EXAMPLE OF FUCHSIAN GROUP OF SMALL CRITICAL EXPONENT BUT WITH A DISCRETE ORBIT OF POSITIVE DENSITY IN $\mathbb{Z}^2$

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Abstract. We construct an example of subgroup Γ of $SL(2,\mathbb{Z})$ of arbitrary small critical exponent, but with an orbit of positive density in $\mathbb{Z}^2$:

$$\liminf_{T\to\infty} \frac{1}{T^2} \# \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \Gamma \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\middle|\; \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| < T \right\} > 0.$$

The result is extended to every group of isometries of a Gromov-hyperbolic space that verify a certain property P.

Contents

Given a subgroup Γ of $SL(2,\mathbb{R})$, one can look at the discrete orbit of a point of $\mathbb{R}^2$. For example it could be the orbit of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under the action of $SL(2,\mathbb{Z})$: in this case the orbit is exactly the set $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \;\middle|\; x \wedge y = 1 \right\}$. And one can look at the growth rate of this orbit. For $SL(2,\mathbb{Z}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ one can check that we have

$$\# \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in SL(2,\mathbb{Z}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\middle|\; \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| \leq R \right\} \underset{R\to+\infty}{\sim} \frac{6R^2}{\pi}.$$

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We are mainly interested in the exponent that appears (2 for $SL(2,\mathbb{Z})$). Let us denote it $\delta^{\mathrm{eucl}}_{\Gamma,v}$. For a vector v, we have more precisely

$$\delta^{\mathrm{eucl}}_{\Gamma,v} = \limsup_{R\to+\infty} \frac{\ln \#\left\{\gamma\in\Gamma \;\middle|\; \|\gamma v\| \le R\right\}}{R}.$$

But there is another natural action for a discrete subgroup Γ of $SL(2,\mathbb{R})$. The group Γ also act on the hyperbolic plane $\mathbb{H}^2$, and we can look at the exponential growth rate of a discrete orbit in $\mathbb{H}^2$. This growth rate is usually called the critical exponent of the group Γ, and we will denote it by δ_Γ. See [Coo] for more details. For any point $o\in\mathbb{H}^2$, we have

$$\delta_\Gamma = \limsup_{R\to+\infty} \frac{\ln \#\left\{\gamma\in\Gamma \;\middle|\; d(o,\gamma o) \le R\right\}}{\ln R},$$

where $d(.,.)$ is the hyperbolic distance of $\mathbb{H}^2$. By triangular inequality, this critical exponent does not depends on $o\in\mathbb{H}^2$. For $\Gamma = SL(2,\mathbb{Z})$, we have $\delta_\Gamma = 1$.
The question is :

Is there a link between the growth rate $\delta^{\mathrm{eucl}}_{\Gamma,v}$ and the critical exponent δ_Γ ?

The following theorem (from [Dal2]) gives a positive answer to this question in some cases :

Theorem 0.1 (Dal'Bo)**.** *If $v\in\mathbb{R}^2$ is a eigenvector of a parabolic element of a discrete subgroup Γ of $SL(2,\mathbb{R})$ such that the orbit Γv is discrete, then we have*

$$\delta^{eucl}_{\Gamma,v} = 2\delta_\Gamma.$$

The example $\Gamma = SL(2,\mathbb{Z})$ and $v = \begin{pmatrix}1\\0\end{pmatrix}$ satisfy this theorem : v is an eigenvector of the parabolic element $\begin{pmatrix}1&1\\0&1\end{pmatrix}\in SL(2,\mathbb{Z})$. This statement can be generalize to groups of isometries acting on a $CAT(-1)$ space and is a corollary of the theorem 4.2 in [Robl]. But we show in this article that the result does not hold anymore if we does not assume that the vector v is an eigenvector for a parabolic element :

Theorem 0.2. *For every $\epsilon>0$, there exists a subgroup Γ of $SL(2,\mathbb{Z})$ such that $\delta_\Gamma\le\epsilon$, but with $\delta^{eucl}_{\Gamma,v} = 2$.*

See below for a more precise statement. The aim of this paper is to prove a stronger version of this theorem and to give a generalization to groups of isometries of a Gromov hyperbolic space. I thanks Pascal Hubert for this question.

0.1. **Organisation of the paper.** This paper is organized as follow : In section 1 we expose our two main results. The first one is about $SL(2,\mathbb{Z})$ acting on $\mathbb{H}^2$, and the other one is a generalization to any group of isometries of a Gromov-hyperbolic space satisfying a certain property P that we present. In the section 2 we give some tools to make our ping-pong strategy works, in order to control the critical exponent. In the section 3 we construct a subgroup Γ of a group G, satisfying a list of properties. In the section 4 we prove the second theorem (the most general one) using the properties of the group Γ and the property P for the group G. The last section 5 is devoted to the proof that $SL(2,\mathbb{Z})$ acting on $\mathbb{H}^2$ satisfy the property P, and this gives a proof of the first theorem as a corollary of the second one.

1. Results

The aim of this paper is to prove the following theorem and a generalization.

Theorem 1.1. *For any $\epsilon > 0$, there exists a subgroup Γ of $SL(2,\mathbb{Z})$ such that the discrete orbit of $\begin{pmatrix}1\\0\end{pmatrix}$ has positive density in $\mathbb{Z}^2$:*

$$\liminf_{T\to\infty} \frac{1}{T^2} \# \left\{ \begin{pmatrix}x\\y\end{pmatrix} \in \Gamma \begin{pmatrix}1\\0\end{pmatrix} \;\middle|\; \left\| \begin{pmatrix}x\\y\end{pmatrix} \right\| < T \right\} > 0,$$

and such that the critical exponent is smaller than ϵ :

$$\limsup_{n\to\infty} \frac{1}{2\ln(n)} \ln\left(\# \left\{\gamma \in \Gamma \;\middle|\; \|\gamma\| < n\right\}\right) \leq \epsilon.$$

This statement can be generalized to any group of isometries of a Gromov-hyperbolic space, with the following correspondance :

$\mathbb{H}^2$	X proper Gromov-hyperbolic space
$SL(2,\mathbb{Z})$	G discrete group of isometries of X
$\Gamma < SL(2,\mathbb{Z})$	$\Gamma < G$
$\begin{pmatrix}1\\0\end{pmatrix}$	P parabolic group fixing a point $\infty \in \partial X$
$\mathbb{Z}^2$	G/P
$\|.\|$	$e^{\frac{1}{2}\beta_\infty(\gamma^{-1}o,o)}$
$\Gamma\begin{pmatrix}1\\0\end{pmatrix}$ discrete	$\forall R \in \mathbb{R}, \left\{\gamma \in G/P \;\middle|\; \beta_\infty(\gamma^{-1}o,o) \leq R\right\}$ is finite.

