Aix–Marseille Université	2018-2019

Mathématiques en anglais 1

THE CLUBS OF ODDTOWN

1 Problem

This problem is *Miniature 3* in Matoušec's book.

There are n citizens living in Oddtown. Their main occupation was forming various clubs, which at some point started threatening the very survival of the city. In order to limit the number of clubs, the city council decreed the following innocent-looking rules :

— Each club has to have an odd number of members.

— Every two clubs must have an even number of members in common.

Show that the city council's rules have the effect to limit the number m of clubs to be at most equal to n.

2 Theory to revise

1. Rank of a matrix. Rank of the product of two matrices.
2. Transpose of a matrix.
3. Positive definite matrix.
4. Definition of determinant using permutations.

This problem can also be used to recall the definition of a field and to introduce the field with 2 elements, as well as the concepts of ring of remainders of type $\mathbb{Z}/k\mathbb{Z}$ and of reduction mod p.

3 Method

Order the citizens from 1 to n and the clubs from 1 to m. Let A be the $m \times n$-matrix defined as follows : the entry ij is equal to 1 if the jth citizen is a member of the ith club and 0 otherwise.

1. Explain why the rank of A is at most n.
2. Consider the $m \times m$ matrix $M = A\,{}^tA$. What do its entries represent ? What is their parity ? (Distinguish the cases $i = j$ and $i \neq j$).
3. Explain why the determinant of M is an integer. Show that this integer is odd and hence non-zero.
4. What is the rank of M ? Deduce a lower bound on the rank of A and reach the desired conclusion.

The proof that the determinant is not zero can be slightly simplified using coefficients in $\mathbb{F}_2$, the field with two elements.

4 Exercise

This is *Miniature 4* in Matoušec's book.

Exploiting similar ideas, prove the *Generalised Fisher inequality* : If C_1, $C_2, \ldots, C_m$ are distinct and nonempty subsets of an n-element set such that all the intersections $C_i \cap C_j$, $i \neq j$, have the same size, then $n \geq m$.