Maximal regularity and applications to PDEs

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Contents

1 Introduction

The purpose of this series of lectures is to give a flavor of the concept of maximal regularity. In the last ten-fifteen years, a lot of progress has been made on this subject. The problem of parabolic maximal L^p-regularity can be stated as follows.

> Let A be an (unbounded) operator on a Banach space X, with domain $D(A)$. Let $p \in]1, \infty[$. Does there exist a constant $C > 0$ such that for all $f \in L^p(0,\infty;X)$, there exists a unique $u \in L^p(0,\infty;D(A)) \cap L^p_1(0,\infty;X)$ solution of $u' + Au = f$ and $u(0) = 0$ verifying
>
> $$\|u'\|_{L^p(0,\infty;X)} + \|Au\|_{L^p(0,\infty;X)} \leq \|f\|_{L^p(0,\infty;X)}?$$

This problem, in its theoretical point of view, has been approached in different manners.

1. If $-A$ generates a semigroup $(T(t))_{t\geq 0}$, the solution u is given by the formula

   $$u(t) = \int_0^t T(t-s)f(s)\,ds, \quad t \geq 0,$$

   and therefore, the question of maximal L^p-regularity is to decide whether the operator R defined by

   $$Rf(t) = \int_0^t AT(t-s)f(s)\,ds, \quad t \geq 0,$$

   for $f \in L^p(0,\infty;X)$, is bounded in $L^p(0,\infty;X)$. In the favorable case where the semigroup is analytic, K has a convolution form with an operator-valued kernel, singular at 0. The study of boundedness of R in $L^p(0,\infty;X)$ leads then to the theory of singular integrals.

2. One way to treat this convolution (in t) operator is to apply the Fourier transform to it. The problem is now to decide whether $M(t) = A(isI + A)^{-1}$, $s \in \mathbb{R}$ is a Fourier multiplier. This has been studied by L. Weis in [40] who gave an equivalent property to maximal regularity of A in terms of bounds of the resolvent of A.

3. Another approach is to see this problem as the invertibility of the sum of two operators $A + B$ where B is the derivative in time. G. Dore and A. Venni in [16] followed this idea, using imaginary powers of A and B.

All these characterizations are not always easy to deal with when concrete examples are concerned. To verify that a precise operator has the maximal L^p-regularity property needs other results. This has been the case for operators with gaussian estimates ([21] and [12]) and more recently for operator with generalized gaussian estimates ([24]). Among a very large literature, let us mention the surveys by W. Arendt [4] and P.C. Kunstmann and L.Weis [25] where the theory of maximal regularity is largely covered.

The first part is dedicated to the study of this problem in a theoretical point of view. In a second part, we will give examples of operators having the maximal L^p-property : generators of contraction semigroups in $L^p(\Omega)$, generators of semigroups having gaussian estimates or generalized gaussian estimates. Applications to partial differential equations, such as the semilinear heat equation of the incompressible Navier-Stokes equations, are given in a third part. Finally, in a last part, we give some results on the non-autonomous maximal L^p-regularity problem.

Many results proved in the autonomous case are also true in the non-autonomous case provided we assume enough regularity (in t) on the operators $A(t)$. This condition may be removed if the operators $A(t)$ have the same domain D, and in that case, only continuity is required. This non-autonomous maximal L^p-regularity is far from being understood, but is nonetheless important for applications to quasi-linear evolution problems.

2 The theoretical point of view

2.1 Statement of the problem

Let X be a Banach space and A be a closed (unbounded) operator with domain $D(A)$ dense in X. Let $f : [0,\infty[\to X$ a measurable function. We consider the problem of existence and regularity of solution to the following equation

$$\left\{\begin{array}{rcl} u'(t)+Au(t) &=& f(t), \quad t\geq 0 \\ u(0) &=& 0. \end{array}\right. \tag{2.1}$$

Definition 2.1. Let $p\in]1,\infty[$. We say that A has the (parabolic) maximal L^p-regularity property if there exists a constant $C>0$ such that for all $f\in L^p(0,\infty;X)$, there is a unique $u\in L^p(0,\infty;D(A))$ with $u'\in L^p(0,\infty;X)$ satisfying (2.1) for almost every $t\in]0,\infty[$ and

$$\|u\|_{L^p(0,\infty;X)}+\|u'\|_{L^p(0,\infty;X)}+\|Au\|_{L^p(0,\infty;X)}\leq C\|f\|_{L^p(0,\infty;X)}. \tag{2.2}$$

2.1.1 Analytic semigroup

Not all operators have the property (2.2). In particular,

Proposition 2.2. *Let A be an operator on a Banach space X with the maximal L^p-regularity property for one $p\in]1,\infty[$. Then $-A$ generates a bounded analytic semigroup on X.*

Proof. Let $z\in\mathbb{C}$ with $\Re e(z)>0$. Define $f_z\in L^p(0,\infty;\mathbb{C})$ by

$$f_z(t)=e^{zt}\quad\text{if}\quad 0\leq t\leq\frac{1}{\Re e(z)}\qquad\text{and}\qquad f_z(t)=0\quad\text{if}\quad t>\frac{1}{\Re e(z)}.$$

Step 1. Let $x\in X$ and denote by u_z the solution of (2.1) for $f=f_z\otimes x$. Define then

$$R_zx=\Re e(z)\int_0^\infty e^{-zt}u_z(t)\,dt.$$

Then the following estimates hold

$$\begin{array}{rcl} \|R_zx\|_X &\overset{(1)}{\leq}& \Re e(z)\|u_z\|_{L^p(0,\infty;X)}\|t\mapsto e^{-zt}\|_{L^{p'}(0,\infty)} \\ &\overset{(2)}{\leq}& \Re e(z)\,C\,\|f_z\|_{L^p(0,\infty;\mathbb{C})}\|x\|_X\|t\mapsto e^{-zt}\|_{L^{p'}(0,\infty)} \\ &\overset{(3)}{\leq}& C\,\dfrac{(e^p-1)^{\frac{1}{p}}}{p'^{\frac{1}{p'}}}\,\|x\|_X. \end{array}$$

The first estimate comes from the Hölder inequality applied to the integral form of R_zx, p' denoting the conjugate exponent of p : $\frac{1}{p}+\frac{1}{p'}=1$. The inequality (2) is obtained by estimating u_z by f via the maximal L^p-regularity property of A ; remark that $\|f\|_{L^p(0,\infty;X)}=\|f_z\|_{L^p(0,\infty;\mathbb{C})}\|x\|_X$. The last estimate comes from the calculations of the different norms of the previous line. By writing R_zx as

$$R_zx=\frac{\Re e(z)}{z}\int_0^\infty e^{-zt}u_z'(t)\,dt$$

(by performing a integration by parts), the same arguments as before give the estimate

$$\|R_z x\|_X \leq \frac{1}{|z|}\, C\, \frac{(e^p-1)^{\frac{1}{p}}}{p'^{\frac{1}{p'}}}\, \|x\|_X.$$

Therefore, we get

$$\|R_z x\|_X \leq \frac{M}{1+|z|}\, \|x\|_X \tag{2.3}$$

with $M = C\, \frac{(e^p-1)^{\frac{1}{p}}}{p'^{\frac{1}{p'}}}$.

Step 2. Let now $x \in D(A)$. We have

$$\begin{aligned} R_z(zI+A)x &\stackrel{(1)}{=} zR_z x + R_z A x \\ &\stackrel{(2)}{=} \Re e(z) \int_0^\infty e^{-zt} u_z'(t)\, dt + \Re e(z) \int_0^\infty e^{-zt} A u_z(t)\, dt \\ &\stackrel{(3)}{=} \Re e(z) \int_0^\infty e^{-zt} f_z(t) x\, dt \stackrel{(4)}{=} x. \end{aligned}$$

The first equality is straightforward. The first term of the second equality comes from the integration by parts as in Step 1, whereas the second term comes from the fact that $Au_z \in L^p(0,\infty;X)$ by the maximal L^p-regularity property of A. The equality (3) comes from (2.1) and equality (4) is obtained by a simple calculation, reminding that $f = f_z \otimes x$.

Step 3. The equality $R_z(zI+A)x = x$ for all $x \in D(A)$ together with (2.3) ensure that R_z is the resolvent of $-A$ in z. Therefore, the spectrum of A $\sigma(A) \subset \mathbb{C}_+ = \{z \in \mathbb{C}; \Re e(z) \geq 0\}$ and there exists $M > 0$ such that for all $z \in \mathbb{C}$ with $\Re e(z) > 0$, we have (2.3). This implies that $-A$ generates a bounded analytic semigroup in X. □

Let us consider for a moment a slightly different problem. We might ask what happens if the initial condition in (2.1) is not equal to zero.

Remark 2.3. Once we know that $-A$ generates a semigroup $(T(t))_{t\geq 0}$ on the Banach space X, we can study the following initial value Cauchy problem

$$\left\{ \begin{array}{rcll} u'(t) + Au(t) &=& 0, & t \geq 0 \\ u(0) &=& u_0. & \end{array} \right. \tag{2.4}$$

It is known that the solution u is given by $u(t) = T(t)u_0$, $t \geq 0$. This solution u belongs to $L^p(0,\infty;X)$ if and only if ([30], Chapter 1) the initial value u_0 belongs to the real interpolation space $(X, D(A))_{\frac{1}{p'},p}$ (where p' is the conjugate of p : $\frac{1}{p} + \frac{1}{p'} = 1$).

2.1.2 Independence with respect to p

Proposition 2.4. *Let A be an operator on a Banach space X with the maximal L^p-regularity property for one $p \in]1,\infty[$. Then A has the maximal L^q-regularity property for all $q \in]1,\infty[$.*

To prove this fact, we need the following auxiliary theorem due to A. Benedek, A.P. Calderón and R. Panzone

Theorem 2.5 (Theorem 2 of [7]). *Let X be a Banach space and let $p \in]1, \infty[$. Let $k : \mathbb{R} \to \mathscr{L}(X)$ be measurable, $k \in L^1_{loc}(\mathbb{R} \setminus \{0\}; \mathscr{L}(X))$. Let $S \in \mathscr{L}(L^p(\mathbb{R}; X))$ be the convolution operator with k, i.e. for all $f \in L^\infty(\mathbb{R}; X)$ with compact support, one can write Sf as follows*

$$Sf(t) = \int_{\mathbb{R}} k(t-s) f(s)\, ds, \quad \forall\, t \notin \operatorname{supp} f. \tag{2.5}$$

Assume that there exists a constant $c > 0$ such that

$$\int_{|t|>2|s|} \|k(t-s) - k(t)\|_{\mathscr{L}(X)} \leq c, \quad \forall\, s \in \mathbb{R}. \tag{2.6}$$

Then $S \in \mathscr{L}(L^q(\mathbb{R}, X))$ for all $q \in]1, \infty[$.

