Operator Theory:
Advances and Applications, Vol. 78

Domain Perturbation for the first Eigenvalue of the Dirichlet Schrödinger Operator

W. Arendt, S. Monniaux

Introduction.

Let $\Omega \subset \mathbb{R}^N$ be an open connected set. We consider the Dirichlet-Schrödinger operator $H = -\Delta_\Omega^d + V$ on $L^2(\Omega)$ (where Δ_Ω^d denotes the Laplacian with Dirichlet boundary conditions and V is a suitable potential).

In a recent paper, F. Gesztesy and Z. Zhao [15] showed that the first eigenvalue $\lambda(H)$ of H is a strictly monotonic function with respect to the domain Ω (up to capacity, see below for the precise statement). Their proof is given with help of probabilistic methods.

The purpose of this article is to give an analytic proof of this result. In fact, we prove a generalization, allowing the potential to vary as well.

Our proof is based on a domination argument for positive irreducible semigroups (Section 2). In the main theorem (Theorem 3.1), the difference of two open sets is measured by capacity. Somé results concerning this notion are established in Section 1. In particular, we give a short proof of the fact that

$$H_0^1(\Omega) = \{u \in H^1(\mathbb{R}^N) \ : \tilde{u} = 0 \ q.e. \text{ on } \Omega^c\}$$

using the characterization of closed order ideals in $H^1(\Omega)$ which has been given recently by Stollmann [24]. This seems to be of independent interest.

Acknoledgement : The motivation for this work comes from a stimulating talk by Michael Demuth during the conference "Partial Differential Equations" held in Holzhau in July 1994.

1. A characterization of $H_0^1(\Omega)$ by capacity.

Let $\Omega \subset \mathbb{R}^N$ be an open set. Let

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_j} \in L^2(\Omega),\ j = 1, \dots, N \right\}$$

the first Sobolev space, which is a Hilbert space under the norm

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^2 dx \right)^{1/2}$$

(see [9, Chapter IX]). We need some basic properties of capacity and refer to Bouleau & Hirsch [8], Fukushima [14] or Ma & Röckner [17] for details.

The **capacity** of a subset A of $\mathbb{R}^N$ is defined by

$$\operatorname{cap}(A) = \inf\left\{\int_{\mathbb{R}^N} |\nabla\varphi|^2 dx + \int_{\mathbb{R}^N} |\varphi|^2 dx : \varphi \in H^1(\mathbb{R}^N), \varphi \geq 1_O \text{ } a.e., \ A \subset O\right\}.$$

Here 1_O denotes the characteristic function of O, open subset of $\mathbb{R}^N$.

One says a property is true **quasi everywhere** in $\mathbb{R}^N$ ($q.e.$) if there exists a set $A \subset \mathbb{R}^N$ of capacity 0 such that the property is true for all $x \in \mathbb{R}^N \setminus A$.

A function $u : \mathbb{R}^N \to \mathbb{R}$ is called **quasi-continuous** if for every $\varepsilon > 0$ there exists an open set $O_\varepsilon \subset \mathbb{R}^N$ such that $\operatorname{cap}(O_\varepsilon) < \varepsilon$ and u is continuous on $\mathbb{R}^N \setminus O_\varepsilon$.

It is well known that every $u \in H^1(\mathbb{R}^N)$ has a quasi-continuous representative, i.e. there exists a quasi-continuous function $\tilde{u} \in H^1(\mathbb{R}^N)$ such that $u(x) = \tilde{u}(x)$ $a.e.$ The function $\tilde{u}$ is unique $q.e.$

By $H_0^1(\Omega)$ we denote the closure of the space of all test functions $C_c^\infty(\Omega)$ in $H^1(\Omega)$ (see [9, Chapitre IX]). It can now be characterized as follows.

