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A perturbation result for bounded imaginary powers

By

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Abstract. We give a new proof of a perturbation result due to J. Prüss and H. Sohr [11]: if an operator A has bounded imaginary powers, then so does $A + w$ ($w \geqq 0$). Instead of Mellin transform on which the proof in [11] is based, we use the functional calculus for sectorial operators developed in particular by A. McIntosh ([8], [3] and [1]). It turns out that our method gives a more general result than the one used in [11].

1. Introduction. Less than ten years ago, G. Dore and A. Venni [5] proved their famous result on maximal regularity on UMD-spaces. J. Prüss and H. Sohr extended this theorem to the case where the considered operators are not necessarily invertible. For that purpose, they proved that if a linear operator A admits bounded imaginary powers, then so does $\varepsilon + A$ for each $\varepsilon > 0$ ([11], Theorem 3). They restricted themselves to operators A for which the type ω_A of the C_0-group $(A^{is})_{s\in\mathbb{R}}$ is less than π, using a functional calculus closely related to the inverse Mellin transform.

The purpose of this paper is to give a different proof of this result based on the functional calculus for sectorial operators (see A. McIntosh [8] and M. Cowling, I. Doust, A. McIntosh, A. Yagi [3]). This approach seems quite easy and elementary. In addition, it gives a more general result. The restrictive assumption on the type made by J. Prüss and H. Sohr is no longer necessary.

Combining these results with an operator-valued functional calculus introduced recently by D. Albrecht and A. McIntosh [1], we obtain a more general perturbation result : given an operator A with bounded imaginary powers and a bounded, invertible, sectorial operator B commuting with the resolvents of A, such that the sum of the two spectral angles of A and B is less than π, the operator $A + B$ admits bounded imaginary powers. A result of this kind was obtained by J. Prüss and H. Sohr for relatively bounded perturbations B which need not commute with the resolvents of A ([12], Proposition 3.1). In the same time as this work was accomplished, another proof of the Prüss-Sohr result has been given by M. Uiterdijk [13].

The paper is organized as follows. In Section 2, we present our main results. We recall the important facts on the functional calculus which we need in Section 3. Finally, the proofs of our main theorems are given in Section 4.

The idea of this paper comes from interesting discussions during January 1996 in Ulm (Germany) with Pr Alan McIntosh from Macquarie University, Australia. The author wishes to thank him here, as well as the referee for having pointed out the papers [12] and [13].

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2. The main theorems. From now on, A will denote a linear operator on a Banach space X; $D(A)$, $N(A)$, $R(A)$, $\rho(A)$, $\sigma(A)$ denote its domain, its kernel, its range, its resolvent set, its spectrum, respectively.

Definition 2.1. The operator A is called *sectorial* on X if it is closed, densely defined, $N(A) = \{0\}$, $\overline{R(A)} = X$, and if it verifies $(-\infty, 0) \subset \rho(A)$ and $\sup\limits_{t>0} \|t(t+A)^{-1}\| < \infty$.

In that case, one has

$$\varphi_A := \inf\left\{\varphi \in (0,\pi];\ \sigma(A) \subset \overline{\Sigma}_\varphi \ \text{ and } \ M_\varphi := \sup_{z \notin \Sigma_\varphi} \|z(z-A)^{-1}\| < \infty\right\} \in (0,\pi),$$

where $\Sigma_\varphi = \{z \in \mathbb{C} \setminus \{0\}\ ;\ |\arg(z)| < \varphi\}$.

The angle φ_A is called the *spectral angle of A*.

Suppose now that A is sectorial. For all $z \in \mathbb{C}$ such that $|\Re(z)| < 1$, we define the complex powers of A by

$$A^z x = \frac{\sin \pi z}{\pi}\left(\frac{x}{z} - \frac{1}{1+z}\ A^{-1}x + \int_0^1 t^{z+1}(t+A)^{-1}A^{-1}x\ dt + \int_1^\infty t^{z-1}(t+A)^{-1}Ax\ dt\right)$$

for all $x \in D(A) \cap R(A)$ (see for instance [6], [7], and [10] pages 212–214, [2] page 157).

It is known that the map $z \mapsto A^z x$ is holomorphic on $\{z \in \mathbb{C};\ |\Re(z)| < 1\}$ with values in X for all $x \in D(A) \cap R(A)$; see [10] page 213, [2] page 154. It can be easily seen that $D(A) \cap R(A)$ is dense in X, and the operator A^z defined on $D(A) \cap R(A)$ is closable. We are now in the position to describe the class $\mathrm{BIP}(X)$.

Definition 2.2. A sectorial operator A is said to admit *bounded imaginary powers* (briefly : $A \in \mathrm{BIP}(X)$) if the closure of $(A^{is}, D(A) \cap R(A))$ is a bounded operator on X for each $s \in \mathbb{R}$ and if $\sup\limits_{s\in[-1,1]} \|A^{is}\| < \infty$.

Remark 2.3. It is known that if $A \in \mathrm{BIP}(X)$ then $(A^{is})_{s\in\mathbb{R}}$ forms a strongly continuous group on X ([2], Theorem 4.7.1, page 162).

Denote by ω_A the type of this group; i.e. $\omega_A := \inf\{\omega \in \mathbb{R};\ \exists\, M\colon\ \|A^{is}\| \leqq Me^{\omega|s|},\ s \in \mathbb{R}\}$. Then it was shown in [11] Th. 2, [10] page 214, [2] page 177 or [9] Cor. 4.4, that $\omega_A \geqq \varphi_A$, where φ_A is the spectral angle of A.

The following theorem is the generalization of a perturbation result for operators in the class $\mathrm{BIP}(X)$ with $\omega_A < \pi$. It was proved by J. Prüss and H. Sohr [11] in the case where the scalar perturbation w is a positive real number. Our proof is completely different than the one given in [11].