Proof. We prove that S is bounded from $L^1(\mathbb{R}; X)$ to $L^1_w(\mathbb{R}; X)$ where L^1_w stands for L^1-weak and is defined as follows

$$L^1_w(\mathbb{R}, X) = \left\{ f : \mathbb{R} \to X \text{ measurable }; \sup_{\alpha > 0} \alpha \cdot \Big| \{t \in \mathbb{R}; \|f(t)\|_X > \alpha\} \Big| < \infty \right\}.$$

Let $f \in L^1(\mathbb{R}; X)$ and fix $\lambda > 0$. By the Calderón-Zygmund decomposition applied to $\mathbb{R} \ni t \mapsto \|f(t)\|_X$ (see Theorem 6.7 below, in the case $n = 1$), we may decompose f into a "good" part g and a "bad" part $b = \sum_k b_k$. We have then $Sf = Sg + \sum_k Sb_k$ and therefore

$$\Big\{ t \in \mathbb{R}; \|Sf(t)\|_X > \lambda \Big\} \subset \Big\{ t \in \mathbb{R}; \|Sg(t)\|_X > \frac{\lambda}{2} \Big\} \cup \Big\{ t \in \mathbb{R}; \|Sb(t)\|_X > \frac{\lambda}{2} \Big\}.$$

The measure of the first set is easy to estimate. Since $g \in L^1 \cap L^\infty$, we have $g \in L^p$ and since S is bounded in L^p, we have

$$\begin{aligned}
\left| \Big\{ t \in \mathbb{R}; \|Sg(t)\|_X > \frac{\lambda}{2} \Big\} \right| &\overset{(1)}{\leq} \frac{\|Sg\|^p_{L^p(\mathbb{R};X)}}{(\frac{\lambda}{2})^p} \\
&\overset{(2)}{\leq} \frac{2^p \|S\|^p_{\mathscr{L}(L^p(\mathbb{R};X))}}{\lambda^p} \|g\|_p^p \\
&\overset{(3)}{\leq} \frac{2^p \|S\|^p_{\mathscr{L}(L^p(\mathbb{R};X))}}{\lambda^p} \Big(\|g\|_1^{\frac{1}{p}} \|g\|_\infty^{1-\frac{1}{p}} \Big)^p \\
&\overset{(4)}{\leq} \frac{2^p \|S\|^p_{\mathscr{L}(L^p(\mathbb{R};X))}}{\lambda^p} \|f\|_1 (2\lambda)^{p-1}
\end{aligned}$$

and therefore

$$\lambda \left| \Big\{ t \in \mathbb{R}; \|Sg(t)\|_X > \frac{\lambda}{2} \Big\} \right| \leq 4^p \|S\|^p_{\mathscr{L}(L^p(\mathbb{R};X))} \|f\|_1. \tag{2.7}$$

Inequality (1) is obvious. Inequality (2) comes from the fact that S is bounded in L^p by hypothesis. Inequality (3) comes from Hölder inequality $\|\cdot\|_p \leq \|\cdot\|_1^{\frac{1}{p}} \|\cdot\|_\infty^{1-\frac{1}{p}}$. Inequality (4) comes from the fact that $\|g\|_1 \leq \|f\|_1$ and $\|g\|_\infty \leq 2\lambda$ by construction in the Calderón-Zygmund decomposition. It remains now to estimate the quantity

$$\left| \Big\{ t \in \mathbb{R}; \|Sb(t)\|_X > \frac{\lambda}{2} \Big\} \right|.$$

We decompose the set as follows

$$\Big\{t \in \mathbb{R}; \|Sb(t)\|_X > \frac{\lambda}{2}\Big\} \subset E \cup \Big\{t \in \mathbb{R} \setminus E; \|Sb(t)\|_X > \frac{\lambda}{2}\Big\}$$

where $E = \bigcup_{k\in\mathbb{N}} \tilde{Q}_k$ ($\tilde{Q}_k$ beeing the double of Q_k. We already have, by the Calderón-Zygmund decomposition, $|E| \leq 2\lambda^{-1}\|f\|_1$. The last term remaining to estimate is the measure of the set

$$\Big\{t \in \mathbb{R} \setminus E; \|Sb(t)\|_X > \frac{\lambda}{2}\Big\},$$

and that is where the assumption (2.6) comes in. We denote by s_k the center of Q_k, $k \in \mathbb{N}$. We have

$$\begin{aligned}
\int_{\mathbb{R}\setminus E} \|Sb(t)\|_X dt &\overset{(1)}{\leq} \sum_{k\in\mathbb{N}} \int_{\mathbb{R}\setminus E} \Big\| \int_{Q_k} k(t-s) b_k(s)\, ds \Big\|_X dt \\
&\overset{(2)}{\leq} \sum_{k\in\mathbb{N}} \int_{\mathbb{R}\setminus E} \Big\| \int_{Q_k} (k(t-s) - k(t-s_k)) b_k(s)\, ds \Big\|_X dt \\
&\overset{(3)}{\leq} \sum_{k\in\mathbb{N}} \int_{\mathbb{R}\setminus \tilde{Q}_k} \int_{Q_k} \|k(t-s) - k(t-s_k)\|_{\mathscr{L}(X)} \|b_k(s)\|_X ds\, dt \\
&\overset{(4)}{\leq} \sum_{k\in\mathbb{N}} c \int_{Q_k} \|b_k(s)\|_X ds \overset{(5)}{\leq} 2c\,\|f\|_1
\end{aligned}$$

The first inequality comes from (2.5) since $t \in \mathbb{R} \setminus E$ and therefore, for $t \notin \operatorname{supp} b_k = Q_k$. The second inequality is in fact an equality since $\int_{Q_k} b_k = 0$ by construction of the b_k's. The third inequality is obvious and inequality (4) comes from the fact that, for $t \in \mathbb{R} \setminus \tilde{Q}_k$ and $s \in Q_k$, we have $|(t - s_k)| > 2|(t - s) - (t - s_k)|$. We can then apply (2.6). The last inequality is obvious taking the Calderón-Zygmund into account. Therefore, we have

$$\begin{aligned}
\Big|\Big\{t \in \mathbb{R}; \|Sb(t)\|_X > \frac{\lambda}{2}\Big\}\Big| &\leq |E| + \Big|\Big\{t \in \mathbb{R} \setminus E; \|Sb(t)\|_X > \frac{\lambda}{2}\Big\}\Big| \\
&\leq 2\lambda^{-1}\|f\|_{L^1(\mathbb{R};X)} + \frac{1}{\frac{\lambda}{2}} \int_{\mathbb{R}\setminus E} \|Sb(t)\|_X dt \\
&\leq \frac{2(1+2c)}{\lambda} \|f\|_{L^1(\mathbb{R};X)}.
\end{aligned}$$

Together with (2.7), this gives

$$\lambda \Big|\Big\{t \in \mathbb{R}; \|Sf(t)\|_X > \lambda\Big\}\Big| \leq C\|f\|_{L^1(\mathbb{R};X)} \tag{2.8}$$

where

$$C = 4^p \|S\|^p_{\mathscr{L}(L^p(\mathbb{R};X))} + 2(1+2c),$$

which means that S is of weak type $(1,1)$. By Marcinkiewicz interpolation theorem (see Theorem 6.5 below), the operator S is of strong type (q,q) for all $q \in]1,p[$. Moreover, it is easy to see that S', the adjoint operator of S is of the same form as S : S' is bounded in $L^{p'}(\mathbb{R};X)$ (where $\frac{1}{p} + \frac{1}{p'} = 1$) and of weak type $(1,1)$ by the same arguments as before, which implies by the Marcinkiewicz interpolation theorem that S' is of strong type (q,q) for all $q \in]1,p'[$, and therefore, by duality, S is of strong type (q,q) for all $q \in]p,\infty[$. We have then proved that S is a bounded operator in $L^q(\mathbb{R};X)$ for all $q \in]1,\infty[$. $\square$

Proof of Proposition 2.4. Let A be an unbounded operator on a Banach space X such that $-A$ generates an analytic semigroup $(T(t))_{t\geq 0}$. Define $k : \mathbb{R} \to \mathscr{L}(X)$ by

$$k(t) = AT(t) \text{ if } t > 0 \quad \text{ and } \quad k(t) = 0 \text{ if } t \leq 0.$$

Then k is measurable, $k \in L^1_{loc}(\mathbb{R} \setminus \{0\}; \mathscr{L}(X))$. Pick any $s \in \mathbb{R}$; we have

$$\begin{aligned}
\int_{|t|>2|s|} \|k(t-s) - k(t)\|_{\mathscr{L}(X)} dt &\overset{(1)}{=} \int_{t>2|s|} \| \int_t^{t-s} A^2 T(\tau)\, d\tau \|_{\mathscr{L}(X)} dt \\
&\overset{(2)}{\leq} C \int_{t>2|s|} \Big| \int_t^{t-s} \frac{1}{\tau^2}\, d\tau \Big|\, dt \\
&\overset{(3)}{=} C \int_{t>2|s|} \Big| \frac{1}{t-s} - \frac{1}{t} \Big|\, dt \\
&\overset{(4)}{\leq} C \ln 2.
\end{aligned}$$

The first equality comes from the fact that k vanishes on $]-\infty, 0[$ and that an analytic semigroup is differentiable on $]0, +\infty[$, its derivative at a point $t > 0$ beeing equal to $AT(t)$. The second inequality is due to the following property of analytic semigroups : for all $n \in \mathbb{N}$, $\sup_{t>0} \|t^n A^n T(t)\|_{\mathscr{L}(X)} < \infty$. As for equality (3), it is obtained by integrating $\int_t^{t-s} \frac{1}{\tau^2}\, d\tau$, the last inequality comes from the exact integration of the integral

$$\int_{t>2|s|} \Big| \frac{1}{t-s} - \frac{1}{t} \Big|\, dt,$$

which gives $\ln 2$ if $s > 0$, 0 if $s = 0$ and $\ln \frac{3}{2}$ if $s < 0$. Therefore, we can apply Theorem 2.5 and conclude then that the property of maximal L^p-regularity is independent of $p \in]1, \infty[$. □

2.2 Necessary and sufficient conditions

2.2.1 Maximal regularity in Hilbert spaces

In the special case where the Banach space X is actually a Hilbert space, the reverse statement of Proposition 2.2 is true.

Theorem 2.6 (de Simon, 1964)**.** *Let $-A$ be the generator of a bounded analytic semigroup in a Hilbert space H. Then A has the maximal L^p-regularity property for all $p \in]1, \infty[$.*

Proof. This theorem is due to de Simon [15]. Denote by $(T(t))_{t\geq 0}$ the semigroup generated by $-A$ and let $f \in L^2(0, \infty; D(A))$. Then it is easy to see that u given by

$$u(t) = \int_0^t T(t-s) f(s)\, ds, \quad t \geq 0 \tag{2.9}$$

is the solution of (2.1), and Au has the form

$$Au(t) = \int_{\mathbb{R}} k(t-s) f(s)\, ds, \quad t \geq 0,$$

where we have extended f by 0 on $]-\infty, 0[$ and $k(t) = AT(t)$ if $t > 0$, $k(t) = 0$ if $t \leq 0$. Applying the Fourier transform (in t) $\mathscr{F}$, we obtain for all $x \in \mathbb{R}$

$$\begin{aligned}
\mathscr{F}(Au)(x) &= \int_{\mathbb{R}} e^{-itx} Au(t)\, dt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-itx} k(t-s) f(s)\, ds\, dt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(t+s)x} k(t) f(s)\, ds\, dt \\
&= \int_0^{\infty} e^{-itx} T(t) \Big(\int_{\mathbb{R}} e^{-isx} Af(s)\, ds \Big)\, dt \\
&= (ix + A)^{-1} A \mathscr{F}(f)(x) \;=\; A(ix+A)^{-1} \mathscr{F}(f)(x).
\end{aligned}$$

Since $-A$ generates a bounded analytic semigroup, we have

$$\sup_{x \in \mathbb{R}} \|A(ix + A)^{-1}\|_{\mathscr{L}(H)} < \infty,$$

and therefore

$$\|\mathscr{F}(Au)\|_{L^2(\mathbb{R};H)} \leq c\, \|\mathscr{F}(f)\|_{L^2(\mathbb{R};H)}.$$

Since $\mathscr{F}$ is an isomorphism on $L^2(\mathbb{R}; H)$, this implies that

$$\|Au\|_{L^2(\mathbb{R};H)} \leq c\, \|f\|_{L^2(\mathbb{R};H)}.$$

This proves that A has the maximal L^2-regularity property. By Proposition 2.4, the proof is complete. □

2.2.2 $UMD-$spaces

The question now arises, whether all negative generator of bounded analytic semigroup in any Banach space X has the property of maximal L^p-regularity. This question, posed by Haïm Brézis, was first partially answered by T. Coulhon and D. Lamberton in [13]. To describe their result, we need to define the notion of $UMD-$space. Actually, we give here a property of $UMD-$spaces equivalent to the original definition. For more on this subject, see [10] and [9]. The Hilbert transform $\mathcal{H}f$ of a measurable function f is, whenever it exists, the limit as $\varepsilon \to 0^+$ and $T \to +\infty$ of

$$\mathcal{H}_{\varepsilon,T} f(t) = \frac{1}{\pi} \int_{\varepsilon \leq |s| \leq T} \frac{f(t-s)}{s}\, ds, \quad t \in \mathbb{R}.$$

Definition 2.7. A Banach space X is said to be of class UMD if the Hilbert transform $\mathcal{H}$ is bounded in $L^p(\mathbb{R}; X)$ for all (or equivalently for one) $p \in]1, \infty[$.

Example 2.8.
1. A Hilbert space is in the class UMD.
2. If X is a Banach space in the $UMD-$class, then $L^p(\Omega; X)$, for $\Omega \subset \mathbb{R}^d$ and $p \in]1, \infty[$, is also in the $UMD-$class.