Theorem 1.1. One has

$$H_0^1(\Omega) = \{u \in H^1(\mathbb{R}^N) : \ \tilde{u}(x) = 0 \ q.e. \text{ on } \mathbb{R}^N \setminus \Omega\}.$$

This characterization is well-known to potential analysts. A proof is given by Deny [13, Theorem 2, p. 143] (another is contained in [16, Theorem 3.1, p. 241], or [14, Example 3.3.2, p.81]). Here we give a short proof based on a recent result of Stollmann [24] characterizing closed ideals in $H^1(\mathbb{R}^N)$. Recall that $H^1(\mathbb{R}^N)$ is stable under the operation of taking the absolute value, i.e.

$$u \in H^1(\mathbb{R}^N) \text{ implies } |u| \in H^1(\mathbb{R}^N) \text{ and } \frac{\partial |u|}{\partial x_j} = \operatorname{sign}(u)\frac{\partial u}{\partial x_j}, \ j = 1, \ldots, N \tag{1.1}$$

(see [11, Chap.IV, §7, p.934] ; a proof by semigroup theory is given in [1, Section 2]).

A subspace J of $H^1(\mathbb{R}^N)$ is called an **ideal** if for $u \in J$, $v \in H^1(\mathbb{R}^N)$,

$$|v| \leq |u| \ a.e. \text{ implies } v \in J.$$

Theorem 1.2. (Stollmann [24]). A subspace J of $H^1(\mathbb{R}^N)$ is a closed ideal of $H^1(\mathbb{R}^N)$ if and only if there exists a Borel set Y in $\mathbb{R}^N$ such that

$$J = \{u \in H^1(\mathbb{R}^N) : \ \tilde{u} = 0 \ q.e. \text{ on } Y^c\}$$

Remark. The ideal property is important for the characterization of domination for semigroups defined by forms, see Stollmann and Voigt [25] for a special case and Ouhabaz [19], [20] for a general investigation. We refer to Schaefer [22] and Batty & Robinson [6] for basic properties of ordered Banach spaces.

In order to prove Theorem 1.1, we first show

Lemma 1.3. $H_0^1(\Omega)$ is a closed ideal in $H^1(\mathbb{R}^N)$.

Proof. *(a)* By [9, Lemme IX.5, p.111], one has $u \in H_0^1(\Omega)$ whenever $u \in H^1(\mathbb{R}^N)$ such that $\operatorname{supp}(u)$ is compact and included in Ω.

(b) It is easy to see (and also follows from an abstract result by Borwein & Yost [7, Corollary 1]) that $u \mapsto |u|$ is a continuous mapping on $H^1(\Omega)$. Consequently, for $u \in H^1(\mathbb{R}^N)$ also $v \mapsto u \wedge v$, $v \mapsto u \vee v$ are continuous mappings.

(c) Let $u \in H_0^1(\Omega)$, $v \in H^1(\mathbb{R}^N)$, $0 \le |v| \le |u|$. Let $\varphi_n \in C_c^\infty(\Omega)$ such that $\varphi_n \to u$ in $H_0^1(\Omega)$. Then $v_n = (v \vee -|\varphi_n|) \wedge |\varphi_n| \in H_0^1(\Omega)$ by *(a)* and $v_n \to v$ by *(b)*. Thus $v \in H_0^1(\Omega)$. □

We need the following well-known properties of capacity.

$$\operatorname{cap}(A \cup B) \le \operatorname{cap}(A) + \operatorname{cap}(B) \text{ for all Borel sets } A,\, B \subset \mathbb{R}^N\,; \tag{1.2}$$

$$\lim_{n\to\infty} \operatorname{cap}(A_n) = \operatorname{cap}\Big(\bigcup_{n\in\mathbb{N}} A_n\Big)$$
whenever $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence of Borel sets. (1.3)

If $Y \subset \mathbb{R}^N$ is a Borel set, we let

$$D_0(Y) = \{u \in H^1(\mathbb{R}^N) : \ \tilde{u} = 0 \ q.e. \text{ on } Y^c\}.$$

Proposition 1.4. If $C_c^\infty(\Omega) \subset D_0(Y)$ then $\operatorname{cap}(\Omega \setminus Y) = 0$.