This correspondance comes from the lemma 5.1 and lemma 5.2.

Remark 1.2. *For $\gamma \in G/P$, the notation $\beta_\infty(\gamma^{-1}o,o)$ makes sense because P stabilize ∞.*

But to generalize the theorem, we will need some hypothesis on the groupe G that are satisfied for $SL(2,\mathbb{Z})$.We could prove the following weaker statement, using a result of T. Roblin :

Theorem 1.3. *Let X be a $CAT(-1)$ space and G be a group of isometries of X satisfying the hypothesis of theorem 4.2 in* [Robl]*, and such that the convergence in this theorem is toward a non-atomic measure having finite and non-zero mass. Let $o \in X$ and $\infty \in \partial X$. Assume that the stabilizer of o is trivial and that the stabilizer P of ∞ is non-trivial and that we have*

$$\forall R \in \mathbb{R},\ \left\{\gamma \in G/P \;\middle|\; \beta_\infty(\gamma^{-1}o,o) \leq R\right\} \text{ is finite.}$$

Then, for every $\epsilon > 0$, there exists a subgroup $\Gamma < G$ such that $\delta_\Gamma \leq \epsilon$ and such that

$$\limsup_{R\to\infty} \frac{\#\left\{\gamma \in \Gamma/P \;\middle|\; \beta_\infty(\gamma^{-1}o,o) \leq R\right\}}{\#\left\{\gamma \in G/P \;\middle|\; \beta_\infty(\gamma^{-1}o,o) \leq R\right\}} \geq 1-\epsilon,$$

where $\Gamma/P := \{\gamma P | \gamma \in \Gamma\} \subseteq G/P$ denote the orbit of P under Γ.

This theorem gives a weaker conclusion (lim sup in place of lim inf) but is satisfied for a large class of groups thanks to T. Roblin. We will not prove this result in this paper, but it can be done by modifying slightly the prove of the theorem 1.6 below. In order to have the stronger conclusion, we will replace what is given by the theorem of Roblin by the following property :

Property 1.4 (Property P)**.** *Let X be a proper Gromov-hyperbolic space, $o \in X$, and $\infty \in \partial X$. Let G be a discrete group of isometries of X, and P be the maximal parabolic subgroup of G stabilizing ∞. We say that (X, G, ∞) has property P if we have*

$$\forall R \in \mathbb{R},\ \left\{\gamma \in G/P \;\middle|\; \beta_\infty(\gamma^{-1}o, o) \leq R\right\} \text{ is finite,}$$

and if there exists $o : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{\infty\}$ such that for all $\xi \in \partial X$, there exists $\mu_\xi : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{\infty\}$ such that $\forall r \in \mathbb{R}^+,\ \forall R \in \mathbb{R}^+$,

$$\frac{\#\left\{\gamma \in G/P \;\middle|\; \gamma\infty \in B(\xi, r),\ \beta_\infty(\gamma^{-1}o, o) \leq R\right\}}{\#\left\{\gamma \in G/P \;\middle|\; \beta_\infty(\gamma^{-1}o, o) \leq R\right\}} \leq \mu_\xi(r) + o(R)$$

where $B(\xi, r) := \left\{y \in \overline{X} \;\middle|\; (\xi|y) \geq -\ln(r)\right\}$, and with

$$\lim_{R\to\infty} o(R) = 0 \quad \text{ and } \quad \forall \xi \in \partial X,\ \lim_{r\to 0} \mu_\xi(r) = 0.$$

The notation $\beta_\infty(.,.)$ stands for the Busemann function at ∞, and $(\xi|y) = (\xi|y)_o$ is the Gromov product of ξ and y with base point o.

This property says that the number of elements of $G\infty$ in small balls is small but with an uniform control. The function μ_ξ can be compared to a measure (we extends it to any $Y \subseteq \overline{X}$ below), and the sentence $\forall \xi \in \partial X,\ \lim_{r\to 0} \mu_\xi(r) = 0$ corresponds to say that the measure has no atom. If we replace the $o(R)$ by an $o(R, \xi, r)$ that tends to 0 for every $\xi \in \partial X$ and $r > 0$, the property can be verified for many groups of isometries of $CAT(-1)$ spaces using the result of T.Roblin. But here we need more : we need this uniformity in order to do a limits exchange to prove that the density of the orbit under the constructed group is big.

Remark 1.5. *We can assume that the function o in the property P is strictly decreasing even if it means replacing it by $R \mapsto 1/R + \sup_{r>R} o(r)$. For a given $\epsilon > 0$, the function $R \mapsto \frac{\epsilon}{2o(R)}$ is then strictly increasing and tends to $+\infty$. Therefore this function is injective, and there exists a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\forall k \in \mathbb{R}^+,\ g(k) \leq R \iff k \leq \frac{\epsilon}{2o(R)}$.*

This will be usefull to construct a group with a big orbit.

Notation . *For $Y \subseteq \overline{X}$, we define*

$$\mu(Y) := \inf\left\{\mu_\xi(r) \;\middle|\; \xi \in \partial X \text{ and } r > 0 \text{ such that } Y \subseteq B(\xi, r)\right\}.$$

Then we have for all $R > 0$,

$$\frac{\#\left\{\gamma \in G/P \;\middle|\; \gamma\infty \in Y,\ \beta_\infty(\gamma^{-1}o, o) \leq R\right\}}{\#\left\{\gamma \in G/P \;\middle|\; \beta_\infty(\gamma^{-1}o, o) \leq R\right\}} \leq \mu(Y) + o(R).$$

The following theorem is the main result of this paper. It is a generalization of the theorem 1.1 since $SL(2, \mathbb{Z})$ verifies the property P, as we show in section 5.