Theorem 2.9 (Coulhon-Lamberton, 1986)**.** *If the negative generator of the Poisson semigroup on $L^2(\mathbb{R}; X)$ has the maximal L^p-property, then the Hilbert transform is bounded in $L^2(\mathbb{R}; X)$.*

This theorem implies that if X is a Banach space with the property that every negative generator of a bounded analytic semigroup has the maximal L^p-property, then necessarily X is of class UMD. The converse was an open problem until the work of N. Kalton and G. Lancien [22] where it was proved that such a Banach space is “essentially” a Hilbert space.

Theorem 2.10 (Kalton-Lancien, 2000). *On every Banach lattice which is not isomorphic to a Hilbert space, there are generators of analytic semigroups without the maximal L^p-regularity property.*

Proof of Theorem 2.9. The Poisson semigroup $(P(t))_{t\geq 0}$ on $L^2(\mathbb{R};X)$ is defined as follows

$$(P(t)f)(x) = \int_{\mathbb{R}} \frac{t}{\pi(y^2+t^2)}\, f(x-y)\, dy, \quad x \in \mathbb{R}, t>0, \quad f \in L^2(\mathbb{R};X).$$

This semigroup is bounded analytic and its generator $-A$ satisfies

$$(AP(t)f)(x) = \int_{\mathbb{R}} \frac{t^2-y^2}{\pi(y^2+t^2)^2}\, f(x-y)\, dy, \quad x \in \mathbb{R}, t>0, \quad f \in L^2(\mathbb{R};X).$$

This relation is obtained by taking the derivative in t of $P(t)f$. As we have already seen, the assumption that A has the maximal L^p-regularity property in $L^2(\mathbb{R};X)$ implies that the operator

$$f \mapsto \Big( ]0,\infty[\times\mathbb{R} \ni (t,x) \mapsto \int_0^t \Big(\int_{\mathbb{R}} \frac{s^2-y^2}{\pi(y^2+s^2)^2}\, f(t-s,x-y)\, dy \Big)\, ds \Big)$$

is bounded in $L^2(0,\infty;L^2(\mathbb{R};X)) = L^2(]0,\infty[\times\mathbb{R};X)$. By the change of variables $y-s=u$, $y+s=v$, $x-t=u'$ and $x+t=v'$, and the change of function $F(u,v) = f(\frac{v-u}{2},\frac{v+u}{2})$ the operator K defined by

$$(KF)(u',v') = \int_{\mathbb{R}} \Big(\int_u^{+\infty} \frac{-uv}{(v^2+u^2)^2}\, F(u'-u,v'-v)\, dv \Big)\, du, \quad (u',v') \in E,$$

is bounded in $L^2(E;X)$, where $E = \{(u,v) \in \mathbb{R}^2; v>u\}$. It means then that there exists a constant $C>0$ such that

$$\int_{\mathbb{R}} \Big(\int_{u'}^{+\infty} \|KF(u',v')\|_X^2 dv' \Big)\, du' \leq C \int_{\mathbb{R}} \Big(\int_u^{+\infty} \|F(u,v)\|_X^2 dv \Big)\, du, \tag{2.10}$$

for all $F \in L^2(E;X)$. Let now $a>0$ and $\phi \in L^2(\mathbb{R};X)$ and take

$$F(u,v) = \phi(u)\chi_{0<v-u<1}.$$

Then we have

$$\int_{\mathbb{R}} \Big(\int_u^{+\infty} \|F(u,v)\|_X^2 dv \Big)\, du = \|\phi\|_{L^2(\mathbb{R};X)}. \tag{2.11}$$

Computing KF, we get

$$KF(u',v') = \int_{\mathbb{R}} \Big(\int_{\max\{u,u+(v'-u')-1\}}^{v'-u'+u} \frac{-uv}{(v^2+u^2)^2}\, dv \Big) \phi(u'-u)\, du$$

and therefore, if $0 < v' - u' < 1$ we have

$$\begin{aligned} KF(u',v') &= \int_{\mathbb{R}} \Big(\frac{u}{2[(v'-u'+u)^2+u^2]^2} - \frac{1}{4u} \Big) \phi(u'-u)\, du \\ &= \left(\int_{\mathbb{R}} \frac{u}{2[(v'-u'+u)^2+u^2]^2}\, \phi(u'-u)\, du \right) - \frac{\pi}{4}\, \mathcal{H}(\phi)(u'), \end{aligned}$$

or equivalently if $0 < v' - u' < 1$

$$\mathcal{H}(\phi)(u') = -\frac{4}{\pi}\, KF(u',v') + \left(\int_{\mathbb{R}} \frac{2u}{\pi[(v'-u'+u)^2+u^2]^2}\, \phi(u'-u)\, du \right).$$

We take the norm in X, square it and then integrate in v' between $u' + \frac{1}{2}$ and $u'+1$ and $u' \in \mathbb{R}$ we obtain

$$\frac{1}{2} \int_{\mathbb{R}} \|\mathcal{H}(\phi)(u')\|_X^2 du' \leq 2 \left(\frac{16}{\pi^2} \|KF\|^2_{L^2(E;X)} + c^2 \|\phi\|^2_{L^2(\mathbb{R};X)} \right) \tag{2.12}$$

since

$$\int_{\mathbb{R}} \frac{2u}{\pi[(v'-u'+u)^2+u^2]^2}\, \phi(u'-u)\, du = (a_{v'-u'} * \phi)(u')$$

where

$$a_w(u) = \frac{2u}{\pi[(w+u)^2+u^2]^2}, \quad u \in \mathbb{R} \ :$$

$a_w \in L^1(\mathbb{R})$ for all $w \in [\frac{1}{2}, 1]$. Putting (2.10), (2.11) and (2.12) together, we conclude that the Hilbert transform is bounded in $L^2(\mathbb{R};X)$ and therefore X is of the $UMD-$class. $\square$

2.2.3 $R-$boundedness

All the details about the following results can be found in [40]. For X and Y Banach spaces, we denote by $\mathscr{L}(X,Y)$ the space of bounded linear operators from X to Y, $\mathscr{S}(\mathbb{R},X)$ denoting the rapidly decreasing functions from $\mathbb{R}$ to X. As before, $\mathscr{F}$ denotes the Fourier transform (in t).

Definition 2.11. A function $M : \mathbb{R} \setminus \{0\} \to \mathscr{L}(X,Y)$ is said to be a Fourier multiplier on $L^p(\mathbb{R};X)$ if the expression $Rf = \mathscr{F}^{-1}\Big(M\, \mathscr{F}(f) \Big)$ is well-defined for $f \in \mathscr{S}(\mathbb{R};X)$ and R extends to a bounded operator $R : L^p(\mathbb{R};X) \to L^p(\mathbb{R};Y)$.

It has been observed by G. Pisier that the converse of Theorem 6.6 is true : if $X = Y$ and all M satisfying for some constant $C > 0$,

$$\|M(t)\|_{\mathscr{L}(X,Y)} \leq C \quad \text{and} \quad \|tM'(t)\|_{\mathscr{L}(X,Y)} \leq C \quad \text{for all } t \in \mathbb{R} \setminus \{0\}$$

are Fourier multipliers in $L^2(\mathbb{R};X)$, then X is isomorphic to a Hilbert space. Therefore, to decide whether a particular M is a Fourier multiplier, some additional assumptions are needed. This was done by L. Weis in [40] in 2001. His result gives also an equivalent property to maximal regularity in terms of $R-$boundedness of the resolvent of the operator.

Definition 2.12. A set $\tau \subset \mathscr{L}(X,Y)$ is called $R-$bounded if there is a constant $C > 0$ such that for all $n \in \mathbb{N}$, $T_1, ..., T_n \in \tau$ and $x_1, ..., x_n \in X$,

$$\int_0^1 \Big\| \sum_{j=1}^n r_j(s) T_j x_j \Big\|_Y ds \leq C \int_0^1 \Big\| \sum_{j=1}^n r_j(s) x_j \Big\|_X ds, \tag{2.13}$$

where $(r_j)_{j=1,\dots,n}$ is a sequence of independent $\{-1,1\}-$valued random variables on $[0,1]$; for example, the Rademacher functions $r_j(t) = \mathrm{sign}(\sin(2^j\pi t))$. The $R-$bound of τ is

$$R(\tau) = \inf\{C > 0; (2.13) \text{ holds}\}.$$

Remark 2.13. If X and Y are Hilbert spaces, a set $\tau \subset \mathscr{L}(X,Y)$ is $R-$bounded if and only if it is bounded.

We have already seen that the maximal L^p-regularity property of an operator A on a Banach space X is equivalent to the boundedness in $L^p(0,\infty;X)$ of the operator R given by

$$Rf(t) = \int_0^t AT(t-s)f(s)\,ds, \quad t \in [0,\infty[, \quad f \in L^p(0,\infty;X). \tag{2.14}$$

When taking (formally) the Fourier transform of Rf, we get

$$\mathscr{F}(Rf)(\sigma) = A(i\sigma I + A)^{-1}\mathscr{F}(f)(\sigma), \quad \sigma \in \mathbb{R}.$$

Therefore, if M denotes the operator valued function $M(\sigma) = A(i\sigma I + A)^{-1}$, our problem is now to find conditions on M (and therefore on A and its resolvent) assuring that M is a Fourier multiplier in $L^p(\mathbb{R};X)$.

Theorem 2.14 (L. Weis, 2001)**.** *Let X and Y be $UMD-$Banach spaces. Let*

$$M : \mathbb{R}\setminus\{0\} \to \mathscr{L}(X,Y)$$

be a differentiable function such that the sets

$$\Big\{M(t), t \in \mathbb{R}\setminus\{0\}\Big\} \quad \text{and} \quad \Big\{tM'(t), t \in \mathbb{R}\setminus\{0\}\Big\}$$

are $R-$bounded. Then M is a Fourier multiplier on $L^p(\mathbb{R};X)$ for all $p \in]1,\infty[$.

References for the proof. This theorem can be found in [40], Theorem 3.4. □

Applying this theorem to our problem of maximal L^p-regularity, we get the following result.

Corollary 2.15 (L. Weis, 2001)**.** *Let X be a $UMD-$Banach space and A be the negative generator of an analytic semigroup on X. Then A has the maximal L^p-regularity property if and only if the set $\Big\{i\sigma(i\sigma I + A)^{-1},\ \sigma \in \mathbb{R}\Big\}$ is $R-$bounded.*

Idea of the proof. This can be found in [40], Corollary 4.4. Denoting as before

$$M(\sigma) = A(i\sigma I + A)^{-1}, \quad \sigma \in \mathbb{R},$$

we have $M(\sigma) = i\sigma(i\sigma I + A)^{-1} - I$. Therefore, if

$$M_0 : \sigma \mapsto i\sigma(i\sigma I + A)^{-1}$$

is a Fourier multiplier, so is M. The first part of Theorem 2.14 applied to M_0 holds by the assumption that

$$\Big\{i\sigma(i\sigma I + A)^{-1},\ \sigma \in \mathbb{R}\Big\}$$

is $R-$bounded. For the second part, we must show that $\{\sigma M_0'(\sigma), \sigma \in \mathbb{R}\}$ is also $R-$bounded. For that purpose, remark that $\sigma M_0'(\sigma) = M_0(\sigma)(I - M_0(\sigma))$. □

3 Examples

In this section, we give three classes of operators in L^q-spaces having the maximal L^p-regularity property. We consider analytic semigroups in $L^2(\Omega, \mu)$ which can be extended to $L^p(\Omega, \mu)$ ((Ω, μ) being a measure space) for p in a subinterval of $]1, \infty[$ containing 2.

3.1 Contraction semigroups

3.1.1 The abstract result

The result presented here is due to D. Lamberton in [28].

Theorem 3.1 (D. Lamberton, 1987)**.** *Let (Ω, μ) be a measure space and A the negative generator of an analytic semigroup of contractions $(T(t))_{t\geq 0}$ in $L^2(\Omega, \mu)$. Assume that for all $q \in [1, \infty]$, the estimate*

$$\|T(t)f\|_q \leq \|f\|_q \text{ holds for all } t \geq 0 \text{ and all } f \in L^2(\Omega) \cap L^q(\Omega).$$

Then the operator A has the maximal L^p-regularity property in $L^q(\Omega)$.