Proof. Assume that $\operatorname{cap}(\Omega \setminus Y) > 0$. Let $(K_n)_{n\in\mathbb{N}}$ be an increasing set of compacts such that

$$\bigcup_{n\in\mathbb{N}} K_n = \Omega.$$

It follows from (1.3) that $\operatorname{cap}(K_n \setminus Y) > 0$ for some $n \in \mathbb{N}$.

Let $0 \le 1_{K_n} \le \varphi \in C_c^\infty(\Omega)$. Then $\varphi \notin D_0(Y)$. □

Proof of Theorem 1.1. By Theorem 1.2, there exists a Borel set Y such that $H_0^1(\Omega) = D_0(Y)$. It follows from Proposition 1.4 that $\operatorname{cap}(\Omega \setminus Y) = 0$.

This implies that $D_0(\Omega) \subset D_0(Y)$. In fact, let $u \in D_0(\Omega)$.

Then $N = \{x \in \Omega^c : \tilde{u}(x) \neq 0\}$ is of capacity 0.

Since $\{x \in Y^c : \tilde{u}(x) \neq 0\} \subset (\Omega \setminus Y) \cup N$, one has

$$\operatorname{cap}\{x \in Y^c : \tilde{u}(x) \neq 0\} \le \operatorname{cap}(\Omega \setminus Y) + \operatorname{cap}(N) = 0.$$

Thus $u \in D_0(Y)$.

Conversely, since $C_c^\infty(\Omega) \subset D_0(\Omega)$ and $D_0(\Omega)$ is closed in $H^1(\mathbb{R}^N)$, it follows that $D_0(Y) = H_0^1(\Omega) \subseteq D_0(\Omega)$. □

Corollary 1.5. Let Λ, $\Omega \subset \mathbb{R}^N$ be open sets. Then $H_0^1(\Omega) = H_0^1(\Lambda)$ if and only if $\operatorname{cap}(\Omega \triangle \Lambda) = 0$.

Proof. Assume that $\operatorname{cap}(\Omega \triangle \Lambda) \neq 0$. Then $\operatorname{cap}(\Omega \setminus \Lambda) \neq 0$ or $\operatorname{cap}(\Lambda \setminus \Omega) \neq 0$.

If $\operatorname{cap}(\Omega \setminus \Lambda) \neq 0$, then by Proposition 1.4 there exists $\varphi \in C_c^\infty(\Omega)$, such that $\varphi \notin D_0(\Lambda) = H_0^1(\Lambda)$. In the other case, $C_c^\infty(\Lambda) \not\subset H_0^1(\Omega)$.

The converse implication follows directly from Theorem 1.1. □

2. Domination and eigenvalues for positive irreducible semigroups.

Let (Ω,ν) be a σ-finite measure space, let $\Lambda \subset \Omega$ be a measurable subset and let $p \in [1,\infty)$. We identify $L^p(\Lambda) = L^p(\Lambda,\nu)$ with a subspace of $L^p(\Omega) = L^p(\Omega,\nu)$, extending functions in $L^p(\Lambda,\nu)$ by 0 on $\Omega \setminus \Lambda$.

The following theorem is a generalization of [2, Theorem 1.3] (where it is assumed that $\Lambda = \Omega$). The argument is similar, but for the sake of completeness we include the proof.

Theorem 2.1. Let T be a bounded irreducible positive C_0-semigroup on $L^p(\Omega)$ with generator A, and let S be an irreducible C_0-semigroup on $L^p(\Lambda)$ with generator B. Assume that

(a) $0 \leq S(t)f \leq T(t)f \;\; (0 \leq f \in L^p(\Lambda),\, t \geq 0)$

(b) $\ker B \neq 0$

Then $\nu(\Omega \setminus \Lambda) = 0$ (so that $L^p(\Omega) = L^p(\Lambda)$) and $S(t) = T(t)$ $(t \geq 0)$.