Theorem 1.6. *Let (X, G, ∞) satisfying the property P. Let $o \in X$. Assume that the stabilizer of o is trivial and that the stabilizer P of ∞ is non-trivial. Then, for every $\epsilon > 0$, there exists a subgroup $\Gamma < G$ such that $\delta_\Gamma \leq \epsilon$ and such that*

$$\liminf_{R\to\infty} \frac{\#\left\{\gamma \in \Gamma/P \;\middle|\; \beta_\infty(\gamma^{-1}o, o) \leq R\right\}}{\#\left\{\gamma \in G/P \;\middle|\; \beta_\infty(\gamma^{-1}o, o) \leq R\right\}} \geq 1 - \epsilon,$$

where $\Gamma/P := \{\gamma P | \gamma \in \Gamma\} \subseteq G/P$ denotes the orbit of P under Γ.

If we compare this theorem to the theorem 1.1, the last inequality generalizes the fact that the orbit has positive density in $\mathbb{Z}^2$. The generalization of the linear action of $SL(2,\mathbb{Z})$ on $\mathbb{R}^2$ is the action of the group G on the set of horoballs of the space X (or equivalently on G/P, for our matter), and the norm of vectors of $\mathbb{R}^2$ becomes the exponential of the Busemann function.

For $X = \mathbb{H}^2$ and $G = SL(2,\mathbb{Z})$, all the hypothesis of this last theorem are satisfied, as shown in section 5, and this gives the first theorem.

In all the following we assume that the hypothesis of this last theorem are verified: X is a proper Gromov-hyperbolic space and ∂X will denote its boundary. $\infty \in \partial X$ and $o \in X$ are set, and $G < \mathrm{Isom}(X)$ is a discrete subgroup such that the stabilizer of o is trivial and such that the stabilizer P of ∞ is non-trivial. And we assume that (X, G, ∞) satisfies the property P. In particular, we have

$$\exists C,\ \forall \gamma \in G/P,\ \beta_\infty(\gamma^{-1}o, o) \geq C.$$

This guarantees that the stabilizer P of ∞ is a parabolic group, because for every $\gamma \in P$, we have

$$\beta_\infty(\gamma^n o, o) = n\beta_\infty(\gamma o, o), \quad \forall n \in \mathbb{Z},$$

hence for every $\gamma \in P\backslash\{id\}$, we must have $\beta_\infty(\gamma o, o) = 0$, therefore γ is parabolic.

2. Ping-pong

In this section we give tools that permits to measure the level of a ping-pong, in order to approach the triangular equality. This will permit to control the critical exponent of the constructed group.

Notation . *We denote by $\overline{X} = X \cup \partial X$ the Gromov-hyperbolic space with its boundary. For $x, y \in \overline{X}$, we denote by $(x|y)$ the Gromov product from o. For $x, y \in X$, we have $(x|y) = (x|y)_o = \frac{1}{2}\left(d(o,x) + d(o,y) - d(x,y)\right)$ by definition. For $x, y \in X$ and $\xi \in \partial X$, we denote by $\beta_\xi(x,y) := (\xi|x)_y - (\xi|y)_x$ the Busemann function (that is the algebraic distance between two horocycles based at point ξ : one passing through x and the other one passing through y).*

For $\alpha \in \mathbb{R}$ and $x \in X$, let

$$X_x^\alpha := \left\{ y \in \overline{X} \;\middle|\; (x|y) \geq \tfrac{1-\alpha}{2} d(o,x) \right\}.$$

and for $\gamma \in G$, let $X_\gamma^\alpha := X_{\gamma o}^\alpha$. As we will see, these sets are usefull to make a ping-pong.

Remark 2.1. *We have $\gamma\left(\overline{X}\backslash X_{\gamma^{-1}}^\alpha\right) \subseteq X_\gamma^{-\alpha}$.*

Definition 2.2. *Let $\gamma \in G$, and let $X^- \subseteq \overline{X}$ and $X^+ \subseteq \overline{X}$. We say that (γ, X^-, X^+) is a **ping-pong player of level** $\alpha \in \mathbb{R}$ if we have the inclusions $X_{\gamma^{-1}}^\alpha \subseteq X^-$, $X_\gamma^\alpha \subseteq X^+$ and $X^- \cap X^+ = \emptyset$.*

Properties 2.3. *(1) The level of a ping-pong player is necessarily less than 1.*

(2) If the level of a ping-pong player (γ, X^-, X^+) is positive or equal to zero, then we have the inclusion $\gamma(X\backslash X^-) \subseteq X^+$.

(3) *Let (γ, X^-, X^+) be a ping-pong player of level α and let $y \in X \backslash X^-$. Then we have*

$$\begin{aligned} d(\gamma y, o) &= d(\gamma o, o) + d(y, o) - 2(\gamma^{-1} o|y) \\ &\geq \alpha d(\gamma o, o) + d(y, o). \end{aligned}$$

(4) *The inverse (γ^{-1}, X^+, X^-) of a ping-pong player (γ, X^-, X^+) of level α is also a ping-pong player of level α.*

Definition 2.4. *We say that ping-pong players $\left((\gamma_i, X_i^-, X_i^+)\right)_{i \in I}$ are playing together if the sets $\overline{X_i^-}$ and $\overline{X_i^+}$ are all pairwise disjoint.*

Remark 2.5. *A group generated by ping-pong players of non-negative level playing together is usually called a* ***Schottky group****. It is a free group and it contains only hyperbolic isometries.*

We will see in section 4.1 that if moreover the levels of ping-pong players playing together are greater than a given constant $C > 0$ and that the isometries are big enough, then the critical exponent is small.