Theorem 2.4. *Let A be a sectorial operator on X with spectral angle φ_A. Assume that A admits bounded imaginary powers, and denote by ω_A the type of the C_0–group $(A^{is})_{s\in\mathbb{R}}$. Consider a complex number $w \in \mathbb{C} \setminus (-\infty, 0)$ such that $|\arg w| + \varphi_A < \pi$. Then the operator $w + A$ defined on $D(A)$ admits bounded imaginary powers in X, and the type of $((w+A)^{is})_{s\in\mathbb{R}}$ is at most* $\max\{\omega_A, |\arg w|\}$.

This theorem may be extended to a more general case.

Corollary 2.5. *Let A be as in the previous theorem. Assume that B is a bounded invertible sectorial operator with spectral angle φ_B, commuting with the resolvents of A. If $\varphi_A + \varphi_B < \pi$, then the operator $A + B$, defined on $D(A)$, admits bounded imaginary powers in X and the type of the group $((A+B)^{is})_{s \in \mathbb{R}}$ is at most* $\max\{\omega_A, \varphi_B\}$.

This result appears as a corollary of Theorem 2.4 by using the operator-valued functional calculus discussed in [1]. We recall the facts necessary for the proof in the following section.

Under stronger assumptions on the Banach space X, namely if X has the UMD-property (see [9], Section 1.2 for instance), J. Prüss and H. Sohr [11] obtained the same result for more general operators B, using the theorem of Dore-Venni. More precisely, if A and B are in the class $\mathrm{BIP}(X)$ with $\omega_A + \omega_B < \pi$ and if their resolvents commute, then the operator $A + B$ defined on $D(A) \cap D(B)$ admits bounded imaginary powers, and $\omega_{A+B} \leqq \max\{\omega_A, \omega_B\}$.

3. Functional calculi for sectorial operators. We present in this section some of the results proved in [8], [3] and [1]. We begin with the so-called classical functional calculus for sectorial operators introduced by A. McIntosh [8] in the case of Hilbert spaces and extended to more general Banach spaces by M. Cowling, I. Doust, A. McIntosh and A. Yagi in [3]. This will also give a new definition for the class $\mathrm{BIP}(X)$.

Let $\mu \in (0, \pi]$. Let $\mathscr{H}^\infty(\Sigma_\mu)$ denote the space of all bounded holomorphic functions on Σ_μ (this sector was defined in Definition 2.1). Consider also the following subspace of $\mathscr{H}^\infty(\Sigma_\mu)$, $\mathscr{H}_0^\infty(\Sigma_\mu) := \{f \in \mathscr{H}^\infty(\Sigma_\mu);\ \exists\, \alpha > 0:\ z \mapsto (z^\alpha + z^{-\alpha}) f(z) \in \mathscr{H}^\infty(\Sigma_\mu)\}$.

Let A be a sectorial operator on the Banach space X with spectral angle φ_A. Let $\mu \in (\varphi_A, \pi]$ be fixed. For each $f \in \mathscr{H}_0^\infty(\Sigma_\mu)$, one defines the bounded operator

$$f(A) = \frac{1}{2i\pi} \int_{\Gamma_\vartheta} f(z)(z - A)^{-1}\, dz,$$

where $\vartheta \in (\varphi_A, \mu)$ and $\Gamma_\vartheta = -e^{i\vartheta}(-\infty, 0] \cup e^{-i\vartheta}[0, \infty)$.

This definition is independent of $\vartheta \in (\varphi_A, \mu)$.

Let now $f \in \mathscr{H}^\infty(\Sigma_\mu)$. Then the function $g : z \mapsto \dfrac{z f(z)}{(1+z)^2}$ belongs to $\mathscr{H}_0^\infty(\Sigma_\mu)$. Define then $f(A)x$ for $x \in D(A) \cap R(A)$ by $f(A)x = g(A)(A^{-1}x + 2x + Ax)$. The operator $f(A)$ is closable; we denote the closure of $(f(A), D(A) \cap R(A))$ also by $f(A)$.

Remark 3.1. This procedure is just one way of defining a functional calculus for functions in $\mathscr{H}^\infty(\Sigma_\mu)$; it was shown in [3] and in [8] that any other functional calculus for these functions, subject to the following requirements 1 and 2 agrees with this one.

1. If $f(z) = z^k$, then $f(A) = A^k$ $(k \in \mathbb{N})$.
2. Convergence lemma: if $(f_\alpha)_\alpha$ is a uniformly bounded net of functions in $\mathscr{H}_0^\infty(\Sigma_\mu)$ which converges to a function f in $\mathscr{H}^\infty(\Sigma_\mu)$ uniformly on compact subsets of Σ_μ, such that the operators $f_\alpha(A)$ are uniformly bounded on X, then $(f_\alpha(A)x)_\alpha$ converges to $f(A)x$ for all $x \in X$, and consequently $f(A)$ is a bounded operator on X which verifies $\|f(A)\| \leqq \sup\limits_\alpha \|f_\alpha(A)\|$.

Therefore, in the case of bounded invertible operators A, this functional calculus agrees in particular with the Dunford functional calculus (see [14] VIII.7 for this notion).