We clarify some notations. If $f : \Omega \to \mathbb{R}$ is measurable, we write

$f \geq 0$ if $f(x) \geq 0 \;\; \nu - a.e.$;
$f > 0$ if $f \geq 0$ and $\nu(\{x : f(x) \neq 0\}) > 0$;
$f \gg 0$ if $f(x) > 0 \;\; \nu - a.e.$

If $Q \in \mathcal{L}(L^p(\Omega))$, we write $Q \gg 0$ if $Qf \gg 0$ whenever $0 < f \in L^p(\Omega)$.

The semigroup T is **irreducible** if $R(\mu, A) := (\mu - A)^{-1} \gg 0$ for all (equivalently one) $\mu > 0$ (see [18, p.306]).

By A' we denote the adjoint of A on $L^{p'}(\Omega)$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

Lemma 2.2. Let $0 < u \in L^p(\Omega)$ such that $\lambda R(\lambda, A)u \geq u$ for all $\lambda > 0$. Then $u \in \ker A$, $u \gg 0$ and there exists $\varphi \in \ker(A')$, $\varphi \gg 0$.

Proof. Let $0 \leq \varphi_0 \in L^{p'}(\Omega)$ such that $< u, \varphi_0 >> 0$. Since T is bounded, one has $\sup\{\|\lambda R(\lambda, A)\|, \lambda > 0\} < \infty$. Thus $\lambda R(\lambda, A)'\varphi_0$ has a ω^*-limit point $\varphi \in L^p(\Omega, \nu)'$ as $\lambda \searrow 0$.

Clearly, $\varphi \geq 0$ and $< u, \varphi > \geq < u,\varphi_0 >> 0$. Thus $\varphi > 0$.

Since for $0 < \mu \leq 1$, $0 < \lambda \leq 1$,

$$\mu R(\mu, A)' \lambda R(\lambda, A)' \varphi_0 = \frac{\mu\lambda}{\mu - \lambda}(R(\lambda,A)'\varphi_0 - R(\mu,A)'\varphi_0),$$

it follows that $\varphi \in \ker(A')$.

If $0 < f \in L^p(\Omega)$, then $< f,\varphi >=< \mu R(\mu,A)f,\varphi >> 0$ since $\varphi > 0$ and $\mu R(\mu,A)f \gg 0$. Hence $\varphi \gg 0$.

Finally, $\lambda R(\lambda,A)u - u \geq 0$ and $< \lambda R(\lambda,A)u - u,\varphi >= 0$. Since $\varphi \gg 0$, it follows that $\lambda R(\lambda,A)u - u = 0$, $(\lambda > 0)$; i.e. $u \in \ker(A)$. □

Proof of Theorem 2.1. Let $0 \neq v \in \ker(B) \subset L^p(\Lambda)$, $u = |v|$.

Then $u = |\lambda R(\lambda,B)v| \leq \lambda R(\lambda,B)u \leq \lambda R(\lambda,A)u$, $(\lambda > 0)$. It follows from Lemma 2.2 that $u \in \ker(B) \cap \ker(A)$ and $u \gg 0$ on Ω. Since $u \in L^p(\Lambda)$ so that $u(x) = 0$ *a.e.* on $\Omega \setminus \Lambda$, it follows that $\nu(\Omega \setminus \Lambda) = 0$ and so $L^p(\Omega) = L^p(\Lambda) =: E$.

Let $0 \leq \psi \in E'$. Then

$$(\lambda R(\lambda,A)' - \lambda R(\lambda,B)')\psi \geq 0 \text{ and } < u,(\lambda R(\lambda,A)' - \lambda R(\lambda,B)')\psi >= 0$$

since $u \in \ker(A) \cap \ker(B)$. Thus $(\lambda R(\lambda,A)' - \lambda R(\lambda,B)')\psi = 0$ since $u \gg 0$. Since $\text{span}E'_+ = E'$, it follows that $R(\lambda,A)' = R(\lambda,B)'$ $(\lambda > 0)$ and so $A = B$. □

If B generates a bounded C_0-semigroup and $\ker(B) \neq 0$, it is easy to see that also $\ker(B') \neq 0$. The converse is true if $1 < p < \infty$, but not for $p = 1$. So it is natural to ask whether in Theorem 2.1 it suffices to assume that $\ker(B') \neq 0$. The following example shows that this is not the case.