3. The construction

The idea is to construct a infinitely generated Schottky group with ping-pong players of good level, and with one player for each point of G/P that we want to keep. Using the property P, we can construct this group in such a way that we can keep most of the points of G/P.
Let $(\overline{\gamma_i})_{i \in \mathbb{N}}$ be an injective enumeration of the set G/P such that the sequence $\left(\beta_\infty\left(\overline{\gamma_i}^{-1} o, o\right)\right)_{i \in \mathbb{N}}$ is increasing. Let $p \in P \backslash \{id\}$ be any parabolic element fixing ∞, and let $\epsilon > 0$.

We construct a subset $I \subseteq \mathbb{N}$ (corresponding to a subset of G/P) and a sequence of integers $(k_i)_{i \in I} \in \mathbb{N}^I$ such that the group Γ generated by the elements

$$\gamma_i := \gamma_i' p^{k_i}, \quad i \in I$$

satisfies a list of ten properties (see below), where γ_i' is any element of G that projects to $\overline{\gamma_i}$ in G/P.

Let

$$X_0 := \left\{ x \in \overline{X} \;\middle|\; (x|\infty) \geq c \right\},$$

where $c > 0$ is chosen big enough to have $\mu(X_0) < \epsilon/2$, and

$$X_i^- := X_{\gamma_i^{-1}}^{-3/4},$$

$$X_i^+ := \left\{ x \in \overline{X} \;\middle|\; (x|\gamma_i o) \geq \tfrac{1}{8} d(o, \gamma_i o) - \delta \right\},$$

where the number δ is such that the space X is δ-hyperbolic.

Remark 3.1. *We have $\gamma_i(\overline{X} \backslash X_i^-) \subseteq X_i^+$.*

We construct the sequence $(k_i)_{i \in I}$ and the set I by induction, with the following properties:

(1) $i \in I \Longleftrightarrow \overline{\gamma_i}\infty \notin \bigcup_{\substack{j \in I \\ j < i}} X_j^+ \cup X_0$.

(2) $o \notin \bigcup_{i \in I} X_i^+$ and $\forall i \in I,\ X_i^- \subseteq X_0$.

(3) $\forall i, j \in I,\ i < j \Longrightarrow 2d(\gamma_i o, o) \leq d(\gamma_j o, o)$.

(4) We have the following inequality (strict if I is finite) :
$$\sum_{i\in I} e^{-\frac{\epsilon}{4}d(\gamma_i o,o)} + ie^{-\frac{\epsilon}{8}d(\gamma_i o,o)} \leq 1/4.$$
(5) $\forall i \in I,\ \infty \notin X_i^-$.
(6) The sets $X_i^+,\ i\in I$ and X_0 are all pairwise disjoint, and the sets $X_i^-,\ i\in I$ are pairwise disjoint.
(7) For all $i\in I$, (γ_i, X_0, X_i^+) is a ping-pong players of level $1/4$.
(8) For all $i\neq j\in I$, $(\gamma_i\gamma_j^{-1}, X_j^+, X_i^+)$ is a ping-pong player of level $1/4$.
(9) We have the following inequality (strict if I is finite) :
$$\mu(X_0) + \sum_{i\in I}\mu(X_i^+) \leq \epsilon/2$$
(10) $\forall i\in I,\ \forall\overline{\gamma}\in G/P,\ \overline{\gamma}\infty\in X_i^+ \implies \beta_\infty(\overline{\gamma}^{-1}o,o) > g(i)$ or $\overline{\gamma}=\overline{\gamma_i}$, where g is the function defined on remark 1.5.

The properties 2, 3, 4, 7, 8 and 6 will permit to show that the group generated by the $\gamma_i,\ i\in I$, is a Schottky group of arbitrary small critical exponent. And the properties 1, 9 and 10 will guaranty that the orbit of P under Γ will have a positive density in G/P.

3.1. **Induction.** Assume that the properties 2, 3, 4, 5, 6, 7, 8, 9, and 10 are satisfied, and that the property 1 is true for $i<n$.

(1) If we have
$$\overline{\gamma_n}\infty \in \bigcup_{\substack{i\in I\\ i<n}} X_i^+ \cup X_0$$
then property 1 is also true for $i=n$.

Else assume that $\overline{\gamma_n}\infty \notin \bigcup_{\substack{i\in I\\ i<n}} X_i^+\cup X_0$. Then we add n to the set I : $I\leftarrow I\cup\{n\}$ so the property 1 is verified for every $i\leq n$.
(2) We have $X_n^- \xrightarrow[k_n\to\infty]{} \{\infty\}$, and X_0 is a neighbourhood of ∞. So for k_n large enough we have $X_n^-\subseteq X_0$. Moreover, for k_n large enough we have $o\notin X_n^+$ since $d(\gamma_n o,o)\xrightarrow[k_n\to\infty]{} +\infty$. Hence the property 2 is true for k_n large enough.
(3) We have $d(\gamma_n o,o)\xrightarrow[k_n\to\infty]{} +\infty$, so the property 3 is true as soon as k_n is large enough.
(4) Property 4 is true for large enough k_n because we have
$$\lim_{k_n\to\infty}\sum_{i\in I} e^{-\frac{\epsilon}{4}d(\gamma_i o,o)} + ie^{-\frac{\epsilon}{8}d(\gamma_i o,o)} = \sum_{i\in I\setminus\{n\}} e^{-\frac{\epsilon}{4}d(\gamma_i o,o)} + ie^{-\frac{\epsilon}{8}d(\gamma_i o,o)} < \frac{1}{4}.$$
(5) We have
$$(\infty|\gamma_n^{-1}o) = \frac{1}{2}\left(d(o,\gamma_n o) - \beta_\infty(\gamma_n^{-1}o,o)\right) \leq \frac{1}{2}(d(o,\gamma_n o) - C).$$
So for k_n large enough we have
$$(\infty|\gamma_n^{-1}o) < \frac{1+3/4}{2}d(o,\gamma_n o),$$

so ∞ is not in $X_n^- = X_{\gamma_n^{-1}}^{-3/4}$.