For each $\varepsilon \in (0,1)$, we consider now the operator $A_\varepsilon := (\varepsilon + A)(1+\varepsilon A)^{-1}$. It is bounded, invertible and sectorial of spectral angle at most φ_A. Moreover, for all $\vartheta \in (\varphi_A, \pi)$, $\lambda \in \overline{\Sigma}_{\pi-\vartheta}$, one has $\lambda \in \rho(-A_\varepsilon)$ and $(\lambda + A_\varepsilon)^{-1} = \dfrac{\varepsilon}{1+\varepsilon\lambda} + \dfrac{1-\varepsilon^2}{(1+\varepsilon\lambda)^2}\left(\dfrac{\lambda+\varepsilon}{1+\lambda\varepsilon} + A\right)^{-1}$. Therefore, one obtains $\lim\limits_{\varepsilon\to 0^+} z(z-A_\varepsilon)^{-1} = z(z-A)^{-1}$ in the norm of operators, uniformly in $z \notin \Sigma_\vartheta$. For $x \in D(A) \cap R(A)$, we also have $\lim\limits_{\varepsilon\to 0^+} A_\varepsilon x = Ax$ and $\lim\limits_{\varepsilon\to 0^+} A_\varepsilon^{-1} x = A^{-1}x$ in X. It is also known (see [10] Proposition 8.1) that $\lim\limits_{\varepsilon\to 0^+} A_\varepsilon^z x = A^z x$ for $x \in D(A) \cap R(A)$ and $z \in \mathbb{C}$, $|\Re(z)| < 1$.

The following lemma holds.

Lemma 3.2. (i) *For all $f \in \mathscr{H}_0^\infty(\Sigma_\mu)$, $\lim\limits_{\varepsilon\to 0^+} f(A_\varepsilon) = f(A)$ in the norm of operators;*

(ii) *for all $f \in \mathscr{H}^\infty(\Sigma_\mu)$, for all $x \in D(A) \cap R(A)$, $\lim\limits_{\varepsilon\to 0^+} f(A_\varepsilon)x = f(A)x$ in X.*

Now there is another way to describe the class BIP(X). For each $s \in \mathbb{R}$, let f_s be the function $f_s(z) = z^{is}$ defined on $\mathbb{C} \setminus (-\infty, 0)$. Denote by g_s the function defined on $\mathbb{C} \setminus (-\infty, 0)$ by $g_s(z) = \dfrac{z}{(1+z)^2}\, z^{is}$; it belongs to $\mathscr{H}_0^\infty(\Sigma_\mu)$. For all $x \in D(A) \cap R(A)$, we then have $f_s(A)x = g_s(A)(A^{-1}x + 2x + Ax)$.

On the other hand, one obtains $f_s(A_\varepsilon)x = A_\varepsilon^{is}x = g_s(A_\varepsilon)(A_\varepsilon^{-1}x + 2x + A_\varepsilon x)$ for all $\varepsilon > 0$ by the Dunford functional calculus. Therefore, for all $x \in D(A) \cap R(A)$, one has $f_s(A_\varepsilon)x = A_\varepsilon^{is}x \longrightarrow A^{is}x$ and $f_s(A_\varepsilon)x = g_s(A_\varepsilon)(A_\varepsilon^{-1}x + 2x + A_\varepsilon x) \to g_s(A)(A^{-1}x + 2x + Ax) = f_s(A)$ as $\varepsilon \to 0^+$.

The operators A^{is} and $f_s(A)$ coincide on $D(A) \cap R(A)$. This proves the following proposition.

Proposition 3.3. *A sectorial operator A admits bounded imaginary powers if and only if the closure of $(f_s(A), D(A) \cap R(A))$ is a bounded operator on X for each $s \in \mathbb{R}$ and* $\sup\limits_{s\in[-1,1]} \|f_s(A)\| < \infty$.

The functional calculus just described has been extended to a more general case by D. Albrecht and A. McIntosh [1]. To see how it works, we need the following definitions.

Let A be as before. Denote by $\mathscr{A}$ the second commutant (or bicommutant) of the bounded operator $(\lambda - A)^{-1}$ (for one $\lambda \in \rho(A)$), *i.e.* the set of all operators which commute with all the bounded operators which commute with $(\lambda - A)^{-1}$: $\mathscr{A}$ is an unital commutative Banach subalgebra of the algebra of bounded operators on X and is independent of the choice of $\lambda \in \rho(A)$.

We denote by $\mathscr{H}^\infty(\Sigma_\mu; \mathscr{A})$ the space of all bounded holomorphic functions on Σ_μ with values in $\mathscr{A}$ and by $\mathscr{H}_0^\infty(\Sigma_\mu; \mathscr{A})$ the subspace $\{f \in \mathscr{H}^\infty(\Sigma_\mu; \mathscr{A});\ \exists\, \alpha > 0 : z \mapsto (z^\alpha + z^{-\alpha})f(z) \in \mathscr{H}^\infty(\Sigma_\mu; \mathscr{A})\}$.

Remark 3.4. It is obvious that $\mathscr{H}^\infty(\Sigma_\mu)$ (resp. $\mathscr{H}_0^\infty(\Sigma_\mu)$) can be considered as a subspace of $\mathscr{H}^\infty(\Sigma_\mu; \mathscr{A})$ (resp. $\mathscr{H}_0^\infty(\Sigma_\mu; \mathscr{A})$) by setting $f(\cdot)I$ for functions f in $\mathscr{H}^\infty(\Sigma_\mu)$ (resp. $\mathscr{H}_0^\infty(\Sigma_\mu)$), where I is the identity on X (and belongs to $\mathscr{A}$).

If $f \in \mathscr{H}_0^\infty(\Sigma_\mu; \mathscr{A})$, then we define $f(A) = \frac{1}{2i\pi} \int_{\Gamma_\vartheta} f(z)(z-A)^{-1}\, dz$, where $\vartheta \in (\varphi_A, \mu)$ and $\Gamma_\vartheta = -e^{i\vartheta}(-\infty, 0] \cup e^{-i\vartheta}[0, \infty)$ (see [1], Section 1). This definition is independent of $\vartheta \in (\varphi_A, \mu)$.