Reference for the proof. The proof of this theorem can be found in [28]. The idea of it is to remark first that A has the maximal L^p-regularity property in $L^2(\Omega)$ by Theorem 2.6. The strategy is to show that the convolution operator R defined by

$$Rf(t) = \int_0^t AT(t-s)f(s)\,ds, \quad t \geq 0,\ f \in L^p(0, \infty; L^p(\Omega)) = L^p(]0, \infty[\times\Omega)$$

is bounded in $L^p(]0, \infty[\times\Omega)$. We already know that this is true if $p = 2$. To get the other $p \in]1, \infty[$, D. Lamberton uses Coifman-Weiss transference principle (this is the core of the proof). Once this is proved, by Theorem 2.4, we conclude that A has the maximal L^p-regularity property in $L^q(\Omega)$. □

3.1.2 An application

This theorem can be applied to show that certain operators have the maximal L^p-regularity property, such as the Laplacian in $L^p(\Omega)$ ($\Omega \subset \mathbb{R}^n$ sufficiently regular) with Dirichlet, Neuman or Robin (Fourier) boundary conditions.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain such that the Stokes formula (integration by parts) applies. We denote by ν the outer normal unit at $\partial\Omega$. Let A_j ($j = D, N$ or R) be the unbounded operator defined in $L^2(\Omega)$ by*

$$\begin{aligned} D(A_j) &= \{u \in H^1(\Omega); \Delta u \in L^2(\Omega) \text{ and } b_j(u) = 0 \text{ on } \partial\Omega\} \\ A_j u &= -\Delta u, \end{aligned}$$

where $b_D(u) = u$, $b_N(u) = \partial_\nu u$ and $b_R(u) = \alpha\, u + \partial_\nu u$ for $\alpha \geq 0$.

Proof. Thanks to Theorem 3.1, we only need to show that $-A_j$ generates an analytic semigroup $(T_j(t))_{t\geq 0}$ in $L^2(\Omega)$ and that this semigroup satisfies the estimate

$$\|T_j(t)f\|_q \leq \|f\|_q, \quad t \geq 0,\ f \in L^2(\Omega) \cap L^q(\Omega). \tag{3.1}$$

Case $j = D$. This case corresponds to Dirichlet boundary conditions. The first assumption to verify is that $-A_D$ generates an analytic semigroup $(T_D(t))_{t\geq 0}$ in $L^2(\Omega)$. Let a_D be the sesquilinear form defined by

$$a_D(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx, \quad u,v \in H_0^1(\Omega;\mathbb{C}).$$

This form a_D is sesquilinear, continuous and coercive. It is easy to show that A_D is associated to the form a_D and therefore generates a bounded analytic semigroup. It remains to show that (3.1) holds for $j = D$. Let $q \in [1,\infty[$ and let

$$u(t) = T_D(t)f,\ t \geq 0,$$

be the solution of the Cauchy problem

$$\frac{\partial u}{\partial t} + A_D u(t) = 0,\ u(0) = f \in L^2(\Omega) \cap L^q(\Omega).$$

Multiplying this equation with $v = |u|^{q-2}u\chi_{u\neq 0}$ and integrating on Ω, we obtain for all $t \geq 0$,

$$\begin{aligned}
0 &= \int_\Omega \frac{\partial u}{\partial t}(t)v(t)\, dx - \int_\Omega v(t)\Delta u(t)\, dx \\
&= \frac{1}{q}\frac{d}{dt}\Big(\int_\Omega |u(t)|^q dx\Big) + \int_\Omega |u(t)|^{q-2}|\nabla u(t)|^2\chi_{u\neq 0}\, dx \\
&\quad + \int_\Omega u\nabla(|u(t)|^{q-2})\cdot\nabla u\chi_{u\neq 0}\, dx \\
&= \frac{d}{dt}\Big(\|u\|_q^q\Big)(t) + (q-1)\int_\Omega |u(t)|^{q-2}|\nabla u(t)|^2\chi_{u\neq 0}\, dx
\end{aligned}$$

and therefore

$$\frac{d}{dt}\Big(\|u\|_q^q\Big)(t) \leq 0$$

which implies that $\|u(t)\|_q \leq \|f\|_q$. We can let $q \to \infty$ to obtain $\|u(t)\|_\infty \leq \|f\|_\infty$. This shows (3.1) for $j = D$.

Case $j = N$. This case corresponds to Neumann boundary conditions. It goes more or less as the previous case. The integrations by parts can be performed and give 0 for the boundary terms since $\partial_\nu u = 0$ at the boundary. As before, the first assumption to verify is that $-A_N$ generates an analytic semigroup $(T_N(t))_{t\geq 0}$ in $L^2(\Omega)$. Let a_N be the sesquilinear form defined by

$$a_N(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\, dx, \quad u,v \in H^1(\Omega;\mathbb{C}).$$

This form a_N is sesquilinear, continuous and coercive. It is easy to show that A_N is associated to the form a_N and therefore generates a bounded analytic semigroup. It remains to show that (3.1) holds for $j = N$. As in the previous case for

$$u(t) = T_N(t)f,\ t \geq 0$$

the solution of the Cauchy problem

$$\frac{\partial u}{\partial t} + A_N u(t) = 0,\ u(0) = f \in L^2(\Omega) \cap L^q(\Omega),$$

we have

$$
\begin{aligned}
0 &= \int_\Omega \frac{\partial u}{\partial t}(t)v(t)\,dx - \int_\Omega v(t)\Delta u(t)\,dx \\
&= \frac{1}{q}\frac{d}{dt}\Big(\int_\Omega |u(t)|^q dx\Big) + \int_\Omega |u(t)|^{q-2}|\nabla u(t)|^2\chi_{u\neq 0}\,dx \\
&\quad + \int_\Omega u\nabla(|u(t)|^{q-2})\cdot\nabla u\chi_{u\neq 0}\,dx \\
&= \frac{d}{dt}\Big(\|u\|_q^q\Big)(t) + (q-1)\int_\Omega |u(t)|^{q-2}|\nabla u(t)|^2\chi_{u\neq 0}\,dx
\end{aligned}
$$

and therefore

$$\frac{d}{dt}\Big(\|u\|_q^q\Big)(t) \le 0$$

which implies that $\|u(t)\|_q \le \|f\|_q$. We can let $q \to \infty$ to obtain $\|u(t)\|_\infty \le \|f\|_\infty$. This shows (3.1) for $j = N$.

Case $j = R$. This case corresponds to Robin (also called Fourier) boundary conditions. The first assumption to verify is that $-A_R$ generates an analytic semigroup $(T_R(t))_{t\geq 0}$ in $L^2(\Omega)$. Let a_R be the sesquilinear form defined by

$$a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\,dx + \int_{\partial\Omega} \alpha\, u\overline{v}\,d\sigma, \quad u,v \in H^1(\Omega;\mathbb{C}).$$

This form a_R is sesquilinear, continuous and coercive. It is easy to show that A_R is associated to the form a_R and therefore generates a bounded analytic semigroup. It remains to show that (3.1) holds for $j = R$. As in the two previous cases for

$$u(t) = T_R(t)f,\ t \ge 0$$

the solution of the Cauchy problem

$$\frac{\partial u}{\partial t} + A_R u(t) = 0,\ u(0) = f \in L^2(\Omega)\cap L^q(\Omega),$$

we have

$$
\begin{aligned}
0 &= \int_\Omega \frac{\partial u}{\partial t}(t)v(t)\,dx - \int_\Omega v(t)\Delta u(t)\,dx \\
&= \frac{1}{q}\frac{d}{dt}\Big(\int_\Omega |u(t)|^q dx\Big) + \int_\Omega |u(t)|^{q-2}|\nabla u(t)|^2\chi_{u\neq 0}\,dx \\
&\quad + \int_\Omega u\nabla(|u(t)|^{q-2})\cdot\nabla u\chi_{u\neq 0}\,dx - \int_{\partial\Omega}\partial_\nu u(t)|u(t)|^{q-2}u(t)\,d\sigma \\
&= \frac{d}{dt}\Big(\|u\|_q^q\Big)(t) + (q-1)\int_\Omega |u(t)|^{q-2}|\nabla u(t)|^2\chi_{u\neq 0}\,dx \\
&\quad + \int_{\partial\Omega} \alpha\,|u(t)|^q\,d\sigma
\end{aligned}
$$

and therefore

$$\frac{d}{dt}\Big(\|u\|_q^q\Big)(t) \le 0$$

which implies that $\|u(t)\|_q \le \|f\|_q$. We can let $q \to \infty$ to obtain $\|u(t)\|_\infty \le \|f\|_\infty$. This shows (3.1) for $j = R$.

It suffices now to apply Theorem 3.1 to obtain that the operators A_j ($j = D, N$ or R) have the maximal L^p−regularity property in $L^q(\Omega)$ for all $p, q \in]1, \infty[$ (Proposition 3.2). □

3.2 Gaussian bounds

3.2.1 Pointwise estimates

The result presented here is due first to M. Hieber and J. Prüss ([21]) and was somewhat extended by T. Coulhon and X.T. Duong ([12]). The theorem below is adapted to semigroups with Gaussian estimates (so, not stated in the full generality).

Theorem 3.3 (Hieber-Prüss 1997, Coulhon-Duong 2000)**.** *Let $\Omega \subset \mathbb{R}^n$. Assume that $T(t))_{t\ge 0}$ is an analytic semigroup in $L^2(\Omega)$ with representation for all $f \in L^2(\Omega)$ and $z \in \mathbb{C}$, $|\arg z| < \varepsilon$*

$$T(z)f(x) = \int_\Omega p(z, x, y) f(y)\, dy, \quad x \in \Omega,$$

where the kernel p, for $t > 0$, enjoys the following estimates

$$|p(t, x, y)| \le c g(bt, x, y), \quad x, y \in \Omega, \tag{3.2}$$

with $c, b > 0$. Here $g(t, x, y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}$. Then the semigroup $(T(t))_{t\ge 0}$ can be extended as an analytic semigroup in $L^q(\Omega)$ for all $q \in]1, \infty[$ and its negative generator A_q has the maximal L^p−regularity property for all $p \in]1, \infty[$.

Lemma 3.4. *Let be the hypotheses of Theorem 3.3 hold. Then there exist $\theta \in]0, \frac{\pi}{2}]$ and constants $c_1, b_1 > 0$ such that*

$$|p(z, x, y)| \le c_1 g(b_1 \Re e(z), x, y), \quad x, y \in \Omega,\ z \in \mathbb{C}, |\arg z| < \varepsilon \tag{3.3}$$

and consequently there are two constants $c_2, b_2 > 0$ such that

$$\Big|\frac{\partial p}{\partial t}(t, x, y)\Big| \le \frac{c_2}{t} g(b_2 t, x, y), \quad t > 0,\ x, y \in \Omega. \tag{3.4}$$

Proof. This result is well-known (see e.g. Davies' book [14]). The estimate(3.4) follows from (3.3), using the Cauchy formula for the holomorphic function $z \mapsto p(z, x, y)$. □

Idea of the proof of Theorem (3.3). Let $Q = [0, \infty[\times \Omega$. The space (Q, μ, d), where μ is the Lebesgue measure on $\mathbb{R}^{n+1}$ and d is the quasi-metric defined by

$$d\Big((t, x), (s, y)\Big) = |x - y|^2 + |t - s|,$$

is of homogeneous type (has the doubling property : there exists a constant $C > 0$ such that if we denote $B(\xi, r) = \{\eta \in Q; d(\xi, \eta) < r^2\}$ then $\mu(B(\xi, 2r)) \le c\mu(B(\xi, r))$). Let K be the operator with kernel

$$k((t, x), (s, y)) = \frac{\partial p(t - s, x, y)}{\partial t}, \quad t > 0,\ x, y \in \Omega.$$

We know that the operator K defined by

$$Kf(\xi) = \int_Q k(\xi,\eta) f(\eta)\, d\mu(\eta), \quad a.e.\ \xi = (t,x) \in Q$$

is bounded on $L^2(Q)$: this is only a reformulation of the maximal L^2-regularity property on $L^2(\Omega)$. The strategy is to prove that K is of weak-type $(1,1)$. By interpolation, we can then prove that K is a bounded operator on $L^q(Q)$ for all $q \in]1,2]$. A duality argument is then used to prove that K is bounded on $L^{q'}(Q)$ (where $\frac{1}{q} + \frac{1}{q'} = 1$: $q' \in [2,\infty[$). Therefore, using the $p-$independence of the maximal L^p-regularity property (see Theorem 2.4), we finish the proof. Of course, the core of the proof is to show that K is of weak-type $(1,1)$. For that purpose, since the kernel k has a behavior like (3.4), we have to study a singular integral. We will use a (regularized) Calderón-Zygmund decomposition (see Theorem 6.7) adapted to the problem. For any $f \in L^1(Q)$, for all $\alpha > 0$, there exist $g, b_i \in L^1(Q)$ with the properties

(i) $|g(\xi)| \leq \kappa\alpha$, for $\mu - a.e.\ \xi \in Q$;

(ii) there exist balls $B_i = B(\xi_i, r_i) \subset Q$ ($i.e.$ $(t,x) \in B(\xi_i,r_i)$ if $|x-x_i|^2 + |t-t_i| < r_i^2$ and $\xi_i = (t_i,x_i)$) such that supp $b_i \subset B_i$ and

$$\int_Q |b_i(\xi)|\, d\mu(\xi) \leq \kappa\alpha\mu(B_i);$$

(iii) $\displaystyle\sum_{i=1}^{\infty} \mu(B_i) \leq \frac{\kappa}{\alpha}\|f\|_1$;

(iv) any $\xi \in Q$ belongs to at most N_0 balls B_i.

