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RESEARCH ARTICLE

Semigroup Methods to Solve Non-autonomous Evolution Equations

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Abstract

Under regularity conditions on the family of (unbounded, linear, closed) operators $(L(t))_{t\in(0,T]}$ ($T>0$) on a Banach space X, there exists an evolution family $(V(t,s))_{T\geq t\geq s>0}$ on X such that $U(t,s)x = L(t)^{-1}V(t,s)L(s)x$ is the unique classical solution of the non-autonomous evolution equation

$$(nCP)\begin{cases} u'(t) + L(t)u(t) = 0 , & t\geq s,\\ u(s) = x , & \end{cases}$$

for $x\in D(L(s))$. Moreover, the evolution semigroup associated to the evolution family $(V(t,s))_{T\geq t\geq s>0}$ on $C_0((0,T];X)$, the Banach space of continuous functions f from $[0,T]$ into X satisfying $f(0)=0$, is generated by the closure of $-L(\cdot)(\frac{d}{dt}+L(\cdot))L(\cdot)^{-1}$. An application to parabolic partial differential equations is given.

1. Introduction

Consider the non-autonomous linear abstract Cauchy problem

$$(nCP)\begin{cases} u'(t) + L(t)u(t) = 0 , & t\geq s,\\ u(s) = x & \end{cases}$$

on a Banach space X and suppose that the operators $\{(L(t),D(L(t))), t\in I := (0,T]\}$ are sectorial and invertible, and satisfy the following conditions.

(A1) There exists an angle $\varphi\in(0,\frac{\pi}{2})$ such that for all $\vartheta\in(\varphi,\pi)$ there exists a constant M_ϑ with $\|(\lambda+L(t))^{-1}\|_{\mathcal{L}(X)}\leq\frac{M_\vartheta}{1+|\lambda|}$, for all $\lambda\in\overline{\Sigma}_{\pi-\vartheta}$ and for all $t\in I$.

(A2) There exists an angle $\nu\in(\varphi,\frac{\pi}{2})$ and two powers α,β ($0\leq\alpha<\beta\leq 1$) such that, for $c=\dfrac{\pi\ (\beta-\alpha)\ (\cos\nu)^{1+\alpha}(\sin\nu)^\alpha}{2\ \Gamma(\alpha+1)T^{\beta-\alpha}}$,

$$\|L(t)(\lambda+L(t))^{-1}(L(t)^{-1}-L(s)^{-1})\|_{\mathcal{L}(X)}\leq\frac{c\ |t-s|^\beta}{1+|\lambda|^{1-\alpha}}$$

holds for all $\lambda\in\overline{\Sigma}_{\pi-\nu}$, and for all $t,s\in I$,

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where $\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\}; |\arg(z)| < \theta\}$ for $\theta \in (0,\pi)$.

Condition (A2) was introduced (in a somewhat weaker form) by P. Acquistapace and B. Terreni [2]. It is known that (nCP) has a unique classical solution on the spaces $D(L(t))$, see [1], Theorem 2.3 or [19], Theorem 3.2, (cf. [4], Theorem 2.3.2 and the references therein). In [2], the authors constructed an evolution operator $(U(t,s))_{t\geq s}$ solving (nCP) by means of suitable integral equations, and by using techniques of fractional powers ; Yagi [19] generalized the results of [2]. They obtained also that $(U(t,s))_{t\geq s}$ is of parabolic type, *i.e.*, $U(t,s)X \subset D(L(t))$ for $t > s$ and $\|L(t)U(t,s)\| \leq \frac{C}{t-s}$ for $T \geq t > s \geq 0$.

In this paper we propose to give a “simple” proof of the solvability of (nCP). More precisely, we prove that the closure of the operator $-L(\cdot)(\frac{d}{dt} + L(\cdot))L(\cdot)^{-1}$ on a suitable domain $\mathcal{D}$ generates an evolution semigroup $(T(t))_{t\geq 0}$ on the Banach space $C_0(I;X) = \{f \in C([0,T];X); f(0) = 0\}$, endowed with the sup-norm. By a simple trick we deduce that $L(\cdot)^{-1}T(t-s)L(\cdot)$ gives the classical solution of (nCP). Finally an application to parabolic partial differential equations in $L^1(\Omega)$ is given.

2. The abstract result

In this section, $(X, \|\cdot\|_X)$ denotes a Banach space. We consider a family $(L(t))_{t\in I}$ ($I = (0,T]$ for $T > 0$) of sectorial and invertible operators on X verifiying (A1) and (A2). On the Banach space $C_0(I;X)$, we define the multiplication operator A as follows

$$\begin{aligned} D(A) &= \{f \in C_0(I;X); f(t) \in D(L(t)) \text{ for all } t \in I \;\&\; L(\cdot)f(\cdot) \in C_0(I;X)\}, \\ (Af)(t) &= L(t)f(t), \quad \text{for all } t \in I \text{ and } f \in D(A). \end{aligned}$$

We consider also the derivative on $C_0(I;X)$, denoted by B, as follows

$$\begin{aligned} D(B) &= \{f \in C^1([0,T];X) \;;\; f, f' \in C_0(I;X)\} \\ Bf &= f', \quad \text{for all } f \in D(B). \end{aligned}$$

Remark 1. The operator $-A$ generates a bounded analytic C_0-semigroup on the space $C_0(I;X)$.