(6) We have $X_n^+ \xrightarrow[k_n \to \infty]{} \{\overline{\gamma_n}\infty\}$, and $\bigcup_{\substack{j\in I\\ j<n}} X_j^+ \cup X_0$ is a closed set that does not contain $\overline{\gamma_n}\infty$. Therefore for k_n large enough we have $X_n^+ \cap \left(\bigcup_{\substack{j\in I\\ j<n}} X_j^+ \cup X_0 \right) = \emptyset$.

In a similar way, we have $X_n^- \xrightarrow[k_n \to \infty]{} \{\infty\}$ and $\infty \notin \bigcup_{\substack{j\in I\\ j<n}} X_j^-$ by previous property, hence $X_n^- \cap \left(\bigcup_{\substack{j\in I\\ j<n}} X_j^- \right) = \emptyset$ for k_n large enough.

(7) Like for property 2 we have $X_{\gamma_n^{-1}}^{3/4} \subseteq X_0$ for k_n large enough, and we have $X_{\gamma_n}^{3/4} \subseteq X_n^+$ by definition, therefore (γ_n, X_0, X_n^+) is a ping-pong player of level $3/4$ (so also of level $1/4$).

(8) Let us show the inclusion $X_{\gamma_i\gamma_j^{-1}}^{1/4} \subseteq X_i^+$. Let $x \in X_{\gamma_i\gamma_j^{-1}}^{1/4}$. We have

$$(x|\gamma_i\gamma_j^{-1}o) \geq \frac{3}{8} d(o, \gamma_i\gamma_j^{-1}o) \geq \frac{1}{8} d(o, \gamma_i o)$$

by property 3. We have $o \in X \backslash X_j^+$ by property 2 so $\gamma_j^{-1}o \in X_j^-$. Therefore we have $\gamma_j^{-1}o \in X \backslash X_i^-$ by property 6. Then by remark 2.1 we have $\gamma_i\gamma_j^{-1}o \in X_{\gamma_i}^{3/4}$. Using the δ-hyperbolicity of X we have

$$(x|\gamma_i o) \geq \min\left\{(x|\gamma_i\gamma_j^{-1}o), (\gamma_i\gamma_j^{-1}o|\gamma_i o)\right\} - \delta$$

and we have $\min\left\{(x|\gamma_i\gamma_j^{-1}o), (\gamma_i\gamma_j^{-1}o|\gamma_i o)\right\} \geq \frac{1}{8} d(o, \gamma_i o)$. Therefore we have $x \in X_i^+$. The inclusion $X_{\gamma_j\gamma_i^{-1}}^{1/4} \subseteq X_j^+$ is obtained by reversing the role of i and j. Finally we have shown that $(\gamma_i\gamma_j^{-1}, X_j^+, X_i^+)$ is a ping-pong player of level $1/4$.

(9) We can choose $r \in \mathbb{R}^+$ small enough such that

$$\mu(X_0) + \sum_{\substack{i\in I\\ i<n}} \mu(X_i^+) + \mu(B(\overline{\gamma_n}\infty, r)) < \epsilon/2.$$

Then, using that $Y_n^+ \xrightarrow[k_n \to +\infty]{} \{\overline{\gamma_n}\infty\}$, we have $Y_n^+ \subseteq B(\overline{\gamma_n}\infty, r)$ for k_n large enough.

(10) Let $J_n = \left\{ j \in \mathbb{N} \;\middle|\; j \neq n \text{ and } \beta_\infty(\overline{\gamma_j}^{-1}o, o) \leq g(n) \right\}$. The set J_n is finite and does not contain n. Hence the quantity $\sup_{j\in J_n}(\overline{\gamma_j}\infty|\overline{\gamma_n}\infty)$ is finite, and moreover it does not depends on k_n. Therefore we can choose k_n big enough to have

$$\min\left\{ (\gamma_n o|\overline{\gamma_n}\infty) - \delta,\ \frac{1}{8} d(o, \gamma_n o) - 2\delta \right\} > \sup_{j\in J_n} (\overline{\gamma_j}\infty \mid \overline{\gamma_n}\infty).$$

Then, by δ-hyperbolicity we have for all $j \in J_n$,

$$(\overline{\gamma_j}\infty|\overline{\gamma_n}\infty) \geq \min\left\{(\overline{\gamma_j}\infty|\gamma_n o), (\gamma_n o|\overline{\gamma_n}\infty)\right\} - \delta = (\overline{\gamma_j}\infty|\gamma_n o) - \delta,$$

and then we have

$$\frac{1}{8} d(o, \gamma_n o) - 2\delta > (\overline{\gamma_j}\infty|\gamma_n o) - \delta,$$

so $\overline{\gamma_j}\infty \notin X_n^+$.

By induction, we have constructed a set $I \subseteq \mathbb{N}$ and a sequence of integers $(k_n)_{n\in I}$ satisfying all the announced properties. We now define the group Γ as the group generated by the isometries $(\gamma_n)_{n\in I}$.

In the following section, we prove that this group Γ satisfies what we want using the above properties.

4. Proof of the theorem 1.6

In this section, we prove that the group Γ constructed in the previous section satisfy all the properties we wanted : we show that it is an infinitely generated Schottky group of arbitrary small critical exponent and that the orbit of P under Γ has positive density in G/P.

4.1. **Critical exponent.** The elements (γ_n, X_n^-, X_n^+) are not good players enough, because X_n^- is too small. But X_n^- cannot be big enough because it has to avoid ∞. It is the reason why we do not consider directly the Schottky group generated by the elements $\gamma_n,\ n \in I$. We consider the same group with the bigger set of generators

$$S = \left\{\gamma_i \;\middle|\; i \in I\right\} \cup \left\{\gamma_i^{-1} \;\middle|\; i \in I\right\} \cup \left\{\gamma_i\gamma_j^{-1} \;\middle|\; i \neq j \in I\right\},$$

and we consider only products that plays to ping-pong with good ping-pong players.

