal{O}_\beta\) where \(\beta\) is the Tribonnacci number.
Another application of $g$-$\beta$-sets

Let $s$ and $h$ be the substitutions

$s : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 12 \end{cases} 
\quad h : \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 3 \\ 3 \mapsto 4 \\ 4 \mapsto 5 \\ 5 \mapsto 1 \end{cases}$

and let $R_s \subseteq \mathbb{C}$ and $R_h \subseteq \mathbb{C}$ be their Rauzy fractals.

**Proposition**

$R_s$ is a countable union of homothetic transformations of $R_h$, union a set of dimension less than two.
Proof

Projecting a substitution on another with same $\beta$

We define the \textit{projection} of a language on another by

$$\text{Proj}(L, L') = \left\{ u \in L' \mid \sum_{i=0}^{\lfloor |u|^{-1} \rfloor} u_i \beta^i \in Q_L \right\} = Z(p_1(L' \times L^0 \cap L^{rel}))$$

\textbf{Proposition}

There exists regular languages $A$ and $B$ such that

$$\text{Proj}(0^3 L_s, L_h) = AL_h \cup B$$

with spectral radius of $B$ less than $\beta$.

\textbf{Figure – Minimal automata of $L_s$ and $L_h$ respectively}
Computation of the dimension

The box dimension of the part of dimension less than two is

$$\dim_{MB}(\sigma_{-}(Q_{LM})) = 2 \frac{\log(\gamma)}{\log(\beta)} \approx 1.94643460326525\ldots$$

where $\gamma \approx 1.31477860592584\ldots$ is the greatest root of $x^{13} - x^{12} - x^{10} + x^{9} - 2x^{4} + x^{3} - 1$ and $\beta$ is the smallest Pisot number.

Theorem

Let $\overline{\beta}$ be a complex conjugate of the smallest Pisot number $\beta$, and let $L \subseteq \Sigma^{*}$ be a language over the alphabet $\Sigma = \{0, 1\}$ such that the elements of $\sigma_{-}(Q_{L}) = \left\{ \sum_{i=0}^{\lfloor |u|/2 \rfloor} u_{i} \overline{\beta}^{i} \mid u \in L \right\} \subseteq \mathbb{C}$ are uniquely represented for a given length (i.e.

$$\forall u, v \in L, \left( |u| = |v| \text{ and } \sum_{i=0}^{\lfloor |u|/2 \rfloor} u_{i} \overline{\beta}^{i} = \sum_{i=0}^{\lfloor |v|/2 \rfloor} v_{i} \overline{\beta}^{i} \right) \implies u = v.$$

Then we have $\dim_{MB}(\sigma_{-}(Q_{L})) = \frac{\log(\gamma)}{\log(1/|\beta|)} = 2 \frac{\log(\gamma)}{\log(\beta)}$, where $\gamma$ is the spectral radius of the minimal automaton of $L$. 
Zoom in the Rauzy fractal of $s$