Proof. Indeed, the domain of A is dense in $C_0(I;X)$, since the set $\{x \in X \;;\; \exists\, f \in D(A) : f(t) = x\}$ contains $D(L(t))$ for all $t \in I$, $t > 0$ (for $x \in D(L(t))$, consider the function $f : I \to X$ defined by $f(\tau) = \psi(\tau)L(\tau)^{-1}L(t)x$ for a function $\psi \in C_0(I;\mathbb{R})$, with $\psi(t) = 1$), and therefore is dense in X. We can use then Lemma 4.5 of [14] to conclude. Moreover, it follows from the assumption (A1) that, for all $\vartheta \in (\varphi,\pi)$, $\|(\lambda + A)^{-1}\|_{\mathcal{L}(C_0(I;X))} \leq \dfrac{M_\vartheta}{1+|\lambda|}$ holds for all $\lambda \in \overline{\Sigma}_{\pi-\vartheta}$. This implies that $-A$ generates a bounded analytic C_0-semigroup on $C_0(I;X)$ ([15], Theorem 5.2). ∎

We denote now by A_n $(n \geq 1)$ and B_m $(m \geq 1)$ the Yosida approximations of A and B, *i.e.*, $A_n = nA(n+A)^{-1}$ and $B_m = mB(m+B)^{-1}$. Those operators A_n $(n \geq 1)$ and B_m $(m \geq 1)$ are bounded on $C_0(I;X)$ and the following assertions hold

(*i*) $\lim\limits_{n\to\infty} \|A_n f - Af\|_{C_0(I;X)} = 0$ for all $f \in D(A)$ and
$\lim\limits_{m\to\infty} \|B_m f - Bf\|_{C_0(I;X)} = 0$ for all $f \in D(B)$.

(*ii*) $\lim\limits_{n\to\infty} \|e^{-tA_n} f - e^{-tA} f\|_{C_0(I;X)} = 0$ and $\lim\limits_{m\to\infty} \|e^{-tB_m} f - e^{-tB} f\|_{C_0(I;X)} = 0$ uniformly on every compact subset of $[0,\infty)$ for all $f \in C_0(I;X)$.

We define $(S_{n,m}(t))_{t\geq 0}$ as the C_0-semigroup generated by the bounded operator $-(A_n + B_m)$ on $C_0(I;X)$, for all $n, m \geq 1$. We have the following result concerning the convergence of these semigroups.

(*iii*) For $n \geq 1$ fixed, the operator $-(A_n + B)$ defined on $D(B)$ generates a C_0-semigroup $\{S_n(t) = e^{-t(A_n+B)}, t \geq 0\}$ (as a bounded perturbation of $-B$). Moreover, we have
$\lim\limits_{m\to\infty} \|S_{n,m}(t)f - S_n(t)f\|_{C_0(I;X)} = 0$ uniformly on every compact subset of $[0,\infty)$ for all $f \in C_0(I;X)$.

For all $f \in C_0(I;X)$ and $t \geq 0$ we have

$$S_{n,m}(t)f = e^{-tA_n} e^{-tB_m} f + \int_0^t A_n^{-1} K_{n,m}(t-\sigma) A_n S_{n,m}(\sigma) f \, d\sigma,$$

where $K_{n,m}(\sigma) = A_n^2 e^{-\sigma A_n}(e^{-\sigma B_m} A_n^{-1} - A_n^{-1} e^{-\sigma B_m})$ for all $\sigma \geq 0$ (for a fixed $t \geq 0$, we differentiate the function $[0,t] \ni \sigma \mapsto e^{-(t-\sigma)A_n} e^{-(t-\sigma)B_m} S_{n,m}(\sigma) f$, and then integrate between 0 and t).

We let now m go to ∞ and we obtain (using (*ii*) and (*iii*)), for all $n \geq 1$ and $t \geq 0$,

$$S_n(t)f = e^{-tA_n} e^{-tB} f + \int_0^t A_n^{-1} K_n(t-\sigma) A_n S_n(\sigma) f \, d\sigma,$$

where $K_n(\sigma) = A_n^2 e^{-\sigma A_n}(e^{-\sigma B} A_n^{-1} - A_n^{-1} e^{-\sigma B})$ for all $\sigma \geq 0$.

Multiplying this equality with A_n on the left and applying it to $A_n^{-1} f$, we obtain for all $t \geq 0$

$$T_n(t)f = A_n e^{-tA_n} e^{-tB} A_n^{-1} f + \int_0^t K_n(t-\sigma) T_n(\sigma) f \, d\sigma, \tag{1}$$

where $T_n(t) = A_n S_n(t) A_n^{-1}$, $t \geq 0$, $n \geq 1$.

Remark 2. For each $n \geq 1$, $K_n(\sigma) = 0$ on $C_0(I;X)$ for all $\sigma > T$.

We denote by $\mathcal{K}_n$ the convolution by K_n on $C^b([0,\infty);\mathcal{L}_s(C_0(I;X)))$, the space of all bounded continuous functions defined on $[0,\infty)$, with values in the space of bounded operators on $C_0(I;X)$ considered with the strong topology, with its natural norm $U \mapsto \sup\limits_{t\geq 0} \|U(t)\|_{\mathcal{L}(C_0(I;X))}$. That is, we let

$$(\mathcal{K}_n U)(t)f = \int_0^t K_n(t-\sigma) U(\sigma) f \, d\sigma \quad \text{for all } t \geq 0 \text{ and } f \in C_0(I;X)$$

for $U \in C^b([0,\infty);\mathcal{L}_s(C_0(I;X)))$.

Remark 3. It is easy to see that, for all $\lambda \in \Gamma_\nu$ and all $n \geq 1$,

$$\frac{1}{|n-\lambda|} \leq \frac{1}{\sin\nu} \ \min\left\{\frac{1}{n}, \frac{2}{1+|\lambda|}\right\}.$$

Here, Γ_ν denotes the path $(\infty, 0]\ e^{i\nu} \cup e^{-i\nu}[0,\infty)$, where ν was defined in (A2).