We "regularize" the functions b_i by applying an operator R_i defined by a kernel $\rho_i : Q \to \mathbb{R}$ as follows

$$\rho_i(\xi,\eta) = \varphi_i(t-s)\chi_{[(t-r_i)_+,t]}(s) k_{r_i}(x,y), \qquad \text{where } \xi = (t,x), \eta = (s,y)$$

and where $\varphi_i(\sigma) = \frac{1}{r_i}\frac{e}{2(e-1)}\, e^{-\frac{|\sigma|}{r_i}}$. This idea was first applied by X.T. Duong and A. M^{c}Intosh in [17]. We can show that $\displaystyle\sum_{i=1}^{\infty} R_i b_i \in L^2(Q)$ with norm, in $L^2(Q)$ controlled by $\alpha^{\frac{1}{2}}\|f\|_1$. Therefore, if we write

$$Kf = Kg + \sum_{i=1}^{\infty} KR_i b_i + \sum_{i=1}^{\infty}(K - KR_i)b_i,$$

only the last term is still to be investigated, the first two coming directly from the fact that K is bounded in $L^2(Q)$. It remains to show that

$$\mu\Big(\Big\{\xi \in Q; \Big|\sum_{i=1}^{\infty}(K - KR_i)b_i(\xi)\Big| > \alpha\Big\}\Big) \leq cst\, \alpha\|f\|_1.$$

This can be done once we prove that

$$\int_{d(\xi,\eta)\geq cr_i} |k(\xi,\eta) - k_i(\xi,\eta)| d\mu(\xi) \leq cst,$$

where $k_i(\xi,\eta) = \int_Q k(\xi,\zeta)\rho_i(\zeta,\eta)d\mu(\zeta)$ is the kernel of KR_i. Indeed, the proof at this step is very much like the proof of Theorem 2.4, using this last estimate. □

Example 3.5. Consider a divergence form second order operator $L = -\text{div}A\nabla$ with Dirichlet boundary conditions in a domain $\Omega \subset \mathbb{R}^n$ with $A \in L^\infty(\Omega; \mathscr{M}_n(\mathbb{C}))$ with antisymmetric imaginary part. Then it has been proved by E.M. Ouhabaz ([35] and [36]) that $-L$ generates an analytic semigroup in $L^2(\Omega)$ with gaussian estimates. Therefore, L has the maximal L^p-regularity property on $L^q(\Omega)$ for all $q \in]1, \infty[$, by Theorem 3.3.

3.2.2 Generalized Gaussian bounds

The abstract result It is sometimes not clear, or not true, whether a semigroup in $L^2(\Omega)$ has Gaussian estimates of the type (3.2). However, it is sometimes possible to prove a weaker form, namely a local integrated bound of the following form. To make the notations shorter, we will use

$$\|\cdot\|_{\mathscr{L}(L^{q_0}(\Omega), L^{q_1}(\Omega))} = \|\cdot\|_{q_0 \to q_1}$$

and

$$A(x, \rho, k) = B(x, (k+1)\rho) \setminus B(x, k\rho), \quad x \in \Omega,\ \rho > 0,\ k \in \mathbb{N}.$$

Definition 3.6. Let $\Omega \subset \mathbb{R}^n$ be a domain. Let A be the negative generator of an analytic semigroup $(T(t))_{t\geq 0}$ in $L^{q_0}(\Omega)$. We say that A has generalized Gaussian estimates (q_0, q_1) (where $1 < q_0 \leq q_1 < \infty$) if one of the following properties holds :

(1) the semigroup $(T(t))_{t\geq 0}$ satisfies

$$\|\chi_{B(x,\rho(t))} T(t) \chi_{A(x,\rho(t),k)}\|_{q_0 \to q_1} \leq |B(x, \rho(t))|^{-(\frac{1}{q_0} - \frac{1}{q_1})} h(k) \tag{3.5}$$

for $t > 0$, $x \in \Omega$, $k \in \mathbb{N}$, $\rho :]0, \infty[\to]0, \infty[$, and $(h(k))_{k\geq 1}$ satisfying

$$h(k) \leq c_\delta (k+1)^{-\delta} \text{ for some } \delta > \frac{n}{q_0} + \frac{1}{q_0'} \ ;$$

(2) or the resolvent of A satisfies

$$\|\chi_{B(x,\rho(t))} (I + zA)^{-1} \chi_{A(x,\rho(t),k)}\|_{q_0 \to q_1} \leq |B(x, \rho(t))|^{-(\frac{1}{q_0} - \frac{1}{q_1})} h(k) \tag{3.6}$$

for $z \in \Sigma_\theta = \{w \in \mathbb{C} \setminus \{0\}; |\arg(w)| < \pi - \theta\}$, $t = |z|^{-\frac{1}{2}}$, $x \in \Omega$, $k \in \mathbb{N}$, $\rho :]0, \infty[\to]0, \infty[$, and $(h(k))_{k\geq 1}$ satisfying

$$h(k) \leq c_\delta (k+1)^{-\delta} \text{ for some } \delta > \frac{n}{q_0} + \frac{1}{q_0'}.$$

Remark 3.7. A semigroup satisfying the Gaussian estimates (3.2) satisfies the two bounds above for all $1 < q_0 \leq q_1 < \infty$.

These kinds of bounds for $q_0 = 2$ are easier to prove than the pointwise Gaussian estimates (3.2), more particularly the second one (3.6). Indeed, it is, for instance for divergence form elliptic operators, only a matter of partial integration, as we will see below for the Lamé operator (see Theorem 3.10).

Theorem 3.8 (Kunstmann, 2008)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain. Let A be the negative generator of an analytic semigroup $(T(t))_{t\geq 0}$ in $L^2(\Omega)$ satisfying* (3.6) *with $q_1 > q_0 = 2$, then A has the maximal L^p-regularity property in $L^q(\Omega)$ for all $q \in [2, q_1[$.*

References for the proof. This result is due to P.C. Kunstmann [24]. The original statement uses (3.5) instead of (3.6), and is presented in a more general context : instead of $\Omega \subset \mathbb{R}^n$, Ω is assumed to be a space of homogeneous type. □

An example We apply here the previous result to the Lamé operator, which appears in the linearization of the compressible Navier-Stokes equations.

Definition 3.9. The Lamé operator with Dirichlet boundary conditions, denoted by L, is defined on $L^2(\Omega;\mathbb{R}^n)$ as the operator generated by the following sesquilinear form

$$\ell(u,v)=\mu\int_\Omega \nabla u\cdot\overline{\nabla v}\,dx+(\lambda+\mu)\int_\Omega \operatorname{div}u\,\overline{\operatorname{div}v}\,dx,\quad u,v\in H_0^1(\Omega;\mathbb{R}^3),$$

where $\mu>0$ and $\mu+\lambda\ge 0$. Since ℓ is continuous, coercive, the operator L is self-adjoint, generates a bounded analytic semigroup in $L^2(\Omega;\mathbb{R}^n)$.

Theorem 3.10. *Let Ω be a domain in $\mathbb{R}^n$ ($n\ge 3$). Then the Lamé operator with Dirichlet boundary conditions has the maximal L^p-regularity property in $L^q(\Omega)$ for all $q\in]\frac{2n}{n+2},\frac{2n}{n-2}[$.*

Proof. Fix an arbitrary point $x\in\Omega$, $z\in\Sigma_\theta$, where

$$\Sigma_\theta=\{w\in\mathbb{C}\setminus\{0\};|\arg(w)|<\pi-\theta\},$$

$t=|z|^{-\frac{1}{2}}$ and an arbitrary partition of unity $\{\eta_j,j\in\mathbb{N}\}$ of $\mathbb{R}^n$ such that

$$\begin{aligned}&\eta_0\in\mathscr{C}_c^\infty(B(x,2t);\mathbb{R}),\quad \eta_j\in\mathscr{C}_c^\infty\Big(B(x,2^{j+1}t)\setminus B(x,2^{j-1}t);\mathbb{R}\Big),\\ &0\le\eta_j\le 1,\quad |\nabla\eta_j|\le\tfrac{1}{2^{j-1}t},\quad \textstyle\sum_{j=0}^\infty\eta_j=1,\end{aligned}\tag{3.7}$$

where $B(x,r)$ is the ball in $\mathbb{R}^n$ with center at $x\in\mathbb{R}^3$ and radius $r>0$ and decompose $f\in L^2(\Omega,\mathbb{R}^n)$ as follows

$$f=\sum_{j=0}^\infty f_j,\quad f_j=\eta_jf;\qquad u=\sum_{j=0}^\infty u_j,\quad u_j=(zI+L)^{-1}f_j\in D(L).\tag{3.8}$$

We will prove that for all $p\in[2,\frac{2n}{n-2}]$, there exists two constants $C,c>0$ such that

$$|z|\Big[\int_{\Omega\cap B(x,t)}|u_j|^pdy\Big]^{\frac{1}{p}}\le Ce^{-c2^j}t^{n(\frac{1}{p}-\frac{1}{2})}\Big[\int_\Omega|f_j|^2dy\Big]^{\frac{1}{2}}\quad\forall\, j\in\mathbb{N}.\tag{3.9}$$

This will be done in three steps.

Step 1. Pick a new family of functions $(\xi_j)_{j\ge1}$ such that $\xi_j\in\mathscr{C}_c^\infty(B(x,2^{j-1}t);\mathbb{R})$. Taking the L^2-pairing of $\xi_j^2\overline{u_j}$ with both sides of $zu_j+Lu_j=f_j$, and keeping in mind that $\xi_jf_j=0$ for each $j\ge1$ we may write, based on integration by parts that

$$\begin{aligned}z\int_\Omega\xi_j^2|u_j|^2dy+\mu\int_\Omega\xi_j^2|\nabla u_j|^2dy+(\lambda+\mu)\int_\Omega\xi_j^2|\operatorname{div}u_j|^2dy\\ =\int_\Omega\mathcal{O}\Big(|\nabla\xi_j||u_j||\xi_j|\Big[\mu|\nabla u_j|+(\lambda+\mu)|\operatorname{div}u_j|\Big]\Big)dy.\end{aligned}\tag{3.10}$$

From this, via Cauchy-Schwarz inequality and a standard trick that allows us to absorb like-terms with small coefficients in the left-hand side, we get

$$|z|\int_\Omega\xi_j^2|u_j|^2dy\le C\int_\Omega|\nabla\xi_j|^2|u_j|^2dy\tag{3.11}$$

and, since $\lambda+\mu\geq 0$,

$$\int_\Omega \xi_j^2|\nabla u_j|^2dy \leq C\int_\Omega |\nabla\xi_j|^2|\nabla u_j|^2dy. \tag{3.12}$$

Step 2. Much as in [6], we now replace the cutoff function ξ_j in (3.11) by another cutoff function $e^{\alpha_j\xi_j}-1$ (which has the same properties as ξ_j), with

$$\alpha_j=\frac{\sqrt{|z|}}{2\sqrt{C}\|\nabla\xi_j\|_\infty},\ j\geq 2.$$

In a first stage, this yields

$$\int_\Omega |u_j|^2|e^{\alpha_j\xi_j}-1|^2dy \leq \frac{1}{4}\int_\Omega |u_j|^2|e^{\alpha_j\xi_j}|^2dy,$$

then further

$$\int_\Omega |u_j|^2|e^{\alpha_j\xi_j}|^2dy \leq 4\int_\Omega |u_j|^2dy, \tag{3.13}$$

in view of the generic, elementary implication

$$\|f-g\|\leq\frac{1}{2}\|f\| \implies \|f\|\leq 2\|g\|.$$

If we now assume that the original cutoff functions $(\xi_j)_{j\geq 2}$ also satisfy

$$0\leq\xi_j\leq 1,\quad \xi_j=1 \text{ on } B(x,t) \quad\text{ and }\quad \|\nabla\xi_j\|_\infty\leq\frac{\kappa}{2^jt},$$

it follows from the definition of α_j that $\alpha_j\geq c2^j$ and from (3.13) that

$$|e^{\alpha_j}|^2\int_{\Omega\cap B(x,t)}|u_j|^2dy\leq 4\int_\Omega|u_j|^2dy\leq\frac{\mathrm{cst}}{|z|^2}\int_\Omega|f_j|^2dy,$$

the second inequality coming from the fact that $-L$ generates an analytic semigroup. This gives then

$$|z|^2\int_{\Omega\cap B(x,t)}|u_j|^2dy\leq Ce^{-c2^j}\int_\Omega|f_j|^2dy, \tag{3.14}$$

The same procedure allows to estimate ∇u on $B(x,t)$ using (3.12) as follows

$$|z|\int_{\Omega\cap B(x,t)}|\nabla u_j|^2dy\leq Ce^{-c2^j}\int_\Omega|f_j|^2dy. \tag{3.15}$$

Those two estimates are also valid if $j=0$ since the resolvent of L is bounded in $L^2(\Omega;\mathbb{R}^n)$.