For $f \in \mathscr{H}_0^\infty(\Sigma_\mu; \mathscr{A})$, let $g: z \mapsto \frac{zf(z)}{(1+z)^2}$. The function g belongs to $\mathscr{H}_0^\infty(\Sigma_\mu; \mathscr{A})$. Now define the operator $f(A)$ by $f(A)x = g(A)(A^{-1}x + 2x + Ax)$ for $x \in D(A) \cap R(A)$. The operator $f(A)$ is closable ; its closure in X is also denoted by $f(A)$.

This $\mathscr{A}$-valued functional calculus can be considered as a generalization of the complex-valued functional calculus discussed at the beginning of this section. Moreover, it has a convergence lemma and a uniqueness theorem similar to those for the classical functional calculus.

Example 3.5. In addition to the assumptions formulated above, let A, B be bounded, invertible and their resolvents commute. Let γ_A be a bounded positively oriented contour in Σ_μ which surrounds $\sigma(A)$ and γ_B a bounded positively oriented contour which surrounds $\sigma(B)$. Assume that F is a bounded holomorphic function in two variables in $\Sigma_\mu \times \rho(B)$. Then we define $F(A, B)$ with help of the Dunford functional calculus by

$$F(A, B) = \frac{1}{(2i\pi)^2} \int_{\gamma_A} \int_{\gamma_B} F(z, w)(z-A)^{-1}(w-B)^{-1}\, dw\, dz.$$

By setting $f(z) = \frac{1}{2i\pi} \int_{\gamma_B} F(z, w)(w-B)^{-1}\, dw$, we may consider the previous formula as an application of the $\mathscr{A}$-valued functional calculus, since $f \in \mathscr{H}^\infty(\Sigma_\mu)$. One has

$$F(A, B) = \frac{1}{2i\pi} \int_{\gamma_A} f(z)(z-A)^{-1}\, dz = \frac{1}{2i\pi} \int_{\Gamma_\vartheta} f(z)(z-A)^{-1}\, dz.$$

We now have the whole material needed to prove Theorem 2.4 and Corollary 2.5.

4. Proof of the perturbation results. Let A be a sectorial operator in X with spectral angle φ_A. Assume that A admits bounded imaginary powers and denote by ω_A the type of the strongly continuous group $(A^{is})_{s \in \mathbb{R}}$. Let $w \in \mathbb{C} \setminus (-\infty, 0]$ be fixed (the case $w = 0$ in Theorem 2.4 is clear), and denote by φ the modulus of the argument of w. We also assume that $\varphi_A + \varphi < \pi$. It is clear that the operator $A + \omega$ is sectorial. The only thing to show is that this operator admits bounded imaginary powers.

For all $\varepsilon \in (0, 1)$, we denote by A_ε the bounded invertible operator $(\varepsilon + A)(1 + \varepsilon A)^{-1}$.

In order to prove Theorem 2.4, choose $\mu \in (\varphi_A, \pi - \varphi)$ and fix $s \in \mathbb{R}$. The function ψ_s defined on $\Sigma_{\pi-\varphi}$ by $\psi_s(z) = (w+z)^{is}$ belongs to $\mathscr{H}^\infty(\Sigma_\mu)$. By Lemma 3.2 (ii), one has $\lim_{\varepsilon \to 0^+} \psi_s(A_\varepsilon)x = \psi_s(A)x$ for all $x \in D(A) \cap R(A)$. By the uniqueness of the limit, we know then that $\psi_s(A)x = (w+A)^{is}x$ for all $x \in D(A) \cap R(A)$.

On the other hand, one has for all $z \in \Sigma_{\pi-\varphi}$,

$$\psi_s(z) = \frac{z}{w+z}\ ((w+z)^{is} - z^{is}) + \frac{z}{w+z}\ z^{is} + \frac{w}{w+z}\ (w+z)^{is}.$$

Denote by ψ_s^1, ψ_s^2, ψ_s^3 the functions defined on $\Sigma_{\pi-\varphi}$ by $z \mapsto \frac{z}{w+z}\ ((w+z)^{is} - z^{is})$, $z \mapsto \frac{z}{w+z}\ z^{is}$, $z \mapsto \frac{w}{w+z}\ (w+z)^{is}$ respectively. We will show in the following that they are in $\mathscr{H}^\infty(\Sigma_\mu)$, that $\psi_s^1(A)$, $\psi_s^2(A)$ and $\psi_s^3(A)$ are bounded operators and finally, that $\psi_s(A) = \psi_s^1(A) + \psi_s^2(A) + \psi_s^3(A)$.

Lemma 4.1. *The function ψ_s^1 defined on $\Sigma_{\pi-\varphi}$ by $\psi_s^1(z) = \dfrac{z}{w+z}\ ((w+z)^{is} - z^{is})$ belongs to $\mathscr{H}_0^\infty(\Sigma_\mu)$.*

Proof. (o) For $z = re^{i\vartheta} \in \mathbb{C}$ $(r > 0,\ \vartheta \in (-\pi, \pi))$ and for $\tau \in (0, \infty)$, we have the estimate: $|\tau + z|^2 \geqq \left(\dfrac{1+\cos\vartheta}{2}\right)(\tau + r)^2$. This can be proved directly as follows.

$$\begin{aligned}
|\tau + z|^2 &= \tau^2 + 2\ \tau r \cos\vartheta + r^2 \\
&= \left(\frac{1+\cos\vartheta}{2}\right)(\tau + r)^2 + \left(1 - \frac{1+\cos\vartheta}{2}\right)(\tau^2 + r^2) \\
&\quad + \left(\cos\vartheta - \frac{1+\cos\vartheta}{2}\right)(2\ \tau r) \\
&= \left(\frac{1+\cos\vartheta}{2}\right)(\tau + r)^2 + \left(\frac{1-\cos\vartheta}{2}\right)(\tau^2 + r^2) - \left(\frac{1-\cos\vartheta}{2}\right)(2\ \tau r) \\
&= \left(\frac{1+\cos\vartheta}{2}\right)(\tau + r)^2 + \left(\frac{1-\cos\vartheta}{2}\right)(\tau - r)^2 \\
&\geqq \left(\frac{1+\cos\vartheta}{2}\right)(\tau + r)^2, \qquad \text{since} \quad \frac{1-\cos\vartheta}{2} \geqq 0.
\end{aligned}$$

In the following, we denote by c_ϑ the constant $\left(\dfrac{2}{1+\cos\vartheta}\right)^{\frac{1}{2}}$ for all $\vartheta \in (-\pi, \pi)$.