Lemma 4. *For $n \geq 1$ and $U \in C^b([0,\infty);\mathcal{L}_s(C_0(I;X)))$, for all $f \in C_0(I;X)$, we have*

$$\sup_{t\geq 0} \|(\mathcal{K}_n U)(t)f\|_{C_0(I;X)} \leq \frac{1}{2} \sup_{t\geq 0} \|U(t)f\|_{C_0(I;X)}$$

Proof. For $f \in C_0(I;X)$, we have for all $n \geq 1$ and $t \in I$

$$\begin{aligned}(K_n(s)f)(t) \ &= \ \frac{1}{2i\pi} \int_{\Gamma_\nu} \lambda e^{-s\lambda} L_n(t)(\lambda - L_n(t))^{-1} \cdot \\ &\qquad\qquad (L_n(t-s)^{-1} - L_n(t)^{-1}) f(t-s)\, \chi_I(t-s)\ d\lambda,\end{aligned}$$

where $L_n(\sigma) = nL(\sigma)(n+L(\sigma))^{-1}$ for $\sigma \in I$ and $n \geq 1$. This gives then

$$\begin{aligned}(K_n(s)f)(t) \ &= \ \frac{1}{2i\pi} \int_{\Gamma_\nu} \frac{\lambda e^{-s\lambda}}{1-\frac{\lambda}{n}}\ L(t) \left(\frac{\lambda}{1-\frac{\lambda}{n}} - L(t)\right)^{-1} \cdot \\ &\qquad\qquad (L(t-s)^{-1} - L(t)^{-1}) f(t-s)\, \chi_I(t-s)\ d\lambda.\end{aligned}$$

Taking the condition (A2) and Remark 3 into account, we obtain for all $s > 0$

$$\begin{aligned}\sup_{t\in I} \|(K_n(s))f(t)\|_X \ &\leq \ \frac{1}{2\pi} \left(\int_{\Gamma_\nu} \frac{|\lambda e^{-s\lambda}|}{|1-\frac{\lambda}{n}|}\ \frac{c\ s^\beta}{1+\left|\frac{\lambda}{1-\frac{\lambda}{n}}\right|^{1-\alpha}}\ |d\lambda|\right)\ \|f\|_{C_0(I;X)} \\ &\leq \ \frac{1}{\pi} \ \left(\int_0^\infty \frac{c\ s^\beta\ re^{-rs\cos\nu}}{(\sin\nu)^\alpha (r^{1-\alpha} + (\sin\nu)^{1-\alpha})}\ dr\right)\ \sup_{t\in I} \|f\|_{C_0(I;X)}.\end{aligned}$$

Therefore, we have for all $s > 0$, using the expression for c in (A2),

$$\|K_n(s)f\|_{C_0(I;X)} \leq \frac{\beta-\alpha}{2\ T^{\beta-\alpha}}\ s^{\beta-\alpha-1}\ \|f\|_{C_0(I;X)}. \tag{2}$$

This gives the following estimate for $n \geq 1$, $U \in C^b([0,\infty);\mathcal{L}_s(C_0(I;X)))$, $f \in C_0(I;X)$ and $t \geq 0$

$$\begin{aligned}\|(\mathcal{K}_n U)(t)f\|_{C_0(I;X)} \ &\leq \ \int_0^t \|K_n(t-\sigma)\|_{\mathcal{L}(C_0(I;X))} \|U(\sigma)f\|_{C_0(I;X)} d\sigma \\ &\leq \ \left(\int_0^T \|K_n(\sigma)\|_{\mathcal{L}(C_0(I;X))} d\sigma\right)\ \sup_{\sigma\geq 0} \|U(\sigma)f\|_{C_0(I;X)} \\ &\leq \ \left(\int_0^T \frac{\beta-\alpha}{2\ T^{\beta-\alpha}}\ \sigma^{\beta-\alpha-1}\ d\sigma\right)\ \sup_{\sigma\geq 0} \|U(\sigma)f\|_{C_0(I;X)} \\ &= \ \frac{1}{2}\ \sup_{\sigma\geq 0} \|U(\sigma)f\|_{C_0(I;X)},\end{aligned}$$

where we have used Remark 2. This completes the proof. ∎

The operators $\mathcal{K}_n$ defined on $C^b([0,\infty);\mathcal{L}_s(C_0(I;X)))$ are uniformly bounded (with respect to $n\geq 1$) with norm less than or equal to $\frac{1}{2}$. Then, for all $n\geq 1$, $(1-\mathcal{K}_n)^{-1}=\sum_{p=0}^{\infty}\mathcal{K}_n^p$ is a bounded operator with norm less than or equal to 2. Moreover, if we denote by K the following family of bounded operators on $C_0(I;X)$: $K(0)=0$, and $K(\sigma)=A^2e^{-\sigma A}(e^{-\sigma B}A^{-1}-A^{-1}e^{-\sigma B})$ for $\sigma>0$, and by $\mathcal{K}$ the convolution by K on $C^b([0,\infty);\mathcal{L}_s(C_0(I;X)))$, we obtain as in Lemma 4

$$\|K(s)f\|_{C_0(I;X)}\leq \frac{\beta-\alpha}{2T^{\beta-\alpha}}\,s^{\beta-\alpha-1}\|f\|_{C_0(I;X)} \tag{3}$$

for $f\in C_0(I;X)$ and

$$\sup_{t\geq 0}\|(\mathcal{K}U)(t)\|_{\mathcal{L}(C_0(I;X))}\leq \frac{1}{2}\ \sup_{t\geq 0}\|U(t)\|_{\mathcal{L}(C_0(I;X))}$$

for $U\in C^b([0,\infty);\mathcal{L}_s(C_0(I;X)))$. We have

$$\begin{aligned}((K_n(\sigma)-K(\sigma))f)(t) &= \\ &\frac{1}{2i\pi}\int_{\Gamma_\nu}\lambda^2e^{-\sigma\lambda}\left((\lambda-L_n(t))^{-1}-(\lambda-L(t))^{-1}\right)\left(L(t-\sigma)^{-1}-L(t)^{-1}\right)\cdot \\ &\qquad\qquad\qquad\qquad\qquad\qquad\chi_I(t-\sigma)f(t-\sigma)\,d\lambda \\ &= \frac{-1}{2i\pi}\int_{\Gamma_\nu}\frac{\lambda^2e^{-\sigma\lambda}}{n-\lambda}\,L(t)\left(\frac{\lambda}{1-\frac{\lambda}{n}}-L(t)\right)^{-1}\cdot \\ &\qquad L(t)(\lambda-L(t))^{-1}(L(t-\sigma)^{-1}-L(t)^{-1})\ \chi_I(t-\sigma)f(t-\sigma)\,d\lambda.\end{aligned}$$