Step 3. Let $p=2^*=\frac{2n}{n-2}$. Sobolev's embedding in a (Lipschitz) domain $D\subset\mathbb{R}^n$ ($n\geq 3$) of diameter $R>0$ for a function $u\in H^1(D)$, after rescaling, reads as follows

$$R^{n(\frac{1}{2}-\frac{1}{p})}\Big(\int_D|u|^pdy\Big)^{\frac{1}{p}}\leq C\Big[\Big(\int_D|u|^2dy\Big)^{\frac{1}{2}}+R\Big(\int_D|\nabla u|^2dy\Big)^{\frac{1}{2}}\Big]. \tag{3.16}$$

Combining this inequality with (3.14) and (3.15), and keeping in mind that $|z| = \frac{1}{t^2}$, we have for all $j \in \mathbb{N}$

$$|z|\Big(\int_{\Omega\cap B(x,t)} |u_j|^{\frac{2n}{n-2}} dy\Big)^{\frac{n-2}{2n}} \leq C\, t^{-1} e^{-c2^j} \Big(\int_{\Omega} |f_j|^2 dy\Big)^{\frac{1}{2}}. \tag{3.17}$$

By interpolation, using (3.14) and (3.17), the generalized Gaussian bound (3.9) is proved for all $2 \leq p \leq \frac{2n}{n-2}$.

By Theorem 3.8, we may conclude that L has the maximal L^p-regularity property in the space $L^q(\Omega;\mathbb{R}^n)$ for all $q \in [2, \frac{2n}{n-2}[$. By duality (since L is self-adjoint), we can prove also that L has the maximal L^p-regularity property in $L^q(\Omega;\mathbb{R}^n)$ for all $q \in]\frac{2n}{n+2}, 2]$, which proves Theorem 3.10. □

4 Applications to partial differential equations

In this section, we will apply maximal L^p-regularity results to show the uniqueness of solutions of certain partial differential equations. We start with a toy problem, studied by F. Weissler in [41]. This will lead the way to prove uniqueness of mild solutions of the incompressible Navier-Stokes system.

4.1 A semilinear initial value problem

We are interested in the following equation

$$\begin{array}{rclll} \frac{\partial u}{\partial t} - \Delta u & = & u^2 & \text{in} &]0,T[\times\Omega \\ u(t,x) & = & 0 & \text{on} &]0,T[\times\partial\Omega \\ u(0) & = & u_0 & \text{in} & \Omega, \end{array} \tag{4.1}$$

where $T > 0$ and $\Omega \subset \mathbb{R}^n$ is a domain with no particular regularity at the boundary. We assume that $n \geq 4$. The critical space where we are looking for solutions is $L^p(\Omega)$ with $p = \frac{n}{2}$. This space is critical in the sense that if $p > \frac{n}{2}$, then the nonlinearity u^2 is a "small" perturbation of the linear part Δu and if $p < \frac{n}{2}$, then the nonlinearity "wins" and the methods applied here are not appropriate. Our purpose is to show that the solution $u \in \mathscr{C}([0,T]; L^p(\Omega))$ of (4.1) (in an integral sense defined below) is unique in the space $\mathscr{C}([0,T]; L^{\frac{n}{2}}(\Omega))$.

Definition 4.1. We say that u is an integral solution of (4.1) with $u_0 \in L^{\frac{n}{2}}(\Omega)$ on $[0,\tau]$ if $u \in \mathscr{C}([0,T]; L^{\frac{n}{2}}(\Omega))$ and u satisfies

$$u(t) = T(t)u_0 + \int_0^t T(t-s)(u(s)^2)\, ds, \quad t \in [0,\tau],$$

where $(T(t))_{t\geq 0}$ is the semigroup generated by the Dirichlet-Laplacian in $L^{\frac{n}{4}}(\Omega)$, which we denote by $-A$.

4.1.1 Existence

Theorem 4.2 (F. Weissler, 1981)**.** *Let $n \geq 4$. For any initial condition $u_0 \in L^{\frac{n}{2}}(\Omega)$, there exists $\tau \in]0,T]$ and $u \in \mathscr{C}([0,\tau]; L^{\frac{n}{2}}(\Omega))$ integral solution of (4.1) in $[0,\tau]$. If $\|u_0\|_{\frac{n}{2}}$ is small enough, then $\tau = T$.*

Proof. We will show the local existence of an integral solution of (4.1) via a fixed point method. We reformulate the problem as to find a Banach space $\mathcal{E}_T$ containing $\mathscr{C}([0,T]; L^{\frac{n}{2}}(\Omega))$ such that $a = T(\cdot)u_0 \in \mathcal{E}_T$ and there exists $u \in \mathcal{E}_T$ verifying

$$u = a + B(u,u),$$

where B is the bilinear operator defined by

$$B(u,v)(t) = \int_0^t T(t-s)(u(s)v(s))\, ds, \quad t \in [0,T].$$

We need B to be continuous on $\mathcal{E}_T \times \mathcal{E}_T$. We choose

$$\mathcal{E}_T = \left\{ u \in \mathscr{C}([0,T]; L^{\frac{n}{2}}(\Omega)); t \mapsto t^{\frac{1}{4}} u(t) \in \mathscr{C}([0,T]; L^{\frac{2n}{3}}(\Omega)) \right\},$$

and we define the norm in this space to be

$$\|u\|_{\mathcal{E}_T} = \sup_{0<t<T} \|u(t)\|_{\frac{n}{2}} + \sup_{0<t<T} t^{\frac{1}{4}} \|u(t)\|_{\frac{2n}{3}} \quad u \in \mathcal{E}_T.$$

This space $\mathcal{E}_T$ endowed with its norm is a Banach space. We remark first that $(T(t))_{t\geq 0}$ is a bounded analytic semigroup in $L^p(\Omega)$ for all $p \in]1, \infty[$, which implies in particular that for all $\alpha \geq 0$, there exists a constant $c_{p,\alpha} > 0$ such that

$$\|t^\alpha A^\alpha T(t)\|_p \leq c_{p,\alpha}, \quad t > 0.$$

Therefore, it is easy to check that $a \in \mathcal{E}_T$. We will show next that

$$B : \mathcal{E}_T \times \mathcal{E}_T \to \mathcal{E}_T$$

is continuous. Let $u, v \in \mathcal{E}_T$. Then we have

$$t \mapsto t^{\frac{1}{2}} u(t)v(t) \in C([0,T]; L^{\frac{n}{3}}(\Omega)$$

with norm bounded by $\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}$ and therefore

$$\|t^{\frac{1}{2}} A^{-\frac{1}{2}}(u(t)v(t))\|_{\frac{n}{2}} \leq c\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}$$

for all $t \in [0,T]$ since $W^{1,\frac{n}{3}} \subset L^{\frac{n}{2}}$ in dimension n by Sobolev embedding. We have then

$$\begin{aligned} B(u,v)(t) &\overset{(1)}{=} \int_0^t T(t-s)(u(s)v(s))\, ds \\ &\overset{(2)}{=} \int_0^t \sqrt{t-s}\, A^{\frac{1}{2}} T(t-s) A^{-\frac{1}{2}}(\sqrt{s}\, u(s)v(s))\, \frac{1}{\sqrt{t-s}\sqrt{s}}\, ds \end{aligned}$$

which gives the estimate for all $t \in [0,T]$

$$\begin{aligned} \|B(u,v)(t)\|_{\frac{n}{2}} &\overset{(1)}{\leq} c\, c_{\frac{n}{2},\frac{1}{2}} \|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T} \Big(\int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}}\, ds \Big) \\ &\overset{(2)}{\leq} \pi c\, c_{\frac{n}{2},\frac{1}{2}} \|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}. \end{aligned}$$

since

$$\int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}}\, ds = \int_0^1 \frac{1}{\sqrt{1-\sigma}\sqrt{\sigma}}\, d\sigma = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{2}{\sqrt{1-4r^2}}\, dr = \pi.$$

The same arguments are used to estimate $t^{\frac{1}{4}} \|B(u,v)(t)\|_{\frac{n}{3}}$. For $u, v \in \mathcal{E}_T$, we have

$$\|t^{\frac{1}{2}} A^{-\frac{3}{4}}(u(t)v(t))\|_{\frac{2n}{3}} \leq c\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}$$

since we have already seen that $t^{\frac{1}{2}} A^{-\frac{1}{2}}(u(t)v(t)) \in L^{\frac{n}{2}}(\Omega)$ and by the Sobolev embedding $W^{\frac{1}{2},\frac{n}{2}} \subset L^{\frac{2n}{3}}$ in dimension n. Therefore, we have

$$\begin{aligned} \|t^{\frac{1}{4}} B(u,v)(t)\|_{\frac{2n}{3}} &\overset{(1)}{\leq} c\, c_{\frac{2n}{3},\frac{3}{4}} \|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T} \Big(t^{\frac{1}{4}} \int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{1}{2}}\, ds \Big) \\ &\overset{(2)}{\leq} c\, c_{\frac{n}{2},\frac{1}{2}} \|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T} \Big(\int_0^1 (1-\sigma)^{-\frac{3}{4}} \sigma^{-\frac{1}{2}}\, d\sigma \Big). \end{aligned}$$

We can conclude the proof of the existence theorem by applying Picard fixed point theorem as long as $\|a\|_{\mathcal{E}_T} \leq \frac{1}{4\|B\|}$ and this is the case if $\|u_0\|_{\frac{n}{2}}$ is small enough. The argument must be adapted a little if $\|u_0\|_{\frac{n}{2}}$ is not small by adjusting T so that the result remains true. $\square$

4.1.2 Uniqueness

Theorem 4.3 (F. Weissler, 1981)**.** *Assume that $n \geq 5$. Let $u_1, u_2 \in \mathscr{C}([0,T]; L^{\frac{n}{2}}(\Omega))$ be two integral solutions of* (4.1) *for the same initial value $u_0 \in L^{\frac{n}{2}}(\Omega)$. Then $u_1 = u_2$ on $[0,T]$.*

Proof. With the same notations as in the previous proof, u_1 and u_2 are both solutions of the equation

$$u = a + B(u,u).$$

If we denote by v the difference between u_1 and u_2, then v must satisfy the equation

$$v = B(v, u_1 + u_2),$$

with $u_1, u_2, v \in \mathscr{C}([0,T]; L^{\frac{n}{2}}(\Omega))$ and $u_1(0) = u_2(0) = u_0$, $v(0) = 0$. To prove that $v = 0$ on a small interval $[0,\tau]$ (which implies then that $v = 0$ on the whole interval $[0,T]$), we need the following auxiliary lemma, which proof lies below, and this is where we use the maximal regularity property for the Dirichlet-Laplacian in $L^{\frac{n}{2}}(\Omega)$.

Lemma 4.4. *The bilinear operator*

$$B : L^q(0,T; L^{\frac{n}{2}}(\Omega)) \times \mathscr{C}([0,T]; L^{\frac{n}{2}}(\Omega)) \to L^q(0,T; L^{\frac{n}{2}}(\Omega))$$

is bounded for all $q \in]1, \infty[$.