Example 2.3. Let $N \geq 3$, $0 < m \in C_c(\mathbb{R}^N)$. Denote by Δ the Laplacian on $L^1(\mathbb{R}^N) = E$. Then there exists $\mu_2 > 0$ such that

$$\sup\{\|e^{t(\Delta+\mu_2 m)}\|_{\mathcal{L}(L^1)},\ t \geq 0\} < \infty$$

(see [23, B5.2]). Let $0 < \mu_1 < \mu_2$, $S(t) = e^{t(\Delta+\mu_1 m)}$, $T(t) = e^{t(\Delta+\mu_2 m)}$. Then $0 \leq e^{t\Delta} \leq S(t) \leq T(t)$. In particular, both semigroups are irreducible. By [5, Remark 3.9], there exists $0 \ll \varphi \in \ker(B')$, $B = \Delta + \mu_1 m$. However, $S \neq T$. It follows from Theorem 2.1 that $\ker(B) = 0$. □

3. Strict monotonicity of the bottom of the spectrum.

Let $\Omega \subset \mathbb{R}^N$ be an open set and denote by Δ^d_Ω the Dirichlet Laplacian on $L^2(\Omega)$, i.e. $-\Delta^d_\Omega$ is associated with the Dirichlet form

$$a(u,v) = \int_\Omega \nabla u \nabla v\ dx,\ D(a) = H^1_0(\Omega).$$

We consider a potential $V \in L^1_{loc}(\mathbb{R}^N)$ such that V^- is relatively bounded with respect to the form a with form bound less than 1 ; i.e. we assume that there exists $0 \leq \alpha < 1$, $\beta \geq 0$ such that

$$\int_\Omega V^- u^2 dx \leq \alpha \int_\Omega |\nabla u|^2 dx + \beta \int_\Omega u^2 dx \tag{3.1}$$

for all $u \in H^1_0(\Omega)$.

Then we can define the self-adjoint operator $H = -\Delta_\Omega^d + V$ as usual to be the operator associated with the closed symmetric lower bounded form b on $L^2(\Omega)$ given by

$$D(b) = \{u \in H_0^1(\Omega) : \int_\Omega V^- u^2 dx < \infty\},$$

$$b(u,v) = \int_\Omega \nabla u \nabla v \; dx + \int_\Omega Vuv \; dx.$$

By $\lambda(H) = \inf\{\sigma(H)\}$, we denote the bottom of the spectrum of H.

Now let $\Lambda \subset \mathbb{R}^N$ be another open set and $U \in L^1_{loc}(\mathbb{R}^N)$ another potential satisfying (3.1). Let $\tilde{H} = -\Delta_\Lambda^d + U$. Assume that

$$\Lambda \subset \Omega \text{ and } V \leq U. \tag{3.2}$$

Then it is not difficult to see that

$$\lambda(\tilde{H}) \geq \lambda(H), \tag{3.3}$$

(see the remark following Proposition 3.3 below).

Our aim is to prove the following result on strict monotonicity in (3.3).

Theorem 3.1. Assume in addition to (3.2) that

(a) Λ, Ω are connected;

(b) $\lambda(\tilde{H})$ is an eigenvalue of $\tilde{H}$.

Then $\lambda(\tilde{H}) = \lambda(H)$ if and only if $\text{cap}(\Omega \setminus \Lambda) = 0$ and $U = V$ *a.e.* on Ω.

This theorem has been proved by Gesztesy and Zhao [15] by probabilistic methods in the case where $U = V$. We give an analytic proof based on domination (see Section 2).

Before giving the proof we mention that in Theorem 3.1, one cannot replace condition *(b)* by the condition that $\lambda(H)$ is an eigenvalue of H. This can be seen by the following example.