By a similar computation as in the proof of Lemma 4 one can see that

$$\lim_{n\to\infty}\|K_n(\sigma)-K(\sigma)\|_{\mathcal{L}(C_0(I;X))}=0\quad\text{ for every }\sigma\geq 0.$$

So by (2), (3), Remark 2 and the Lebesgue dominated convergence theorem we obtain, for $f\in C_0(I;X)$ and $U\in C^b([0,\infty);\mathcal{L}_s(C_0(I;X)))$,

$$\lim_{n\to\infty}\sup_{t\geq 0}\|(\mathcal{K}_nU)(t)f-(\mathcal{K}U)(t)f\|_{C_0(I;X)}=0.$$

This implies

$$\lim_{n\to\infty}\sup_{t\geq 0}\|((I-\mathcal{K}_n)^{-1}U)(t)f-((I-\mathcal{K})^{-1}U)(t)f\|_{C_0(I;X)}=0 \tag{4}$$

for $U\in C^b([0,\infty);\mathcal{L}_s(C_0(I;X)))$ and $f\in C_0(I;X)$.

Consider now $\{U_n(t)=A_ne^{-tA_n}e^{-tB}A_n^{-1},\ t\geq 0\}$ for $n\geq 1$. For each $n\geq 1$, we have $U_n\in C^b([0,\infty);\mathcal{L}_s(C_0(I;X)))$.

Lemma 5. *There exists a constant $M\geq 0$ such that $\sup_{t\geq 0}\|U_n(t)\|_{\mathcal{L}(C_0(I;X))}\leq M$ for all $n\geq 1$. Moreover, denote by U the following family of bounded operators on $C_0(I;X)$: $U(0)=1$ and $U(t)=Ae^{-tA}e^{-tB}A^{-1}$ for $t>0$. Then we have $U\in C^b([0,\infty);\mathcal{L}_s(C_0(I;X)))$ and*

$$\lim_{n\to\infty}\sup_{t\geq 0}\|U_n(t)f-U(t)f\|_{C_0(I;X)}=0\quad \textit{for all } f\in C_0(I;X).$$

Proof. Using the same methods as in Lemma 4, we have for all $n \geq 1$ and $f \in C_0(I;X)$,

$$\begin{aligned}
\|U_n(t)f - e^{-tA_n}e^{-tB}f\|_{C_0(I;X)} &= \|A_n^{-1}K_n(t)f\|_{C_0(I;X)} \\
&\leq \frac{1}{2\pi}\left(\int_{\Gamma_\nu} \frac{|e^{-t\lambda}|}{|1-\frac{\lambda}{n}|}\ \frac{c\ t^\beta}{1+\left|\frac{\lambda}{1-\frac{\lambda}{n}}\right|^{1-\alpha}}\ |d\lambda|\right)\ \|f\|_{C_0(I;X)} \\
&\leq \frac{1}{2}\ (\beta-\alpha)\ \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)}\ \|f\|_{C_0(I;X)} \quad \text{for all } t \in [0,T]
\end{aligned}$$

and $U_n(t) = 0$, $e^{-tA_n}e^{-tB} = 0$ on $C_0(I;X)$ for all $t > T$. Therefore, we obtain

$$\sup_{t\geq 0} \|U_n(t) - e^{-tA_n}e^{-tB}\|_{\mathcal{L}(C_0(I;X))} \leq \frac{1}{2}\ (\beta-\alpha)\ \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)}.$$

Since $\sup\{\|e^{-tA_n}e^{-tB}\|_{\mathcal{L}(C_0(I;X))}, t \geq 0, n \geq 1\} < \infty$, we have

$$\sup\{\|U_n(t)\|_{\mathcal{L}(C_0(I;X))}, t \geq 0, n \geq 1\} < \infty.$$

On the other hand, since

$$\begin{aligned}
\left(A^{-1}K(t)f\right)(\tau) &= \frac{1}{2i\pi}\int_{\Gamma\nu} e^{-t\lambda}L(\tau)(\lambda - L(\tau))^{-1}(L(\tau-t)^{-1} - L(\tau)^{-1}) \cdot \\
&\qquad \chi_I(\tau - t)\ f(\tau - t)\ d\lambda
\end{aligned}$$

for $t \geq 0$ and $s \in I$, it follows from (A1) and (A2) that the function $A^{-1}K(\cdot)f$ is continuous on $[0,\infty)$ for every $f \in C_0(I;X)$ and, as above, we have

$$\sup\{\|A^{-1}K(t)\|,\ t \geq 0\} < \infty.$$

This implies that $[0,\infty) \ni t \mapsto U(t) = A^{-1}K(t) + e^{-tA}e^{-tB}$ is a strongly continuous and bounded function. Since we have, for all $f \in C_0(I;X)$, $U_n(t)f - U(t)f = (A_n^{-1}K_n(t)f - A^{-1}K(t)f) + (e^{-tA_n}e^{-tB}f - e^{-tA}e^{-tB}f)$ for $t \in [0,T]$ and $U_n(t)f - U(t)f = 0$ for $t > T$, it suffices to prove that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|A_n^{-1}K_n(t) - A^{-1}K(t)\|_{\mathcal{L}(C_0(I;X))} = 0.$$