To simplify the notations, let

— $Y_{\gamma_n}^- := X_0$
— $Y_{\gamma_n}^+ := X_n^+$
— $Y_{\gamma_n^{-1}}^- := X_n^+$
— $Y_{\gamma_n^{-1}}^+ := X_0$
— $Y_{\gamma_n\gamma_m^{-1}}^+ = X_n^+$
— $Y_{\gamma_n\gamma_m^{-1}}^- = X_m^+$

for all $n \neq m \in I$. By properties 7 and 8 of the construction, we have that $\forall \gamma \in S,\ (\gamma, Y_\gamma^-, Y_\gamma^+)$ is a ping-pong player of level 1/4. And by property 6, we have that $\forall (\gamma,\gamma') \in S^2,\ Y_\gamma^+ \cap Y_{\gamma'}^- \neq \emptyset \Longleftrightarrow Y_\gamma^+ = Y_{\gamma'}^-$.

Definition 4.1. *Let $w = w_1w_2...w_n \in S^*$ be a word over the alphabet S. We say that the word w is **reduced** if for all $1 \le i < n$ we have $Y_{w_i}^- \cap Y_{w_{i+1}}^+ = \emptyset$.*

In others words, a reduced word is a product of elements such that every consecutive elements plays to ping-pong. The following lemma says that every element of Γ can be written like that.

Lemma 4.2. *Every element of the group Γ can be written as a reduced word.*

Proof. Any element of Γ is obtained as a word over the alphabet $\left\{\gamma_i \;\middle|\; i \in I\right\} \cup \left\{\gamma_i^{-1} \;\middle|\; i \in I\right\}$ such that there is no subword of the form $\gamma_i\gamma_i^{-1}$ or $\gamma_i^{-1}\gamma_i$ (it is the usual definition of reduced word). We associate to such a word, a word over the alphabet S, by replacing every occurrence of $\gamma_i\gamma_j^{-1}$ for $i \neq j \in I$ by a single letter of S. Let us show that the word obtained by this process is reduced. Let $w_1w_2...w_n$ such a word. Then, for every $1 \le i < n$ we have two cases

— If $Y_{w_{i+1}}^+ = X_0$, then we have $w_{i+1} = \gamma_k^{-1}$ for a $k \in I$. The letter w_i cannot be equal to some γ_l for a $l \in I$ otherwise the word could be reduced. Therefore we have $Y_{w_i}^- \neq X_0$, hence $Y_{w_{i+1}}^+ \cap Y_{w_i}^- = \emptyset$.

— If $Y^+_{w_{i+1}} = X^+_k$ for some $k \in I$, then we have $w_{i+1} = \gamma_k$ or $w_{i+1} = \gamma_k \gamma_l^{-1}$ for some $l \in I \backslash \{k\}$. The letter w_i is not equal to γ_k^{-1} neither to $\gamma_j \gamma_k^{-1}$ for some $j \in I$, otherwise the word could be reduced. Therefore we have $Y^-_{w_i} \neq X^+_k$, hence $Y^-_{w_i} \cap Y^+_{w_{i+1}} = \emptyset$.

□

Now, using the fact that good ping-pong players are playing together, we get the following

Proposition 4.3. *Let $g_1 g_2 ... g_n \in S^*$ be a reduced word. Then we have*

$$d(g_1 g_2 ... g_n o, o) \geq \frac{1}{4} \left(d(g_1 o, o) + d(g_2 o, o) + ... + d(g_n o, o) \right).$$

Proof. We show by induction on $k \in \mathbb{N}$ that

$$d(g_1 g_2 ... g_k y, o) \geq \frac{1}{4} \left[d(g_1 o, o) + d(g_2 o, o) + ... + d(g_k o, o) \right] + d(y, o)$$

for all $y \in X \backslash Y^-_{g_k}$.

If $k = 0$, the result is obvious.

Assume that the result is true for a $k < n$. For any $y \in X \backslash Y^-_{g_k}$, we have $d(g_{k+1} y, o) \geq \frac{1}{4} d(g_{k+1} o, o) + d(y, o)$ because $(g_{k+1}, Y^-_{g_{k+1}}, Y^+_{g_{k+1}})$ is a ping-pong player of level $1/4$. We have $g_{k+1} y \in Y^+_{g_{k+1}}$, and we have $Y^+_{g_{k+1}} \cap Y^-_{g_k} = \emptyset$ because the word is assumed reduced. Hence, we have $g_{k+1} y \notin Y^-_{g_k}$. Therefore we can apply the induction hypothesis and we get

$$\begin{aligned} d(g_1 g_2 ... g_k g_{k+1} y, o) &\geq \frac{1}{4} [d(g_1 o, o) + d(g_2 o, o) + ... + d(g_k o, o)] + d(g_{k+1} y, o) \\ &\geq \frac{1}{4} [d(g_1 o, o) + d(g_2 o, o) + ... + d(g_k o, o) + d(g_{k+1} o, o)] + d(y, o). \end{aligned}$$

The induction is proven, and by taking $y = o$, we end the proof of the lemma. □

Corollary 4.4. *We have $\delta_\Gamma \leq \epsilon$.*

Proof. Let us show that the Poincaré series converges at ϵ. We have

$$\begin{aligned} \sum_{\gamma \in \Gamma} e^{-\epsilon d(o, \gamma o)} &= \sum_{\substack{g_1 g_2 \cdots g_n \in S^* \\ \text{reduced word} \\ n \in \mathbb{N}}} e^{-\epsilon d(o, g_1 g_2 \ldots g_n o)} \\ &\leq \sum_{\substack{g_1 g_2 \cdots g_n \in S^* \\ n \in \mathbb{N}}} e^{-\frac{\epsilon}{4} (d(o, g_1) + d(o, g_2 o) + \ldots + d(o, g_n o))} \\ &= \sum_{i=0}^{+\infty} \left(\sum_{g \in S} e^{-\frac{\epsilon}{4} d(o, g)} \right)^n. \end{aligned}$$