(i) Note that $\dfrac{1}{|w| + |z|} \leqq \dfrac{1}{|w + z|} \leqq \dfrac{c(\mu, \varphi)}{|w| + |z|}$ holds for all $z \in \overline{\Sigma}_\mu$, where $c(\mu, \varphi) = \max\{c_{\mu+\varphi}, c_{\mu-\varphi}\}$. Note also that $|tw + z| \geqq \sin(\mu + \varphi)\ |z|$ for all $z \in \overline{\Sigma}_\mu$, for all $t \geqq 0$.

(ii) Fix $z \in \overline{\Sigma}_\mu \setminus \{0\}$. Apply the mean value theorem on the intervall $\mathscr{I} = \{tw + z\ ;\ t \in [0, 1]\}$. The intervall $\mathscr{I}$ stays in $\overline{\Sigma}_{\max\{\mu,\varphi\}}$, and we have

$$\begin{aligned}
|(w+z)^{is} - z^{is}| &\leqq \sup_{t\in[0,1]} |is\ w\ (tw + z)^{is-1}| \\
&\leqq |s||w|\ e^{\max\{\mu,\varphi\}|s|} \sup_{t\in[0,1]} \left(\frac{1}{|tw + z|}\right) \\
&\leqq |s||w|\ e^{\max\{\mu,\varphi\}|s|}\ \frac{1}{\sin(\mu + \varphi)}\ \frac{1}{|z|}.
\end{aligned}$$

So, one obtains the following estimate for all $z \in \overline{\Sigma}_\mu \setminus \{0\}$:

$$|\psi_s^1(z)| \leqq \frac{|w|\ c(\mu, \varphi)}{\sin(\mu + \varphi)}\ |s|\ e^{\max\{\mu,\varphi\}|s|}\ \frac{1}{|w| + |z|}.$$

(iii) On the other hand, one also has

$$\begin{aligned}
|(w+z)^{is} - z^{is}| &\leqq e^{\max\{\mu,\varphi\}|s|} + e^{\mu|s|} \\
&\leqq 2\ e^{\max\{\mu,\varphi\}|s|}.
\end{aligned}$$

Finally, one obtains for all $z \in \overline{\Sigma}_\mu$,

$$|\psi_s^1(z)| \leqq \frac{c(\mu,\varphi)}{|w|+|z|}\ e^{\max\{\mu,\varphi\}|s|} \min\left\{2|z|, \frac{|s||w|}{\sin(\mu+\varphi)}\right\}.$$

This proves that $\psi_s^1 \in \mathscr{H}_0^\infty(\Sigma_\mu)$ and for all $\delta > 0$, there exists a constant k_1 such that

$$|\psi_s^1(z)| \leqq k_1\ (1+|w|)\ e^{(\max\{\mu,\varphi\}+\delta)|s|}\ \frac{\min\{|z|,1\}}{|w|+|z|}$$

(k_1 is independent of $w \in \overline{\Sigma}_\mu \setminus \{0\}$, and of $s \in \mathbb{R}$). $\square$

Corollary 4.2. *The operator $\psi_s^1(A)$ is bounded; moreover, for all $\delta > 0$, for all $\vartheta \in (\varphi_A, \mu)$, there exists a constant K_1 such that $\|\psi_s^1(A)\| \leqq K_1\ e^{(\max\{\mu,\varphi\}+\delta)|s|}$.*

Proof. The functional calculus developed in Section 3 gives for any $\vartheta \in (\varphi_A, \mu)$,

$$\psi_s^1(A) = \frac{1}{2i\pi} \int_{\Gamma_\vartheta} \psi_s^1(z)(z-A)^{-1}\ dz.$$

The previous lemma combined with this formula gives for all $\delta > 0$,

$$\|\psi_s^1(A)\| \leqq \frac{1}{2\pi}\ k_1\ (1+|w|)\ e^{(\max\{\mu,\varphi\}+\delta)|s|} \int_{\Gamma_\vartheta} \frac{\min\{|z|,1\}}{|w|+|z|}\ \frac{M_\vartheta}{|z|}\ |dz|,$$

where M_ϑ was defined in Definition 2.1. $\square$

Lemma 4.3. *The function ψ_s^2, defined on $\Sigma_{\pi-\varphi}$ by $\psi_s^2(z) = \dfrac{z}{w+z}\ z^{is}$, belongs to $\mathscr{H}^\infty(\Sigma_\mu)$, and the operator $\psi_s^2(A)$ is bounded; it is given by $\psi_s^2(A) = A(w+A)^{-1}A^{is}$.*

Proof. For all $z \in \overline{\Sigma}_\mu$, one has $|\psi_s^2(z)| \leqq |z|\ \dfrac{c(\mu,\varphi)}{|w|+|z|}\ e^{\mu|s|} \leqq c(\mu,\varphi)\ e^{\mu|s|}$; *i.e.* $\psi_s^2 \in \mathscr{H}^\infty(\Sigma_\mu)$.

For each $\varepsilon > 0$, the operator $\psi_s^2(A_\varepsilon)$ is given by the Dunford integral, which gives $\psi_s^2(A_\varepsilon) = A_\varepsilon(w+A_\varepsilon)^{-1}A_\varepsilon^{is}$. By Lemma 3.2 (ii), one has for all $x \in D(A) \cap R(A)$, $\psi_s^2(A_\varepsilon)x \longrightarrow \psi_s^2(A)x$ as $\varepsilon \to 0^+$ in X.