Let $\varepsilon > 0$ be fixed. Choose $u_{0,\varepsilon} \in \mathscr{C}_c^\infty(\Omega)$ such that

$$\|u_0 - u_{0,\varepsilon}\|_{\frac{n}{2}} < \varepsilon.$$

We decompose $B(v, u_1 + u_2)$ into three parts :

$$B(v, u_1 + u_2) = B(v, u_1 + u_2 - 2u_0) + 2B(v, u_0 - u_{0,\varepsilon}) + 2B(v, u_{0,\varepsilon}).$$

We can estimate the first two parts thanks to Lemma 4.4. This gives then for any $\tau \in]0,T]$

$$\begin{aligned} & \|B(v, u_1 + u_2 - 2u_0)\|_{L^q(0,\tau; L^{\frac{n}{2}}(\Omega))} \\ \leq \;& C\|v\|_{L^q(0,\tau;L^{\frac{n}{2}}(\Omega))} \Big(\|u_1 - u_0\|_{C([0,\tau];L^{\frac{n}{2}}(\Omega))} + \|u_2 - u_0\|_{C([0,\tau];L^{\frac{n}{2}}(\Omega))} \Big) \end{aligned} \tag{4.2}$$

and

$$\|B(v, u_0 - u_{0,\varepsilon})\|_{L^q(0,\tau;L^{\frac{n}{2}}(\Omega))} \leq C\varepsilon \|v\|_{L^q(0,\tau;L^{\frac{n}{2}}(\Omega))}. \tag{4.3}$$

Since $\|u_i - u_0\|_{C([0,\tau];L^{\frac{n}{2}}(\Omega))} \to 0$ as $\tau \to 0$ for $i = 1,2$, it remains to estimate the last part $B(v, u_{0,\varepsilon})$. Since $u_{0,\varepsilon} \in \mathscr{C}_c^\infty(\Omega)$, then $vu_0 \in L^q(0,\tau; L^{\frac{n}{2}}(\Omega))$ and

$$\|B(v, u_{0,\varepsilon})(t)\|_{\frac{n}{2}} \leq M t^{1-\frac{1}{q}} \|v\|_{L^q(0,\tau;L^{\frac{n}{2}}(\Omega))} \|u_{0,\varepsilon}\|_\infty.$$

It is now obvious that $\|B(v, u_{0,\varepsilon})\|_{L^q(0,\tau;L^{\frac{n}{2}}(\Omega))} \to 0$ as $\tau \to 0$. Combining this last result with the estimates (4.2) and (4.3), we obtain that for ε and τ small enough,

$$\|B(v, u_1 + u_2)\|_{L^q(0,\tau;L^{\frac{n}{2}}(\Omega))} \leq \frac{1}{2} \|v\|_{L^q(0,\tau;L^{\frac{n}{2}}(\Omega))}.$$

Since v is solution of $v = B(v, u_1 + u_2)$ on $[0,T]$, and then in particular on $[0,\tau]$, this implies that $v = 0$ almost everywhere on $[0,\tau]$. Since moreover v is continuous on $[0,\tau]$, we conclude that $v = 0$ (everywhere) on $[0,\tau]$. These arguments show that $\{t \in [0,T]; v = 0\}$ is an open set in $[0,T]$ and by continuity of v, this set is also closed. Since it is not empty, it is necessarily equal to the whole interval $[0,T]$ by connexity. $\square$

Proof of Lemma 4.4. For $u \in L^q(0,T;L^{\frac{n}{2}}(\Omega))$ and $v \in \mathscr{C}([0,T];L^{\frac{n}{2}}(\Omega))$ the product uv is in $L^q(0,T;L^{\frac{n}{4}}(\Omega))$. Therefore,

$$f : t \mapsto A^{-1}(u(t)v(t)) \in L^q(0,T;L^{\frac{n}{2}}(\Omega)).$$

Since the Dirichlet-Laplacian enjoys the maximal L^q-regularity (see Proposition 3.2) in $L^{\frac{n}{2}}(\Omega)$, we have

$$\begin{aligned} \|B(u,v)\|_{L^q(0,T;L^{\frac{n}{2}}(\Omega))} &= \left\| t \mapsto A\int_0^t AT(t-s)f(s)\,ds \right\|_{L^q(0,T;L^{\frac{n}{2}}(\Omega))} \\ &\leq C\|u\|_{L^q(0,T;L^{\frac{n}{2}}(\Omega))}\|v\|_{C([0,T];L^{\frac{n}{2}}(\Omega))}, \end{aligned}$$

the constant C coming from the maximal L^q-regularity property of A in $L^{\frac{n}{2}}(\Omega)$. $\square$

4.2 Uniqueness for the incompressible Navier-Stokes system

The (incompressible) Navier-Stokes system in the whole space $\mathbb{R}^n$ reads as follows

$$\begin{aligned} \tfrac{\partial u}{\partial t} - \Delta u + \nabla\pi + (u\cdot\nabla)u &= 0 \quad \text{in} \quad]0,T[\times\mathbb{R}^n \\ \operatorname{div} u &= 0 \quad \text{in} \quad]0,T[\times\mathbb{R}^n \\ u(0) &= u_0 \quad \text{in} \quad \mathbb{R}^n, \end{aligned} \tag{4.4}$$

where

$$u : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n \quad \text{and} \quad \pi : [0,T]\times\mathbb{R}^n \to \mathbb{R}$$

denote the velocity of a fluid and its pressure. The notation $(u\cdot\nabla)v$ for u and v vector fields stands for $\sum_{i=1}^n u_i\partial_i v$. We assume that there is no external force. We can reformulate this system in a functional analysis setting as follows. Let A be the operator with domain $W^{2,p}(\mathbb{R}^n;\mathbb{R}^n)$ (for a $p \in]1,\infty[$),

$$Au = \mathbb{P}(-\Delta u) \text{ where } \mathbb{P} = I + \nabla(-\Delta)^{-1}\text{div},$$

$\mathbb{P}$ is the Leray projection and is bounded in $L^p(\mathbb{R}^n;\mathbb{R}^n)$ since the Riesz projections are bounded. As in the case of the semilinear heat equation, there is a critical space for (NS). In the scale of Lebesgue spaces, the critical space here is $L^n(\mathbb{R}^n;\mathbb{R}^n)$, which means that if $p > n$, then the nonlinearity $(u\cdot\nabla)u$ appears as a "small" perturbation of the linear part $\mathbb{P}(-\Delta u)$ and if $p < n$, then the nonlinearity "wins".

The existence of integral solutions of (4.4) (see Definition 4.5 below) for an initial condition $u_0 \in L^n(\mathbb{R}^n;\mathbb{R}^n)$ has been proved by T. Kato in 1984 in [23]. The proof is similar to the proof of the existence of solutions for the semilinear heat equation (Theorem 4.2). A good reference for this problem is the book by P.G. Lemarié-Rieusset [29].

Definition 4.5. We say that u is an integral solution of (4.4) with $u_0 \in L^n(\mathbb{R}^n;\mathbb{R}^n)$ with $\operatorname{div} u_0 = 0$ on $[0,\tau]$ if $u \in \mathscr{C}([0,T];L^n(\mathbb{R}^n;\mathbb{R}^n))$ and u satisfies

$$u(t) = e^{t\Delta}u_0 - \mathbb{P}\int_0^t e^{(t-s)\Delta}\nabla\cdot\Big(u(s)\otimes u(s)\Big)\,ds, \quad t\in[0,\tau],$$

where $(e^{t\Delta})_{t\geq 0}$ is the semigroup generated by the Laplacian denoted by Δ.

Remark 4.6. (*i*) Since u is a divergence-free vector field, we have

$$(u \cdot \nabla)u = \sum_{i=1}^{n} u_i \partial_i u = \sum_{i=1}^{n} \partial_i (u_i u) = \nabla \cdot (u \otimes u),$$

which explains the form of u in Definition 4.5.

(*ii*) In the previous definition, we want the semigroup $(e^{t\Delta})_{t\geq 0}$ to act on a distribution $\nabla(u \otimes u) \in W^{-1,\frac{n}{2}}$. This makes sense since in the case of the whole space $\mathbb{R}^n$, the heat semigroup acts on all $W^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$, $s \in \mathbb{R}$, $p \in]1, \infty[$.

Theorem 4.7 (T. Kato, 1984)**.** *Let* $u_0 \in L^n(\mathbb{R}^n; \mathbb{R}^n)$ *satisfy* $\operatorname{div} u_0 = 0$*. Then there exists* $T > 0$ *and* $u \in \mathscr{C}([0, T]; L^n(\mathbb{R}^n; \mathbb{R}^n)$ *integral solution of* (4.4)*. If* $\|u_0\|_n$ *is small enough, then the solution* u *is global (i.e. we can take* $T = \infty$*).*

Idea of the proof. The proof follows the line of the proof of Theorem 4.2 by working on the space

$$\begin{aligned} \mathcal{E}_T \; = \; & \Big\{ u \in \mathscr{C}([0, T]; L^n(\mathbb{R}^n; \mathbb{R}^n); \operatorname{div} u = 0 \text{ in } \mathbb{R}^n \text{ and} \\ & \quad t \mapsto \sqrt{t} \nabla u(t) \in \mathscr{C}([0, T]; L^n(\mathbb{R}^n; \mathscr{M}_n(\mathbb{R})) \Big\}, \end{aligned}$$

endowed with the norm

$$\|u\|_{\mathcal{E}_T} = \sup_{0<t<T} \|u(t)\|_n + \sup_{0<t<T} \sqrt{t} \|\nabla u(t)\|_n.$$

We are looking for $u \in \mathcal{E}_T$ solution of $u = a + B(u, u)$ where B is defined by

$$B(u, v)(t) = -\mathbb{P} \int_0^t e^{(t-s)\Delta} \Big(\frac{1}{2} \nabla \cdot (u(s) \otimes v(s) + v(s) \otimes u(s)) \Big) \, ds, \quad t \in [0, T]. \tag{4.5}$$

and $a(t) = e^{t\Delta} u_0$, $t \in [0, T]$. □

Theorem 4.8 (G. Furioli, P.G. Lemarié-Rieusset, E. Terraneo, 2000)**.** *Let* u_1, u_2 *be two integral solutions of* (4.4) *for the same initial value* $u_0 \in L^n(\mathbb{R}^n; \mathbb{R}^n)$ *satisfying* $\operatorname{div} u_0 = 0$*. Then* $u_1 = u_2$ *on* $[0, T]$*.*

Proof. This result was first proved by G. Furioli, P.G. Lemarié-Rieusset and E. Terraneo in [18] (see also the very nice review on the subject by M. Cannone [11]). The proof presented here is based on [33] and of the same spirit as the proof of Theorem 4.3. The basic idea is to reformulate the problem of uniqueness as to show that $u = u_1 - u_2$ is equal to zero on the interval $[0, T]$. The function u satisfies the equation $u = B(u, u_1 + u_2)$ where B is defined by (4.5) above. As shown by F. Oru in her PhD-thesis, the bilinear operator

$$B : \mathscr{C}([0, T]; L^n(\mathbb{R}^n; \mathbb{R}^n)) \times \mathscr{C}([0, T]; L^n(\mathbb{R}^n; \mathbb{R}^n)) \to \mathscr{C}([0, T]; L^n(\mathbb{R}^n; \mathbb{R}^n))$$

is not bounded. Had it been continuous, the proof of uniqueness of integral solution of the Navier-Stokes system would have been straightforward. The idea of [18], in dimension $n = 3$, was then to lower the regularity of the space $L^3(\mathbb{R}^3; \mathbb{R}^3)$ and consider a Besov space E instead (or, as shown by Y. Meyer in [31], the weak L^3 space, namely $L^{3,\infty}(\mathbb{R}^3; \mathbb{R}^3)$) to obtain a bounded bilinear operator B in $\mathscr{C}([0, T]; E) \times \mathscr{C}([0, T]; E)$.