So it is enough to prove that $\sum_{g\in S} e^{-\frac{\epsilon}{4}d(o,g)} < 1$. We have

$$\begin{aligned}
\sum_{g\in S} e^{-\frac{\epsilon}{4}d(o,g)} &= 2\sum_{i\in I} e^{-\frac{\epsilon}{4}d(o,\gamma_i)} + 2\sum_{i\in I}\sum_{\substack{j\in I\\ j<i}} e^{-\frac{\epsilon}{4}d(o,\gamma_i\gamma_j^{-1})} \\
&\leq 2\sum_{i\in I} e^{-\frac{\epsilon}{4}d(o,\gamma_i)} + 2\sum_{i\in I}\sum_{\substack{j\in I\\ j<i}} e^{-\frac{\epsilon}{8}d(o,\gamma_i)} \\
&\leq 2\sum_{i\in I} e^{-\frac{\epsilon}{4}d(o,\gamma_i)} + ie^{-\frac{\epsilon}{8}d(o,\gamma_i)} \\
&\leq 1/2
\end{aligned}$$

by properties 3 and 4.

□

This proves that the constructed group has arbitrary small critical exponent since $\epsilon > 0$ can be chosen as small as we want.

4.2. **Positive density of the orbit.** In this subsection, we prove that the orbit of the constructed group Γ has positive density in the orbit of G. More precisely, we show that

$$\liminf_{R\to+\infty} \frac{\#\left\{\gamma\in\Gamma/P \;\middle|\; \beta_\infty(\gamma^{-1}o,o)\leq R\right\}}{\#\left\{\gamma\in G/P \;\middle|\; \beta_\infty(\gamma^{-1}o,o)\leq R\right\}} \geq 1-\epsilon.$$

In order to prove that, let's remarks that we have $\forall i\in I, \gamma_i\infty\in X_i^+$ by property 5. Hence, we can write

$$\Gamma/P = \left\{\gamma\in G/P \;\middle|\; \gamma\infty\notin X_0 \text{ and } \forall i\in I,\ (\gamma=\gamma_i \text{ or } \gamma\infty\notin X_i^+)\right\},$$

and, we have

$$\begin{aligned}
&\frac{\#\left\{\gamma\in\Gamma/P \;\middle|\; \beta_\infty(\gamma^{-1}o,o)\leq R\right\}}{\#\left\{\gamma\in G/P \;\middle|\; \beta_\infty(\gamma^{-1}o,o)\leq R\right\}} \\
\geq\;& 1-\frac{\#\left\{\gamma\in G/P \;\middle|\; \gamma\infty\in X_0,\ \beta_\infty(\gamma^{-1}o,o)\leq R\right\}}{\#\left\{\gamma\in G/P \;\middle|\; \beta_\infty(\gamma^{-1}o,o)\leq R\right\}} \\
&-\sum_{i\in I}\frac{\#\left\{\gamma\in (G/P)\setminus\{\overline{\gamma_i}\} \;\middle|\; \gamma\infty\in X_i^+,\ \beta_\infty(\gamma^{-1}o,o)\leq R\right\}}{\#\left\{\gamma\in G/P \;\middle|\; \beta_\infty(\gamma^{-1}o,o)\leq R\right\}} \\
\geq\;& 1-\mu(X_0)-o(R)-\sum_{\substack{i\in I\\ g(i)\leq R}}\left(\mu(X_i^+)+o(R)\right) \\
\geq\;& 1-\mu(X_0)-\sum_{i\in I}\mu(X_i^+)-\left(\frac{\epsilon}{2o(R)}+2\right)o(R) \\
\geq\;& 1-\epsilon-2o(R).
\end{aligned}$$

So the $\liminf_{R\to+\infty}$ is greater than or equal to $1-\epsilon$.

5. Property P for $SL(2,\mathbb{Z})$

In this section we show that $X=\mathbb{H}^2$ and $G=SL(2,\mathbb{Z})$ verify the property P. This gives a proof of the theorem 1.1, by using the theorem 1.6 and lemma 5.5.

In the following, we identify ∂X with $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. For all $\gamma \in SL(2,\mathbb{Z})$, we have $\gamma\infty \in \mathbb{Q} \cup \{\infty\}$. The parabolic group at ∞ is the subgroup P generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The following lemma permits to characterize the set G/P.

Lemma 5.1. *We have*

$$G/P = SL(2,\mathbb{Z}) \Big/ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \simeq SL(2,\mathbb{Z}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \;\middle|\; x \wedge y = 1 \right\}.$$

Proof. Given a vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2$ such that $x \wedge y = 1$, we get a matrix $\gamma = \begin{pmatrix} x & u \\ y & v \end{pmatrix} \in SL(2,\mathbb{Z})$ (and hence we get an element of G/P) by taking the Bézout coefficients $(u,v) \in \mathbb{Z}^2$ of $x, y : xv - yu = 1$. We get back the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ from an element of G/P by taking the image of the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (which is well defined because P stabilize the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$). $\square$

The following lemma gives a way to compute the Buseman function.

Lemma 5.2. *For all $\gamma \in SL(2,\mathbb{Z})$ and for $o = i \in \mathbb{H}^2$, we have*

$$\beta_\infty(\gamma^{-1}o, o) = 2\ln\left\|\gamma\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\|,$$

where $\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\| = \sqrt{x^2+y^2}$ is the Euclidean norm of $\mathbb{R}^2$.

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. Then we have

$$\gamma^{-1}i = \frac{di - b}{-ci + a} = \frac{-ab - cd + i}{a^2 + c^2}.$$

So we have

$$\beta_\infty(\gamma^{-1}i, i) = \int_{\frac{1}{a^2+c^2}}^{1} \frac{dy}{y} = \ln(a^2+c^2) = 2\ln\left(\left\|\begin{pmatrix} a \\ c \end{pmatrix}\right\|\right) = 2\ln\left(\left\|\gamma\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\|\right).$$

$\square$

The following lemma give a majoration of the number of rationnals in a given intervalle of $\mathbb{R}$.