On the other hand, $A_\varepsilon(w+A_\varepsilon)^{-1} \longrightarrow A(w+A)^{-1}$ as $\varepsilon \to 0^+$, in the norm of operators and $A_\varepsilon^{is}x \longrightarrow A^{is}x$ as $\varepsilon \to 0^+$ in X for all $x \in D(A) \cap R(A)$.

Therefore, by uniqueness of the limit, one obtains $\psi_s^2(A)x = A(w+A)^{-1}A^{is}x$ for all $x \in D(A) \cap R(A)$. The operator $A(w+A)^{-1}A^{is}$ is bounded by assumption, and $D(A) \cap R(A)$ is dense in X. Therefore, the closure of $(\psi_s^2(A), D(A) \cap R(A))$ is a bounded operator equal to $A(w+A)^{-1}A^{is}$ and then, for all $\delta > 0$, there exists a constant K_2 such that $\|\psi_s^2(A)\| \leqq K_2\ e^{(\omega_A+\delta)|s|}$. $\square$

Obviously, the function ψ_s^3 defined on $\Sigma_{\pi-\varphi}$ by $\psi_s^3(z) = \dfrac{w}{w+z}\ (w+z)^{is}$ belongs to the space $\mathscr{H}^\infty(\Sigma_\mu)$.

Lemma 4.4. *The operator $\psi_s^3(A)$ is bounded and for all $\vartheta \in (\varphi_A, \mu)$, there exists a constant K_3 such that $\|\psi_s^3(A)\| \leqq K_3\ e^{\max\{\mu,\varphi\}|s|}$.*

Proof. Let $\varepsilon \in (0,1)$ be fixed and choose $R > \|A_\varepsilon\| + |\Re(w)|$. Denote by Γ_ϑ^R the contour $-e^{i\vartheta}[-R,0] \cup e^{-i\vartheta}[0,R] \cup Re^{i[-\vartheta,\vartheta]}$. By the Dunford functional calculus, one has

$$\begin{aligned}\psi_s^3(A_\varepsilon) &= \frac{1}{2i\pi} \int\limits_{\Gamma_\vartheta^R} \psi_s^3(z)(z-A_\varepsilon)^{-1}\ dz \\ &= \frac{1}{2i\pi} \int\limits_{\Gamma^R_{\vartheta - \frac{|\Re(w)|}{2}}} \frac{w}{w+z}\ (w+z)^{is}(z-A_\varepsilon)^{-1}\ dz, \quad \text{by holomorphy.}\end{aligned}$$

Moreover, one has

$$\lim_{R\to\infty} \left\| \int\limits_{-\vartheta}^{\vartheta} \frac{w}{w+Re^{i\sigma} - \frac{|\Re(w)|}{2}} \left(w + Re^{i\sigma} - \frac{|\Re(w)|}{2}\right)^{is} \cdot \left(Re^{i\sigma} - \frac{|\Re(w)|}{2} - A_\varepsilon\right)^{-1} iRe^{i\sigma}\ d\sigma \right\| = 0.$$

By holomorphy, one then obtains

$$\psi_s^3(A_\varepsilon) = \frac{1}{2i\pi} \int\limits_{\Gamma_{\vartheta - \frac{|\Re(w)|}{2}}} \frac{w}{w+z}\ (w+z)^{is}(z-A_\varepsilon)^{-1}\ dz.$$

It is now clear, by absolutely convergence, that one has for all $x \in X$

$$\psi_s^3(A_\varepsilon) \longrightarrow \frac{1}{2i\pi} \int\limits_{\Gamma_{\vartheta - \frac{|\Re(w)|}{2}}} \frac{w}{w+z}\ (w+z)^{is}(z-A)^{-1}\ dz \quad \text{as} \quad \varepsilon \to 0^+.$$

On the other hand, since $\psi_s^3 \in \mathscr{H}^\infty(\Sigma_\mu)$, by Lemma 3.2 (ii), one has $\psi_s^3(A_\varepsilon)x \longrightarrow \psi_s^3(A)x$ as $\varepsilon \to 0^+$ for all $x \in D(A) \cap R(A)$. Since $D(A) \cap R(A)$ is dense in X, by uniqueness of the limit, one obtains for all $x \in X$ $\psi_s^3(A)x = \frac{1}{2i\pi} \int\limits_{\Gamma_{\vartheta - \frac{|\Re(w)|}{2}}} \frac{w}{w+z}\ (w+z)^{is}(z-A)^{-1} x\ dz$. The operator $\psi_s^3(A)$ is bounded and its norm verifies

$$\begin{aligned}\|\psi_s^3(A)\| &\leqq \frac{1}{2\pi} \int\limits_{\Gamma_{\vartheta - \frac{|\Re(w)|}{2}}} \frac{c(\mu,\varphi)\ |w|}{|w|+|z|}\ e^{\max\{\vartheta,\varphi\}|s|}\ \frac{M_\vartheta}{|z|}\ |dz| \\ &\leqq K_3\ e^{\max\{\mu,\varphi\}|s|}, \quad \text{since } \vartheta \leqq \mu. \qquad \square\end{aligned}$$

We are now in position to prove Theorem 2.4. For all $\mu \in (\varphi_A, \pi - \varphi)$, for all $\vartheta \in (\varphi_A, \mu)$, for all $\delta > 0$, we have shown that there exist three constants K_1, K_2, K_3 independent of $s \in \mathbb{R}$ such that

$$\|\psi_s^1(A) + \psi_s^2(A) + \psi_s^3(A)\| \leqq K_1\ e^{(\max\{\mu,\varphi\}+\delta)|s|} + K_2\ e^{(\omega_A+\delta)|s|} + K_3\ e^{\max\{\mu,\varphi\}|s|}.$$