For this purpose, let $f \in C_0(I;X)$ and $t, \tau \in I$. Then we have

$$\begin{aligned}
&((A_n^{-1}K_n(t) - A^{-1}K(t))f)(\tau) = \\
&\quad \frac{1}{2i\pi}\int_{\Gamma_\nu} \lambda e^{-t\lambda}\left((\lambda - L_n(\tau))^{-1} - (\lambda - L(\tau))^{-1}\right)(L(\tau-t)^{-1} - L(\tau)^{-1}) \cdot \\
&\qquad \chi_I(\tau-t)f(\tau-t)\ d\lambda \\
&= \frac{1}{2i\pi}\int_{\Gamma_\nu} \lambda e^{-t\lambda}L_n(\tau)(\lambda - L_n(\tau))^{-1}(L(\tau)^{-1} - L_n(\tau)^{-1}) \cdot \\
&\qquad L(\tau)(\lambda - L(\tau))^{-1}(L(\tau-t)^{-1} - L(\tau)^{-1})\ \chi_I(\tau-t)f(\tau-t)\ d\lambda \\
&= \frac{-1}{2i\pi}\int_{\Gamma_\nu} \frac{\lambda e^{-t\lambda}}{n-\lambda}\ L(\tau)\left(\frac{\lambda}{1-\frac{\lambda}{n}} - L(\tau)\right)^{-1} \cdot \\
&\qquad L(\tau)(\lambda - L(\tau))^{-1}(L(\tau-t)^{-1} - L(\tau)^{-1})\ \chi_I(\tau-t)f(\tau-t)\ d\lambda.
\end{aligned}$$

So using (A1), (A2) and Remark 3, we obtain

$$
\begin{aligned}
\|(A_n^{-1}K_n(t)-A^{-1}K(t))f)(\tau)\|_X \;\le\; & \\
& \frac{1}{2\pi}\left(\int_{\Gamma_\nu}\frac{|\lambda e^{-t\lambda}|}{|n-\lambda|}\,(M_\nu+1)\,\frac{c\,t^\beta}{1+|\lambda|^{1-\alpha}}\,|d\lambda|\right)\,\|f\|_{C_0(I;X)}\\
& \le \frac{c\,(M_\nu+1)}{\pi\sin\nu}\,t^\beta\left(\int_0^\infty\frac{re^{-tr\cos\nu}}{1+r^{1-\alpha}}\,\min\left\{\frac{1}{n},\frac{2}{1+r}\right\}\,dr\right)\,\|f\|_{C_0(I;X)}\\
& \le \frac{c\,(M_\nu+1)}{\pi\sin\nu}\,t^\beta\left(\int_0^\infty r^\alpha e^{-tr\cos\nu}\min\left\{\frac{1}{n},\frac{2}{1+r}\right\}dr\right)\,\|f\|_{C_0(I;X)}\\
\le & \begin{cases}\frac{2\,c\,(M_\nu+1)}{\pi\sin\nu(\cos\nu)^\alpha}\,t^{\beta-\alpha}\left(\int_0^\infty r^{\alpha-1}e^{-r}dr\right)\,\|f\|_{C_0(I;X)} & \text{if}\quad t\le\frac{1}{n}\\ \frac{c\,(M_\nu+1)}{\pi\sin\nu(\cos\nu)^{1+\alpha}}\,\frac{t^{\beta-\alpha-1}}{n}\left(\int_0^\infty r^\alpha e^{-r}dr\right)\,\|f\|_{C_0(I;X)} & \text{if}\quad \frac{1}{n}<t\le T\end{cases}\\
\le & \begin{cases}\frac{2\,c\,(M_\nu+1)}{\pi\sin\nu(\cos\nu)^\alpha}\,\frac{1}{n^{\beta-\alpha}}\,\Gamma(\alpha)\,\|f\|_{C_0(I;X)} & \text{if}\quad t\le\frac{1}{n}\\ \frac{c\,(M_\nu+1)}{\pi\sin\nu(\cos\nu)^{1+\alpha}}\,\frac{t^{\frac{\beta-\alpha}{2}}}{n^{\frac{\beta-\alpha}{2}}}\,\Gamma(\alpha+1)\,\|f\|_{C_0(I;X)} & \text{if}\quad \frac{1}{n}<t\le T\end{cases}\\
\le & \frac{1}{n^{\frac{\beta-\alpha}{2}}}\,\frac{2\,c\,(M_\nu+1)}{\pi\sin\nu(\cos\nu)^{1+\alpha}}\,(\Gamma(\alpha)+\Gamma(\alpha+1))\max\{T^{\frac{\beta-\alpha}{2}},1\}\|f\|_{C_0(I;X)},
\end{aligned}
$$

and the lemma is proved. ∎

We can now give the main result of this section.