Lemma 5.3. *Let I be a bounded intervalle of $\mathbb{R}$. We have we have for all $n \geq 1$*

$$\#\left\{\tfrac{p}{q} \in \mathbb{Q} \;\middle|\; \tfrac{p}{q} \in I,\ 0 < |q| \leq n\right\} \leq \frac{n(n+1)}{2}|I| + n \leq n^2|I| + n.$$

Proof. For all $q \in \mathbb{N}_{>0}$,

$$\#\left\{p \in \mathbb{Z} \;\middle|\; \tfrac{p}{q} \in I\right\} \leq q|I| + 1.$$

By summing this from $q = 1$ to n we get the result. $\square$

We do the same for the intervalle $X_0 \cap \overline{\mathbb{R}}$:

Lemma 5.4. *For all $c \in \mathbb{R}^+$ and $n \in \mathbb{R}^+$, we have*

$$\#\left\{ \tfrac{p}{q} \in \mathbb{Q} \;\middle|\; \tfrac{p}{q} \in \overline{\mathbb{R}} \backslash [-c, c],\ p^2 + q^2 \leq n^2 \right\} \leq \frac{2n^2}{\sqrt{c^2+1}} + \frac{n}{\sqrt{c^2+1}} \leq \frac{2n^2}{\sqrt{c^2+1}} + n.$$

Proof. We have $\frac{p}{q} \in \overline{\mathbb{R}} \backslash [-c, c] \iff q^2c^2 \leq p^2$. And we have $q^2c^2 \leq p^2$ and $p^2 + q^2 \leq n^2$ that gives $|q| \leq \frac{n}{\sqrt{1+c^2}}$. Then we have $p^2 \leq n^2 - q^2 \leq n^2$, so we obtain the expected inequality. ☐

The following lemma says that the set $SL(2, \mathbb{Z}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ has a positive density in $\mathbb{Z}^2$.

Lemma 5.5. *There exists $C > 0$ such that for all $n \in \mathbb{N}$,*

$$\#\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \;\middle|\; x \wedge y = 1,\ \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| \leq n \right\} \geq Cn^2.$$

This result is well know, but we give a proof for completeness.

Proof. Let us do a coarse sieve. We have the equality

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \;\middle|\; x \wedge y = 1 \right\} = \mathbb{Z}^2 \backslash \bigcup_{p \text{ prime}} p\mathbb{Z}^2.$$

Moreover, we have

$$\begin{aligned} \#\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \;\middle|\; x^2 + y^2 \leq n^2 \right\} &\geq \#\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \;\middle|\; x^2 \leq n^2/2 \text{ and } y^2 \leq n^2/2 \right\} \\ &\geq (\sqrt{2}n - 1)^2, \end{aligned}$$

and we have

$$\begin{aligned} \#\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in p\mathbb{Z}^2 \;\middle|\; \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| \leq n \right\} &= \#\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \;\middle|\; \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| \leq n/p \right\} \\ &\leq \#\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \;\middle|\; x^2 \leq n^2/p^2 \text{ and } y^2 \leq n^2/p^2 \right\} \\ &\leq (2n/p + 1)^2 \end{aligned}$$

So we have

$$\begin{aligned} &\#\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \;\middle|\; x \wedge y = 1,\ \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| \leq n \right\} \\ \geq\ & (\sqrt{2}n - 1)^2 - \sum_{\substack{p \text{ prime} \\ p \leq n}} (2n/p + 1)^2 \\ \geq\ & \left(2 - \sum_{p \text{ prime}} \frac{4}{p^2} \right) n^2 - 4n \sum_{\substack{p \text{ prime} \\ p \leq n}} \frac{1}{p} - \left(2\sqrt{2} + 1 \right) n. \end{aligned}$$

We can verify that

$$2 - \sum_{p \text{ prime}} \frac{4}{p^2} \geq 2 - \sum_{p \in \{2,3,5,7,11,13\}} \frac{1}{p^2} - \sum_{n=17}^{+\infty} \frac{1}{n^2} > 0,$$

and we have
$$\sum_{\substack{p \text{ prime} \\ p\leq n}} \frac{1}{p} \underset{n\to+\infty}{\sim} \ln\ln n = \underset{n\to+\infty}{o}(n).$$
So we can find $C > 0$ small enough to have the inequality that we want for all $n \in \mathbb{N}$. □

Remark 5.6. *In fact we have* $\#\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \;\middle|\; x \wedge y = 1,\ \left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\| \leq n\right\} \sim \frac{6n^2}{\pi}$.

Now we can prove what we wanted :

Proposition 5.7. $(\mathbb{H}^2, \infty, SL(2,\mathbb{Z}))$ *verify the property P.*

Proof. Let $o : R \mapsto \frac{2}{CR}$, where C is the constant of the lemma 5.5. Let $\xi \in \partial X$. If $\xi \neq \infty$, let $\mu_\xi : r \mapsto \frac{2}{C}\left|B(\xi, r)\right|$, and if $\xi = \infty$ let $\mu_\infty : r \mapsto \frac{4}{C\sqrt{1+\frac{1}{4}|\mathbb{R}\setminus B(\infty,r)|^2}}$.

Then we have
$$\begin{aligned}
&\frac{\#\left\{\gamma \in G/P \;\middle|\; \gamma\infty \in B(\xi, r),\ \beta_\infty(\gamma^{-1}o, o) \leq R\right\}}{\#\left\{\gamma \in G/P \;\middle|\; \beta_\infty(\gamma^{-1}o, o) \leq R\right\}} \\
\leq\ & \frac{2\#\left\{\frac{p}{q} \in \mathbb{Q} \;\middle|\; \frac{p}{q} \in B(\xi, r),\ p^2 + q^2 \leq e^{4R}\right\}}{\#\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \;\middle|\; x \wedge y = 1,\ x^2 + y^2 \leq e^{4R}\right\}} \\
\leq\ & \mu_\xi(r) + o(R),
\end{aligned}$$
and we have $\lim_{R\to\infty} o(R) = 0$ and $\forall \xi \in \partial X,\ \lim_{r\to 0} \mu_\xi(r) = 0$. □

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