Example 3.2. Let $\Lambda = \Omega = \mathbb{R}^N$, $N \geq 5$. Let $0 < m \in C_c(\mathbb{R}^N)$. Then $\mu_0 = \sup\{\mu > 0 : \lambda(-\Delta - \mu m) = 0\} \in (0,\infty)$ (see e.g. [5, Remark 2.11 or Section 4]). Thus $\lambda(-\Delta - \mu_0 m) = 0$. Moreover, by [10, Proposition 4.1], 0 is an eigenvalue of $-\Delta - \mu_0 m$. Letting

$$U = -\mu_0 m, \; V = -\frac{\mu_0}{2} m, \text{ and } A = B = \Delta,$$

one sees that the conclusion of Theorem 3.1 is false in this situation. Moreover, Theorem 3.1 implies that 0 is not an eigenalue of $-\Delta - \mu m$ for any $0 \leq \mu < \mu_0$.

For the proof of Theorem 3.1 we identify again $L^2(\Lambda)$ with a subspace of $\Lambda^2(\Omega)$ extending functions by 0, and $\Lambda^2(\Omega)$ with a subspace of $\Lambda^2(\mathbb{R}^N)$. Note that H generates a C_0-semigroup e^{-tH} on $\Lambda^2(\Omega)$. By the spectral theorem one has

$$\|e^{-tH}\| = e^{-t\lambda(H)} \quad (t \geq 0). \tag{3.4}$$

Proposition 3.3. One has

$$0 \le e^{-t\tilde{H}} f \le e^{-tH} f \quad (0 \le f \in L^2(\Lambda),\ t \ge 0). \tag{3.5}$$

Note that $e^{-t\tilde{H}} f$ and $e^{-tH} f$ are both functions defined on $\mathbb{R}^N$ by our convention and the inequality (3.5) holds *a.e.* As a consequence one has $\|e^{-t\tilde{H}}\| \le \|e^{-tH}\|$ $(t \ge 0)$ and so (3.3) follows with help of (3.4).

As in [3] and [4] we denote by Δ_Ω the pseudo-Dirichlet Laplacian on $\Lambda^2(\Omega)$; i.e., $-\Delta_\Omega$ is associated with the form a_Ω given by

$$D(a_\Omega) = \{u_{|\Omega} : u \in H^1(\mathbb{R}^N); u(x) = 0 \ a.e. \text{ on } \Omega^c\},$$

$$a_\Omega(u,v) = \int_\Omega \nabla u \nabla v \, dx.$$

Then by [3, Section 7] or [4],

$$e^{t\Delta_\Omega} f = \lim_{n \to \infty} e^{t(\Delta - n 1_{\Omega^c})} f \ (t \ge 0) \tag{3.6}$$

for all $f \in L^2(\Omega)$, where Δ denotes the Laplacian on $\Lambda^2(\mathbb{R}^N)$. The Dirichlet Laplacian is obtained by a second approximation. Let $\Omega_n \subset \Omega$ be open such that $\overline{\Omega}_n$ is compact, $\overline{\Omega}_n \subset \Omega_{n+1}$ $(n \in \mathbb{N})$ and $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$. Then

$$e^{t\Delta_\Omega^d} f = \lim_{n \to \infty} e^{t\Delta_{\Omega_n}} f \quad (f \in L^2(\Omega),\ t \ge 0). \tag{3.7}$$

Remark. One has $e^{t\Delta_\Omega^d} = e^{t\Delta_\Omega}$ if Ω is of class C^1 (see [9, Chapitre IX]).

Proof of Proposition 3.3.

First step : Domination for the Dirichlet Laplacian .

It follows from the Trotter product formula that

$$e^{t(\Delta - n1_{\Lambda^c})} f \le e^{t(\Delta - n1_{\Omega^c})} f \quad (0 \le f \in L^2(\mathbb{R}^N)\ t \le 0).$$

Thus it follows from (3.6) (and the same formula with Ω replaced by Λ) that

$$e^{t\Delta_\Lambda} f \le e^{t\Delta_\Omega} f \quad (0 \le f \in L^2(\Lambda),\ t > 0). \tag{3.8}$$