Theorem 6. *Under the assumptions (A1) and (A2), with the same notations as those used in this section, the sequence consisting in the bounded C_0-semigroups $(\{T_n(t),t\ge0\})_{n\ge1}$ converges strongly (as n goes to ∞) to a bounded C_0-semigroup $\{T(t),t\ge0\}$ uniformly in $[0,\infty)$ which satisfies, for all $f\in C_0(I;X)$,*

$$T(t)f=U(t)+\int_0^t K(t-\sigma)T(\sigma)\,d\sigma \quad \textit{for all } t\ge0. \tag{5}$$

Proof. We know, by (1), that $T_n=(1-\mathcal{K}_n)^{-1}U_n$. From Lemma 4 we have

$$\sup_{t\ge0}\|((I-\mathcal{K}_n)^{-1}(U_n-U))(t)f\|_{C_0(I;X)}\le 2\sup_{t\ge0}\|U_n(t)f-U(t)f\|_{C_0(I;X)}.$$

Using then Lemma 5 and (4), we obtain, for $f\in C_0(I;X)$,

$$\lim_{n\to\infty}\sup_{t\ge0}\|T_n(t)f-T(t)f\|=0,$$

where $T(t)=((I-\mathcal{K})^{-1}U)(t)$, $t\ge0$, which gives (5). Since $(T_n(t))_{t\ge0}$ is a C_0-semigroup for all $n\ge1$, we can easily see that $(T(t))_{t\ge0}$ is also a C_0-semigroup (the strong continuity follows from the convergence, and the semigroup formula for $T_n(\cdot)$ gives the one for $T(\cdot)$). ∎

Remark 7. The restriction concerning the constant c in (A2) can be weakened. In fact, the solutions of the problem

$$(nCP)_R\begin{cases} u'(t)\;+\;(R+L(t))u(t)\;=\;0\;, & t\ge s,\\ \qquad\qquad u(s)\;=\;x\;, & \end{cases}$$

are the same, modulo a factor $e^{-R(t-s)}$, as the one of (nCP). For $R>0$ large enough, the family $(R+L(t))_{t\in I}$ verifies the conditions (A1) and (A2) (in (A2), the power α is maybe replaced by $\alpha'\in(\alpha,\beta)$). See also [12].

3. Applications to non-autonomous Cauchy problems

In this section we apply our abstract result to the non-autonomous Cauchy problem

$$(nCP)\left\{\begin{array}{rcrcl} u'(t) & + & L(t)u(t) & = & 0 \quad , \qquad t \geq s,\, t \in I, \\ & & u(s) & = & x \quad , \end{array}\right.$$

where $I = (0, T]$, $(L(t))_{t\in I}$ is a family of closed linear densely defined operators in a Banach space X and $x \in D(L(s))$ for a fixed $s \in I$. The section concludes with an application to parabolic partial differential equations.

Recall that $u \in C([s,T];X)$ is called a *classical solution* of (nCP) if $u \in C^1([s,T];X) \cap \{v \in C([s,T];X); v(t) \in D(L(t)), L(\cdot)v(\cdot) \in C([s,T];X)\}$ and satisfies

$$u' + L(\cdot)u = 0 \;\text{ in } [s,T],\; u(s) = x.$$

We prove here the following result.

Proposition 8. *Let X be a Banach space, $(L(t))_{t\in I}$ a family of closed linear densely defined operators in X which is subject to (A1) and (A2), and let $s \in I$ and $x \in D(L(s))$. Then the Cauchy problem (nCP) admits a unique classical solution u. Moreover, u is given by*

$$u(t) = (A^{-1}T(t-s)Af)(t), \quad t \in [s,T],$$

where $(T(t))_{t\geq 0}$ is the C_0-semigroup obtained in Theorem 6 and $f \in D(A)$ with $f(s) = x$.

Proof. For each $n \geq 1$ we consider the generator $G_n := -A_n(A_n+B)A_n^{-1}$, with domain $D(G_n) = \{g \in C_0(I;X) : A_n^{-1}g \in D(B)\}$, of the C_0-semigroup $(T_n(t))_{t\geq 0}$ given in Section 2. Since, for every $t \geq 0$, $T_n(t)g \in D(G_n)$ if $g \in D(G_n)$, we obtain the following

$$\begin{aligned} \frac{d}{d\sigma}(e^{-(t-\sigma)B}A_n^{-1}T_n(\sigma - s)g) &= e^{-(t-\sigma)B}BA_n^{-1}T_n(\sigma-s)g \\ &\qquad + e^{-(t-\sigma)B}A_n^{-1}G_nT_n(\sigma-s)g \\ &= -e^{-(t-\sigma)B}T_n(\sigma-s)g \end{aligned}$$

for $n \geq 1$, $s \leq \sigma \leq t$ and $g \in D(G_n)$. Integrating over $[s,t]$, we obtain

$$A_n^{-1}T_n(t-s)g - e^{-(t-s)B}A_n^{-1}g = -\int_s^t e^{-(t-\sigma)B}T_n(\sigma-s)g\,d\sigma,$$

for all $g \in D(G_n)$. Since $D(G_n)$ is dense in $C_0(I;X)$, this also holds for every $g \in C_0(I;X)$. Since the semigroup $(T_n(t))_{t\geq 0}$ is bounded independently of $n \in \mathbb{N}$, we can pass to the limit as n goes to ∞, and Theorem 6 yields

$$A^{-1}T(t-s)g - e^{-(t-s)B}A^{-1}g = -\int_s^t e^{-(t-\sigma)B}T(\sigma-s)g\,d\sigma$$

for all $t \geq s$ and all $g \in C_0(I;X)$. In particular for $g = Af$ with $f := \varphi(\cdot)L(\cdot)^{-1}L(s)x$ such that $\varphi \in C_c^\infty(I)$ and $\varphi(s) = 1$ for a fixed $s \in I$ and $x \in D(L(s))$, we obtain

$$u(t) := (A^{-1}T(t-s)Af)(t) = x - \int_s^t L(\sigma)u(\sigma)d\sigma.$$

This proves the existence of a classical solution of (nCP).