For each $\varepsilon \in (0,1)$, one also has $\psi_s^1(A_\varepsilon) + \psi_s^2(A_\varepsilon) + \psi_s^3(A_\varepsilon) = \psi_s(A_\varepsilon)$ by the Dunford calculus. Since ψ_s^1, ψ_s^2, ψ_s^3 are in $\mathscr{H}^\infty(\Sigma_\mu)$, one obtains from this that

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \psi_s(A_\varepsilon)x &= \psi_s^1(A)x + \psi_s^2(A)x + \psi_s^3(A)x \ \text{ for all } \ \in D(A)\cap R(A)\\ &= (w+A)^{is}x, \quad \text{by uniqueness of the limit.}\end{aligned}$$

Therefore, the operator $(w+A)^{is}$ is bounded for all $s \in \mathbb{R}$ and the type of the C_0-group $((w+A)^{is})_{s\in\mathbb{R}}$ is less than $\max\{\max\{\mu,\varphi\}+\delta,\ \omega_A+\delta\}$ for all $\mu \in (\varphi_A, \pi-\varphi)$, for all $\delta > 0$; i.e. $\omega_{w+A} \leqq \max\{\omega_A, \varphi\}$ (since $\varphi_A \leqq \omega_A$). $\square$

This proof will be used to show Corollary 2.5. Let B be a bounded invertible sectorial operator with spectral angle φ_B, such that $\varphi_A + \varphi_B < \pi$. Assume that the resolvents of B commute with the resolvents of A. Known results ([4], 3. Sommes commutatives) already imply that the operator $A+B$ defined on $D(A)$ is sectorial. Only the boundedness of the imaginary powers of $A+B$ remains to be shown. Choose $\mu \in (\varphi_A, \pi-\varphi_B)$, fix $s\in\mathbb{R}$ and let γ be a closed positively oriented contour which surrounds $\sigma(B)$ and such that $\gamma \subset \Sigma_{\pi-\mu}$. We will consider the following $\mathscr{A}$–valued functions ($\mathscr{A}$ is the second commutant of $(\lambda-A)^{-1}$ for one $\lambda\in\rho(A)$) ; for all $z\in\Sigma_\mu$

$$\psi_s(z) = \frac{1}{2i\pi}\int_\gamma (w+z)^{is}(w-B)^{-1}\,dw = (z+B)^{is},$$

$$\psi_s^1(z) = \frac{1}{2i\pi}\int_\gamma \frac{z}{w+z}\ ((w+z)^{is}-z^{is})(w-B)^{-1}\,dw = z(z+B)^{-1}((z+B)^{is}-z^{is}),$$

$$\psi_s^2(z) = \frac{1}{2i\pi}\int_\gamma \frac{z}{w+z}\ z^{is}(w-B)^{-1}\,dw = z^{1+is}(z+B)^{-1},$$

$$\psi_s^3(z) = \frac{1}{2i\pi}\int_\gamma \frac{w}{w+z}\ (w+z)^{is}(w-B)^{-1}\,dw = B(z+B)^{-1}(z+B)^{is}.$$

These functions verify $\psi_s(z) = \psi_s^1(z)+\psi_s^2(z)+\psi_s^3(z)$ for all $z\in\Sigma_\mu$. In the following, we will prove that ψ_s, ψ_s^1, ψ_s^2 and ψ_s^3 belong to $\mathscr{H}^\infty(\Sigma_\mu;\mathscr{A})$ and that $\psi_s^1(A)$, $\psi_s^2(A)$, $\psi_s^3(A)$ are bounded operators.

Using Lemma 4.1, one can see that for all $\delta>0$, there is a constant k_1 such that

$$|\psi_s^1(z)| \leqq \frac{1}{2\pi}\ |\gamma|\ k_1\ (1+R_\gamma)\ c(B)\ e^{(\max\{\mu,\pi-\mu\}+\delta)|s|}\ \frac{\min\{|z|,1\}}{r_\gamma+|z|},$$

where $|\gamma| = \int_\gamma |dz|$, $r_\gamma = \inf\{|\Re(w)|\ ;\ w\in\gamma\}$, $R_\gamma = \sup\{|w|;\ w\in\gamma\}$ and $c(B) = \sup\{\|(w-B)^{-1}\|;\ w\in\gamma\}$. This proves that $\psi_s^1\in\mathscr{H}_0^\infty(\Sigma_\mu;\mathscr{A})$. Therefore, the operator $\psi_s^1(A)$ is bounded and for all $\delta>0$, there is a constant K_1 such that $\|\psi_s^1(A)\| \leqq K_1\ e^{(\max\{\mu,\varphi\}+\delta)|s|}$. The constant K_1 is independent of $s\in\mathbb{R}$.

As in Lemma 4.3, it is easy to see that $\psi_s^2\in\mathscr{H}^\infty(\Sigma_\mu;\mathscr{A})$ and that $\psi_s^2(A) = A(A+B)^{-1}A^{is}$. For all $\delta>0$, the operator $\psi_s^2(A)$ is then bounded by $\|\psi_s^2(A)\| \leqq K_2 e^{(\omega_A+\delta)|s|}$, where K_2 is a constant independent of $s\in\mathbb{R}$.

The function ψ_s^3 belongs obviously to $\mathscr{H}^\infty(\Sigma_\mu;\mathscr{A})$. Let now $\varepsilon\in(0,1)$be fixed and choose $R > \|A_\varepsilon\| + r_\gamma$. Denote by Γ_ϑ^R the contour $-e^{i\vartheta}[-R,0]\cup e^{-i\vartheta}[0,R]\cup Re^{i[-\vartheta,\vartheta]}$, for $\vartheta\in(\varphi_A,\mu)$.