Let Λ_n be open sets such that $\overline{\Lambda}_n$ is compact, $\overline{\Lambda}_n \subset \Lambda_{n+1} \subset \Lambda$ $(n \in \mathbb{N})$ and $\bigcup_{n \in \mathbb{N}} \Lambda_n = \Lambda$. Choose Ω_n open such that $\Lambda_n \subset \Omega_n \subset \overline{\Omega}_n \subset \Omega_{n+1} \subset \Omega$ and $\bigcup_{n \in \mathbb{N}} \Omega_n = \Omega$. Then it follows from (3.7) and (3.8) that

$$e^{t\Delta_\Lambda^d} f = \lim_{n \to \infty} e^{t\Delta_{\Lambda_n}} f \le \lim_{n \to \infty} e^{t\Delta_{\Omega_n}} f = e^{t\Delta_\Omega^d} f \quad (0 \le f \in L^2(\Lambda),\ t \ge 0). \tag{3.9}$$

Second step : Perturbation by V^- and U^-

Let $V_n^- = \inf\{V^-,n\}$, $U_n^- = \inf\{U^-,n\}$. Since $V \leq U$ one has $U^- \leq V^-$ and $U_n^- \leq V_n^-$ $(n \in \mathbb{N})$. It follows from the Trotter product formula and (3.9) that

$$e^{t(\Delta_\Lambda^d + U_n^-)} f \leq e^{t(\Delta_\Omega^d + V_n^-)} f \tag{3.10}$$

for all $0 \leq f \in L^2(\Lambda)$, $t \geq 0$. If we show that

$$e^{t(\Delta_\Lambda^d + U^-)} f = \lim_{n\to\infty} e^{t(\Delta_\Lambda^d + U_n^-)} f \tag{3.11}$$

for all $f \in L^2(\Lambda)$, and hence the analogous formula for $\Delta_\Omega^d + V^-$ as well, we can conclude from (3.10) that

$$e^{t(\Delta_\Lambda^d + U^-)} f \leq e^{t(\Delta_\Omega^d + V^-)} f \tag{3.12}$$

for $0 \leq f \in L^2(\Lambda)$, $t \geq 0$.

In order to show (3.11), recall that $\int U_n^- u^2 \leq \int U^- u^2 \leq \alpha \int |\nabla u|^2 + \beta \int |u|^2$ for $u \in H_0^1(\Lambda)$, where $0 < \alpha < 1$. Let $c = \beta + 1$.

Then
$\int |\nabla u|^2 - \int U_n^- u^2 + c \int |u|^2 \geq (1-\alpha) \int |\nabla u|^2 + \int |u|^2 \geq (1-\alpha) \|u\|_{H_0^1(\Lambda)}$.

Denote by b_n the form associated with $-\Delta - U_n^- + cI$ and by b the form associated with $-\Delta - U^- + cI$. Then $b_n \geq b_{n+1}$ and $\lim_{n\to\infty} b_n(u,u) = b(u,u)$ for $u \in D(b) = D(b_n) = H_0^1(\Lambda)$.

Now it follows from [21, Theorem S16, p.373]

$$e^{t(\Delta_\Lambda^d + U^- - c)} f = \lim_{n\to\infty} e^{t(\Delta_\Lambda^d + U_n^- - c)} f, \; f \in L^2(\Lambda).$$

This implies (3.11) and the proof of (3.12) is finished.

Third step : Perturbation by V^+ and U^+.

Let $U_n^+ = \inf\{U^+,n\}$, $V_n^+ = \inf\{V^+,n\}$.
Then $V_n^+ \leq U_n^+$ and thus it follows from (3.12) that

$$e^{t(\Delta_\Lambda^d + U^- - U_n^+)} f \leq e^{t(\Delta_\Omega^d + V^- - V_n^+)} f \tag{3.13}$$

for $0 \leq f \in L^2(\Lambda)$, $t \geq 0$, $n \in \mathbb{N}$. It follows from [21, Theorem S14] that

$$\lim_{n\to\infty} e^{t(\Delta_\Lambda^d + U^- - U_n^+)} f = e^{-t\tilde{H}} f \quad (f \in L^2(\Lambda)),$$

and

$$\lim_{n\to\infty} e^{t(\Delta_\Omega^d + V^- - V_n^+)} f = e^{-tH} f \quad (f \in L^2(\Omega)).$$

Hence passing to the limit in (3.13) yields the claim (3.5). □

Proposition 3.4. Assume that Λ is connected. Then $(e^{-t\tilde{H}})_{t\geq 0}$ is an irreducible semigroup on $L^2(\Lambda)$.