To show the uniqueness we use the same procedure as in [2], p. 56, (cf. [18], p. 257). We consider a classical solution v of (nCP) and set $w(\sigma) := e^{-(t-\sigma)L(t)}v(\sigma)$ for $\sigma \in [s,t]$, where s is fixed in I and $t > s$. Then, for each $\sigma \in [s,t]$,

$$\begin{aligned} w'(\sigma) &= L(t)e^{-(t-\sigma)L(t)}v(\sigma) - e^{-(t-\sigma)L(t)}L(\sigma)v(\sigma) \\ &= L(t)e^{-(t-\sigma)L(t)}(L(\sigma)^{-1} - L(t)^{-1})L(\sigma)v(\sigma). \end{aligned}$$

Therefore, by integrating over $[s,t]$ and applying $L(t)$ to both sides, we obtain

$$L(t)v(t) = L(t)e^{-(t-s)L(t)}x + \int_s^t L(t)^2 e^{-(t-\sigma)L(t)}(L(\sigma)^{-1} - L(t)^{-1})L(\sigma)v(\sigma)d\sigma.$$

¿From the definition of classical solutions of (nCP) we have

$$v, v' = -Av \in C([s,T];X)$$

and then the previous equation can be rewritten as follows

$$(I - \mathcal{K}_s)Av = Ae^{-(\cdot - s)A}x,$$

where $(\mathcal{K}_s\psi)(t) := \int_s^t L(t)^2 e^{-(t-\sigma)L(t)}(L(\sigma)^{-1} - L(t)^{-1})\psi(\sigma)d\sigma$ for all functions $\psi \in C([s,T];X)$ and $t \in [s,T]$. The same computation as in the proof of Lemma 4 implies that

$$\mathcal{K}_s \in \mathcal{L}(C([s,T];X)) \text{ and } \|\mathcal{K}_s\|_{\mathcal{L}(C([s,T];X))} \leq \frac{1}{2}.$$

Therefore, we obtain

$$Av = (I - \mathcal{K}_s)^{-1}(Ae^{-(\cdot - s)A}x) = Au$$

and then the uniqueness of the classical solution of (nCP) follows. ■

We now show that the closure of $-L(\cdot)(\frac{d}{dt} + L(\cdot))L(\cdot)^{-1}$ on a suitable domain is the generator of the semigroup $(T(t))_{t\geq 0}$ given by (5).

From [14], Theorem 6 (see also [16], Theorem 2.4 and [17], Theorem 2.6 for more general situations) it follows that the semigroup $(S_n(t))_{t\geq 0}$ generated by $-(A_n + B)$ is an evolution semigroup. This means that there is a family $(U_n(t,s))_{T\geq t\geq s>0}$ of bounded linear operators satisfying

(i) the function $\{(t,s) \in I \times I : t \geq s\} \ni (t,s) \mapsto U_n(t,s)$ is strongly continuous,

(ii) $\|U_n(t,s)\| \leq M_n$ for a constant $M_n \geq 1$ and $t \geq s$, $(t,s) \in I \times I$,

(iii) $U_n(t,r)U_n(r,s) = U_n(t,s)$ for $T \geq t \geq r \geq s > 0$

such that

$$(S_n(t)f)(\tau) = U_n(\tau, \tau - t)f(\tau - t)\chi_I(\tau - t), \quad t \geq 0, \tau \in I \text{ and } f \in C_0(I;X).$$

Hence, $(T_n(t))_{t\geq 0}$ is also a bounded evolution semigroup and

$$(T_n(t)f)(\tau) = V_n(\tau, \tau - t)f(\tau - t)\chi_I(\tau - t), \quad t \geq 0, s \in I \text{ and } f \in C_0(I;X),$$

where $V_n(t,s) = L_n(t)U_n(t,s)L_n(s)^{-1}$, $T \geq t \geq s > 0$.

By Lemmas 4, 5 and Theorem 6 we have the following assertions:

(a) There is a constant $M \geq 1$ such that $\|V_n(t,s)\| \leq M$ for all $n \geq 1$ and $T \geq t \geq s > 0$.

(b) The semigroup $(T(t))_{t\geq 0}$ is a bounded evolution semigroup, *i.e.*,

$$(T(t)f)(\tau) = V(\tau, \tau - t)f(\tau - t)\chi_I(\tau - t), \quad t \geq 0, s \in I \text{ and } f \in C_0(I;X).$$

(c) $\lim\limits_{n\to\infty} \sup\limits_{(t,s)\in I\times I, t\geq s} \|V_n(t,s)x - V(t,s)x\| = 0$ for every $x \in X$.

(d) The evolution family $(V(t,s))_{T\geq t\geq s>0}$ is given by

$$V(t,s) = L(t)U(t,s)L(s)^{-1},\, T \geq t \geq s > 0,$$

where $U(t,s)$ is the classical solution of (nCP) given by Proposition 8.

Using the same idea as in [11], Proposition 2.9 (see also [17], Proposition 1.13) we obtain the following result.

Corollary 9. *Let $(G, D(G))$ be the generator of the semigroup $(T(t))_{t\geq 0}$ corresponding to the evolution family $(V(t,s))_{T\geq t\geq s>0}$. Set*

$$\begin{aligned} \mathcal{D} \;:=\; & lin\{f \in C_0(I;X) : f(s) = \psi(s)V(s,s_0)x,\ s \in I, \\ & \qquad \text{where } s_0 \in I, x \in X, \psi \in C_c^1(I), \psi(s) = 0 \text{ for } s \leq s_0\}. \end{aligned}$$

Then $\mathcal{D} \subseteq D(A(A+B)A^{-1}) := \{f \in C_0(I;X) : A^{-1}f \in D(B)$ & $(A+B)A^{-1}f \in D(A)\}$ and $\mathcal{D} \subseteq D(G)$. Moreover, $(G, D(G))$ is the closure of $(-A(A+B)A^{-1}, \mathcal{D})$.