By the Dunford functional calculus, one has

$$\begin{aligned}\psi_s^3(A_\varepsilon) &= \frac{1}{2i\pi}\int\limits_{\Gamma_\vartheta^R} \psi_s^3(z)(z-A_\varepsilon)^{-1}\,dz\\ &= \frac{1}{2i\pi}\int\limits_{\Gamma_\vartheta^R-\frac{r_\gamma}{2}} \psi_s^3(z)(z-A_\varepsilon)^{-1}\,dz, \quad \text{by holomorphy.}\end{aligned}$$

Since

$$\lim_{R\to\infty}\left(\int\limits_{-\vartheta}^{\vartheta}\left(\int\limits_{\gamma}\left|\frac{w}{w+Re^{i\sigma}-\frac{r_\gamma}{2}}\right|\,\left|\left(w+Re^{i\sigma}-\frac{r_\gamma}{2}\right)^{is}\right|\|(w-B)^{-1}\|\;|dw|\right)\right.$$
$$\left.\cdot\left\|\left(Re^{i\sigma}-\frac{|r_\gamma|}{2}-A_\varepsilon\right)^{-1}\right\|\,R\,d\sigma\right)=0,$$

one obtains by holomorphy

$$\begin{aligned}\psi_s^3(A_\varepsilon) &= \frac{1}{2i\pi}\int\limits_{\Gamma_\vartheta-\frac{r_\gamma}{2}} \psi_s^3(z)(z-A_\varepsilon)^{-1}\,dz\\ &= \frac{1}{2i\pi}\int\limits_{\Gamma_\vartheta-\frac{r_\gamma}{2}}\left(\frac{1}{2i\pi}\int\limits_\gamma\frac{w}{w+z}\;(w+z)^{is}(w-B)^{-1}\,dw\right)(z-A_\varepsilon)^{-1}\,dz.\end{aligned}$$

By absolute convergence of this integral, one can easily see that for all $x \in X$,

$$\psi_s^3(A_\varepsilon)x \longrightarrow \frac{1}{2i\pi}\int\limits_{\Gamma_\vartheta-\frac{r_\gamma}{2}}\psi_s^3(z)(z-A)^{-1}x\,dz \quad \text{as} \quad \varepsilon\to 0^+.$$

Since $\psi_s^3 \in \mathscr{H}^\infty(\Sigma_\mu;\mathscr{A})$, one has by Lemma 3.2 (ii) extended to the case of $\mathscr{A}$-valued functions, $\psi_s^3(A_\varepsilon)x \to \psi_s^3(A)x$ as $\varepsilon \to 0^+$ for all $x \in D(A)\cap R(A)$. Therefore, the operator $\psi_s^3(A)$ is bounded and is equal to $\frac{1}{2i\pi}\int\limits_{\Gamma_\vartheta-\frac{r_\gamma}{2}}\psi_s^3(z)(z-A)^{-1}x\,dz$; its norm verifies

$$\begin{aligned}\|\psi_s^3(A)\| &\leqq \frac{1}{2\pi}\int\limits_{\Gamma_\vartheta-\frac{r_\gamma}{2}}\left(\frac{1}{2\pi}\int\limits_\gamma\frac{c(\mu,\varphi_B)\;|w|}{|w|+|z|}\;e^{\max\{\vartheta,\varphi_B\}|s|}c(B)\;|dw|\right)\frac{M_\vartheta}{|z|}|dz|\\ &\leqq K_3\;e^{\max\{\mu,\varphi_B\}|s|}, \qquad \text{for a constant } K_3 \text{ independent of } s\in\mathbb{R}.\end{aligned}$$

This shows Corollary 2.5. $\square$

References

[1] D. ALBRECHT and A. MCINTOSH, Holomorphic functional calculi of commuting pairs of operators. Preprint 1996.

[2] H. AMANN, Linear and quasilinear parabolic problems. Basel 1995.

[3] M. COWLING, I. DOUST, A. MCINTOSH and A. YAGI, Banach space operators with a bounded $\mathscr{H}^\infty$ functional calculus. J. Austral. Math. Soc. Ser. A **60,** 51–89 (1996).

[4] G. DA PRATO et P. GRISVARD, Sommes d'opérateurs linéaires et équations différentielles opérationnelles. J. Math. Pures Appl. **54,** 305–387 (1975).
[5] G. DORE and A. VENNI, On the closedness of the sum of two closed operators. Math. Z. **196,** 189–201 (1987).
[6] H. KOMATSU, Fractional powers of operators. Pacific J. Math. **9,** 285–346 (1966).
[7] S. G. KREIN, Linear differential equations in Banach spaces. Transl. Math. Monographs **29,** Amer. Math. Soc. 1972.
[8] A. MCINTOSH, Operators which have an $\mathcal{H}^\infty$ functional calculus. Miniconference on operator theory and partial differential equations. Proc. Centre Math. Anal. Austral. Nat. Univ. **14,** 210–231 (1986).
[9] S. MONNIAUX, Générateur analytique et régularité maximale. Thèse, Université de Franche-Comté, Besançon 1995.
[10] J. PRÜSS, Evolutionary integral equations and applications. Basel 1993.
[11] J. PRÜSS and H. SOHR, On operators with bounded imaginary powers in Banach spaces. Math. Z. **203,** 429–452 (1990).
[12] J. PRÜSS and H. SOHR, Imaginary powers of elliptic second-order differential operators in L^p-spaces. Hiroshima Math. J. **23,** 161–192 (1993).
[13] M. UITERDIJK, A note on the functional calculus for sectorial operators. Reports on the Faculty of Technical Mathematics and Informatics, no. 96–48, Delft 1996.
[14] K. YOSIDA, Functional analysis. Berlin-Heidelberg-New York 1980.

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