Proof. It follows from [12, Theorem 3.3.5] that $e^{t\Delta_\Lambda^d} f \gg 0$ whenever $t > 0$, $0 < f \in L^2(\Lambda)$.

Now we argue as in [5, Proposition 1.3] : we can assume that $U^- = 0$ (since $e^{t(\Delta_\Lambda^d - U^+)} \leq e^{t(\Delta_\Lambda^d - U)}$). It follows from [21, Theorem S16, p.373] that

$$e^{t\Delta_\Lambda^d} f = \lim_{n\to\infty} e^{t(\Delta_\Lambda^d - U^+ + U_n^+)} f \ (f \in L^2(\Lambda)). \tag{3.14}$$

Let $0 < f \in L^2(\Lambda)$. Let $t > 0$. We show that $e^{t(\Delta_\Lambda^d - U^+)} f \gg 0$.
If not, there exists $M \subset \Lambda$ with positive Lebesgue measure and $e^{t(\Delta_\Lambda^d - U^+)} f(x) = 0$ *a.e.* on M.

Since $e^{t(\Delta_\Lambda^d - U^+ + U_n^+)} f \leq e^{tn} e^{t(\Delta_\Lambda^d - U^+)} f$ $(n \in \mathbb{N})$, it follows from (3.14) that $(e^{t\Delta_\Lambda^d} f)(x) = 0$ *a.e.* on M, a contradiction. □

Proof of Theorem 3.1. Since Ω and Λ are connected, both semigroups e^{-tH} and $e^{-t\tilde{H}}$ are irreducible. Assume that $\lambda(H) = \lambda(\tilde{H})$. Replacing U by $U - \lambda(\tilde{H})$ and V by $V - \lambda(H)$ we can assume that $\lambda(H) = \lambda(\tilde{H}) = 0$. It follows from Theorem 2.1 that $(\Omega \setminus \Lambda)$ has zero Lebesgue measure and $H = \tilde{H}$. Denote by b, $\tilde{b}$ the forms associated with H and $\tilde{H}$, respectively. Then $b = \tilde{b}$. In particular :

$$\begin{aligned} \left\{u \in H_0^1(\Omega) : \int_\Omega V^+|u|^2 dx < \infty\right\} &= D(b) \\ &= D(\tilde{b}) = \left\{u \in H_0^1(\Lambda) : \int_\Lambda U^+|u|^2 dx < \infty\right\}. \end{aligned}$$

Assume that $\operatorname{cap}(\Omega \setminus \Lambda) > 0$. Then it follows from Proposition 1.4 that there exists $\varphi \in C_c^\infty(\Omega) \setminus H_0^1(\Lambda)$. Thus $\varphi \in D(b) \setminus D(\tilde{b})$, a contradiction. Thus $\operatorname{cap}(\Omega \setminus \Lambda) = 0$. It follows from Corollary 1.5 that $H_0^1(\Omega) = H_0^1(\Lambda)$.

Since $b = \tilde{b}$ one has

$$\int |\nabla u|^2 dx + \int V u^2 dx = \int |\nabla u|^2 dx + \int U u^2 dx \text{ for all } u \in C_c^\infty(\Lambda)$$

and so

$$\int_\Lambda (V - U) u^2 dx = 0 \text{ for all } u \in C_c^\infty(\Lambda).$$

Since $V - U \geq 0$ *a.e.*, this implies that $V - U = 0$ *a.e.* This completes the proof of the direct implication in Theorem 3.1. The other is obvious. □

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W. Arendt, S. Monniaux
URA CNRS 741, Équipe de Mathématiques,
16 route de Gray, F-25030 BESANÇON Cedex