Proof. Let $\psi \in C_c^1(I)$, $s_0 \in I$, and $x \in X$. Assume that $\psi(s) = 0$ for $s \leq s_0$ and set $f(s) = \psi(s)V(s,s_0)x$, $s \in I$. Then, from (b), it is easy to see that $(T(t)f)(\tau) = \psi(\tau - t)V(\tau, s_0)x$, $\tau \in I$. Hence, $f \in D(G)$ and $(Gf)(\tau) = -\psi'(\tau)V(\tau, s_0)x$, $\tau \in I$. Therefore, $T(t)\mathcal{D} \subseteq \mathcal{D} \subseteq D(G)$ for $t \geq 0$. From (d) we have $(A^{-1}f)(\tau) = \psi(\tau)U(\tau,s_0)L(s_0)^{-1}x$ and since $U(\tau, s_0)$ gives the classical solution of (nCP), we obtain $A^{-1}f \in D(B)$ and

$$(BA^{-1}f)(\tau) = \psi'(\tau)U(\tau,s_0)L(s_0)^{-1} - \psi(\tau)V(\tau,s_0)x, \;\; \tau \in I.$$

This implies that $f \in D(A(A+B)A^{-1})$ and

$$(-A(A+B)A^{-1}f)(\tau) = -\psi'(\tau)V(\tau,s_0)x = (Gf)(\tau).$$

Due to [13], A-I, Proposition 1.9, it remains to show that $\mathcal{D}$ is dense in $C_0(I;X)$. This follows from the strong continuity of the evolution family $(V(t,s))_{t\geq s}$ and by considering a partition of unity. For more details see [17], Proposition 1.13 (cf. [11], Proposition 2.9). ∎

Remark 10. If we replace $C_0(I;X)$ by $L^p(I;X)$ for $1 < p < \infty$ and assume that the Banach space X has the $UMD-$property (for definitions and properties of such spaces see [7], [8] and [6]) then we obtain more. Suppose that $L(t) \in BIP(X)$ for all $t \in I$ and that there are constants $K_A > 0$ and $\varphi_A \in (0, \frac{\pi}{2})$ such that

$$\|L(t)^{is}\| \leq K_A e^{\varphi_A |s|} \quad \text{for all } s \in \mathbb{R},$$

for all $t \geq 0$, $\lambda \in \Sigma_{\pi - \varphi_A}$. Then $A + B$ considered as an operator in $L^p(I; X)$ with $D(A+B) = D(A) \cap D(B)$ is sectorial (see [12], Theorem 1). Therefore one can see that the operator $-A(A+B)A^{-1}$ with

$$D(A(A+B)A^{-1}) = \{f \in C_0(I;X) : A^{-1}f \in D(B) \text{ and } (A+B)A^{-1}f \in D(A)\}$$

is the generator of the evolution semigroup $(T(t))_{t \geq 0}$ given by (5).

Remark 11. Let $u_0 \in \mathcal{D}$ and consider the function $u(t,a) := (T(t)u_0)(a)$, $t \geq 0$ and $a \in I$. By Corollary 9, we obtain that u is the unique solution of the following partial differential equation

$$\left\{ \begin{array}{rcll} \frac{\partial}{\partial t} u(t,a) & = & -L(a)(u(t,a) + \frac{\partial}{\partial a}(L(a)^{-1}u(t,a))) \ , & t \geq 0,\ a \in I, \\ u(0,a) & = & u_0(a) \ , & a \in I. \end{array} \right.$$

Remark 12. The solution of (nCP) satisfies $u(0) = 0$ since we work in the space $C_0(I;X)$. This is not a restriction. Indeed, we extend $L(\cdot)$ to the interval $J := (-1, T]$ by setting $L(t) := L(0)$ for $t \in (-1, 0)$. Clearly, the extension still satisfies (A1) and (A2) with the same constants and one can do the same in $C_0(J;X) := \{f : [-1,T] \to X \text{ continuous and } f(-1) = 0\}$ instead of $C_0(I;X)$.

Example 13. By using Proposition 8, we can solve in $L^1(\Omega)$ the following non-autonomous partial differential equation:

$$(*) \left\{ \begin{array}{rcll} \frac{\partial}{\partial t} u(t,x) & = & div[a(t,x)\nabla u(t,x)], & t \geq s,\ x \in \Omega, \\ \mathcal{B}u|_{\partial\Omega}(t,x) & := & n(x) \cdot (a(t,x)\nabla u(t,x)) + b(t,x)u(t,x) = 0, & t \geq s,\ x \in \partial\Omega, \\ u(s,x) & = & u_0(x), & x \in \Omega, \end{array} \right.$$

where $s \in I$, $\Omega \subset \mathbb{R}^N$ is a bounded domain of class C^2, $n(x)$ denotes the outer normal of Ω at $x \in \partial\Omega$ and u_0 is a given function in $L^1(\Omega)$.
We shall assume the following conditions:

(1) $a : [0,T] \times \overline{\Omega} \to Sym(n)$ satisfies the strong ellipticity condition, *i.e.*, there exist a constant $a_0 > 0$ such that $y \cdot a(t,x)y \geq a_0|y|^2$, for all $t \geq 0$, $x \in \Omega$, $y \in \mathbb{R}^N$.

(2) $a, a_{x_j} \in C^\delta([0,T], C(\overline{\Omega}))$ and $b, b_{x_j} \in C^\delta([0,T], C(\partial\Omega))$ for some $\delta > \frac{1}{2}$.

We denote by $L(t)$, $t \in [0,T]$ the realization of the differential operator $-div[a(t,x)\nabla\cdot]$ in $L^1(\Omega)$ under the boundary conditions $\mathcal{B}u|_{\partial\Omega} = 0$ (cf. [3], Section 9).
From a result of Amann [3] (see also [5], Theorem 3.2 and the references therein) follows that the family $L(\cdot)$ satisfies (A1) and in [18], 6.13, it is proved that (A2) holds. Therefore Proposition 8 implies that $(*)$ has a unique classical solution in $L^1(\Omega)$.

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