The proof of [33] relies on a slightly different idea : instead of weaken the regularity of the space in the $x-$variable, we consider a Lebesgue space L^p in time instead of the space of continuous functions in time. As in the proof of Theorem 4.3, we write

$$u = B(u, u_1 - u_0) + B(u, u_2 - u_0) + 2B(u, u_0 - u_{0,\varepsilon}) + 2B(u, u_{0,\varepsilon})$$

where $u_{0,\varepsilon}$ is chosen in $\mathscr{C}_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ close to u_0 in the L^n-norm. To be able to show that $u = 0$ on a small interval $[0,\tau]$ $(\tau > 0)$ with the same method as in the proof of Theorem 4.3, we only need a result on the same kind as Lemma 4.4, see Lemma 4.9 below. At that point, the argument goes exactly as before. $\square$

Lemma 4.9. *The bilinear operator*

$$B : L^p(0,T;L^n(\mathbb{R}^n;\mathbb{R}^n)) \times \mathscr{C}([0,T];L^n(\mathbb{R}^n;\mathbb{R}^n)) \to L^p(0,T;L^n(\mathbb{R}^n;\mathbb{R}^n))$$

is bounded for all $p \in]1,\infty[$. More precisely, for all $p \in]1,\infty[$, there exists a constant $c_p > 0$ such that for all $u \in L^p(0,T;L^n(\mathbb{R}^n;\mathbb{R}^n))$ and all $v \in \mathscr{C}([0,T];L^n(\mathbb{R}^n;\mathbb{R}^n))$, we have

$$\|B(u,v)\|_{L^p(0,T;L^n(\mathbb{R}^n;\mathbb{R}^n))} \le c_p \|u\|_{L^p(0,T;L^n(\mathbb{R}^n;\mathbb{R}^n))} \|v\|_{\mathscr{C}([0,T];L^n(\mathbb{R}^n;\mathbb{R}^n))}.$$

Proof. To prove this lemma is exactly where the maximal L^p-regularity property of the Laplacian $-\Delta$ comes in. We rewrite $B(u,v)$ as

$$B(u,v)(t) = \mathbb{P}(-\Delta)\int_0^t e^{-(t-s)(-\Delta)}(-\Delta)^{-1} f(s)\, ds$$

where $f = -\frac{1}{2}\nabla\cdot(u\otimes v + v\otimes u)$. The only thing to prove is then that we can estimate $(-\Delta)^{-1}f$ in the $L^p(L^n)$ norm with respect to the norm of u in $L^p(L^n)$ and the norm of v in $\mathscr{C}(L^n)$. Indeed, since we know that $-\Delta$ has the maximal L^p-regularity property, this would imply the result. We have by Sobolev embeddings

$$\begin{aligned} \|(-\Delta)^{-1}f\|_{L^p(L^n)} &\le C\|(-\Delta)^{-1}f\|_{L^p(W^{1,\frac{n}{2}})} \\ &\le C\|f\|_{L^p(W^{-1,\frac{n}{2}})} \\ &\le C\|(u\otimes v + v\otimes u)\|_{L^p(L^{\frac{n}{2}})} \\ &\le C\|u\|_{L^p(L^n)}\|v\|_{\mathscr{C}(L^n)} \end{aligned}$$

and therefore we have proved Lemma 4.9. $\square$

5 Non-autonomous maximal regularity

In this section, we deal with non-autonomous problems of the form

$$\begin{aligned} u'(t) + A(t)u(t) &= f(t) \quad t \geq s \\ u(s) &= u_s, \end{aligned} \tag{5.1}$$

$s \geq 0$. Compared with the problems studied before, the difference is now that the operator A itself depends on the time t. The main consequence is that the operators $\frac{d}{dt}$ and A do not commute anymore. We will present here results without proofs (references for the proofs are however given).

5.1 Coefficients regular in time

We will here assume that the operators $\{A(t), t \in [0,T]\}$ defined on a Banach space X are uniformly (in $t \in [0,T]$) sectorial for all $t \in [0,T]$ (see Definition 6.9 below : this implies in particular that $-A(t)$ is the generator of an analytic semigroup in X for all $t \in [0,T]$) and satisfy the so-called Acquistapace-Terreni condition : there exist constants $c > 0$, $\alpha \in [0,1[$ and $\delta \in]0,1]$ such that

$$\left\| A(t)(\lambda I + A(t))^{-1}[A(t)^{-1} - A(s)^{-1}] \right\|_{\mathscr{L}(X)} \leq \frac{c|t-s|^\delta}{1+|\lambda|^{1-\alpha}}, \tag{5.2}$$

holds for all $t, s \in [0,T]$ and $\lambda \in \Sigma_\theta = \{z \in \mathbb{C} \setminus \{0\}; |\arg(z)| < \pi - \theta\}$. This condition implies in particular Hölder continuity in time of $t \mapsto A(t)^{-1}$. Let us point out that the Acquistapace-Terreni condition allows however the domains $D(A(t))$ to depend on t. This condition has been first used by P. Acquistapace and B. Terreni in [1] (and in a somewhat more abstract form by R. Labbas and B. Terreni in [26] and [27]) to prove the existence of an evolution family $\{U(t,s), t \geq s \geq 0\}$ (for all $t \geq s \geq 0$, $U(t,s)$ is a bounded operator in X) so that $u(t) = U(t,s)u_s$, $t \geq s$, is the solution of (5.1) with $f = 0$. The general form of a solution of (5.1) is then given by the formula

$$u(t) = U(t,s)u_s + \int_s^t U(t,r)f(r)\,dr.$$

Definition 5.1. We say that the family of operators $\{A(t), t \in [0,T]\}$ has the maximal L^p-regularity property if for all $f \in L^p(0,T;X)$, there exists a unique solution u of (5.1) (with $s = 0$ and $u_0 = 0$) satisfying $u' \in L^p(0,T;X)$ and

$$u(t) \in D(A(t)) \text{ for } a.a.\ t \in [0,T] \quad \text{and} \quad t \mapsto A(t)u(t) \in L^p(0,T;X).$$

5.1.1 Independence with respect to p

As much as in the autonomous case (see Proposition 2.4), under the condition (5.2), the property of maximal L^p-regularity is independent of $p \in]1,\infty[$.

Theorem 5.2. *Let X be a Banach space. Let $\mathscr{A} = \{A(t), t \in [0,T]\}$ be a family of uniformly sectorial operators on X satisfying the Acquistapace-Terreni condition* (5.2)*. Assume that $\mathscr{A}$ enjoys the maximal L^p-regularity property for one $p \in]1,\infty[$. Then $\mathscr{A}$ has the maximal L^q-regularity property for all $q \in]1,\infty[$.*

Reference for the proof. The proof of this theorem is due to M. Hieber and S. Monniaux and can be found in [20], Theorem 3.1. The idea of the proof is to show that the condition (5.2) implies a Hörmander-type condition for the operator S defined by

$$Sf(t) = \int_0^t A(t)e^{-(t-s)A(t)} f(s)\, ds, \quad t \in [0,T]$$

which is the singular part of the operator $f \mapsto A(\cdot)u(\cdot)$ for u the solution of (5.1) with $s = 0$ and $u_0 = 0$. Therefore, applying Theorem (2.5), since S is bounded in $L^p(0,T;X)$ for one $p \in]1,\infty[$, it is bounded in $L^q(0,T;X)$ for all $q \in]1,\infty[$. □

5.1.2 The case of Hilbert spaces

Theorem 5.3. *Let $X = H$ be a Hilbert space. Let $\mathscr{A} = \{A(t), t \in [0,T]\}$ be a family of uniformly sectorial operators on H satisfying the Acquistapace-Terreni condition* (5.2). *Then $\mathscr{A}$ enjoys the maximal L^p-regularity property for all $p \in]1,\infty[$.*

Reference for the proof. The proof is due to M. Hieber and S. Monniaux and can be found in [20], Theorem 3.2. It appears as a corollary of Theorem 5.4 and Theorem 5.2 below with the symbol a defined by

$$a(t,\tau) = \begin{cases} A(0)(i\tau I + A(0))^{-1}, & t < 0, \\ A(t)(i\tau I + A(t))^{-1}, & t \in [0,T], \\ A(T)(i\tau I + A(T))^{-1}, t > T. & \end{cases}$$

Indeed, it suffices to show that a satisfies the conditions of Theorem 5.4 to get the maximal L^2-regularity property of $\mathscr{A}$ and it remains to apply Theorem 5.2 to obtain the maximal L^p-regularity property for all $p \in]1,\infty[$. □

Theorem 5.4. *Let H be a Hilbert space and $a \in L^\infty(\mathbb{R}\times\mathbb{R};\mathscr{L}(H))$ such that $\xi \mapsto a(x,\xi)$ has an analytic extension (with values in $\mathscr{L}(H)$) in $S_\theta = \{\pm z \in \mathbb{C}; |\arg(z)| < \theta\}$ for one $\theta \in]0,\frac{\pi}{2}[$ and*

$$\sup_{z\in S_\theta}\sup_{x\in\mathbb{R}} \|a(x,z)\|_{\mathscr{L}(H)} < \infty.$$

Let $u \in \mathscr{S}(\mathbb{R};H)$ and define

$$Op(a)u(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi} a(x,\xi)\mathscr{F}(u)(\xi)\, d\xi, \quad x \in \mathbb{R}.$$

Then $Op(a)$ extends to a bounded operator on $L^2(\mathbb{R};H)$.

Reference for the proof. The proof of this theorem is due to M. Hieber and S. Monniaux, Theorem 2.1 in [20]. This can be viewed as a parameter dependent version of the Fourier-multiplier theorem in Hilbert spaces. □

5.1.3 The case of $UMD-$spaces

A version of Theorem 5.4 was proved by P. Portal and Ž. Štrkalj in $UMD-$spaces [37]. This allows to prove the following result, similar to Theorem 5.3 in the case of $UMD-$spaces (see also [40] in the autonomous case). The idea is to replace boundedness of the resolvent in the case of a Hilbert space with $R-$boundedness of the resolvent in the case of a Banach space with the $UMD-$property.

Definition 5.5. Let X be a Banach space. Let $\mathscr{A} = \{A(t), t \in [0,T]\}$ be a family of uniformly sectorial operators on X. We say that $\mathscr{A}$ is a family of uniformly $R-$sectorial operators on X if it satisfies

$$R\Big(\{(1+|\lambda|)(\lambda I + A)^{-1}; \lambda \in \Sigma_\theta, t \in [0,T]\}\Big) < \infty,$$

where the $R-$bound of a set of bounded operators τ, $R(\tau)$, has been defined in Section 2 (Definition 2.12).

Theorem 5.6 (Portal-Štrkalj, 2006)**.** *Let X be a $UMD-$space. Let $\mathscr{A} = \{A(t), t \in [0,T]\}$ be a family of uniformly $R-$sectorial operators on X satisfying the Acquistapace-Terreni condition* (5.2)*. Then $\mathscr{A}$ enjoys the maximal L^p-regularity property for all $p \in]1,\infty[$.*

Reference for the proof. The proof of this result can be found in [37], Section 5 (Corollary 14). □

5.2 Sufficient conditions

5.2.1 Non commutative Dore-Venni theorem

The first theorem about maximal regularity in the non-autonomous setting was proved by S. Monniaux and J. Prüss in 1997 ([34]) and is a generalization of the theorem of Dore-Venni (see Theorem 6.13 below) when the operator A depends on t and satisfies a condition of the type Acquistapace-Terreni (5.2).

Theorem 5.7. *Let X be a $UMD-$Banach space. Let $\mathscr{A} = \{A(t), t \in [0,T]\}$ be a family of uniformly sectorial operators with bounded imaginary powers on X satisfying the Acquistapace-Terreni condition* (5.2) *such that*

$$\sup_{t\in[0,T]} \sup_{s\in\mathbb{R}} \Big\{\frac{1}{|s|} \ln \|A(t)^{is}\|_{\mathscr{L}(X)}\Big\} \in \Big[0, \frac{\pi}{2}\Big[.$$

Then $\mathscr{A}$ enjoys the maximal L^p-regularity property for all $p \in]1,\infty[$.

Reference for the proof. This result is due to S. Monniaux and J. Prüss and its proof, in a slightly more general setting, can be found in [34], Theorem 1. Remark that in particular, the bound on $\{A(t)^{is}, s \in \mathbb{R}\}$ implies by Theorem 6.13 that every operator $A(t)$ has the property of maximal L^p-regularity. □

5.2.2 Heat-kernel bounds

The result presented here is a generalization of Theorem 3.3 in a non-autonomous setting.

Theorem 5.8. *Let $\Omega \subset \mathbb{R}^n$ and $\mathscr{A} = \{A(t), t \in [0,T]\}$ be a family of unbounded operators defined on $L^2(\Omega)$ such that $-A(t)$ generates an analytic semigroup in $L^2(\Omega)$. We assume moreover that $\mathscr{A}$ satisfies the Acquistapace-Terreni condition* (5.2) *in $L^2(\Omega)$ and that for all $t \in [0,T]$, the semigroup $(e^{-sA(t)})_{s\geq 0}$ has a kernel $p_t(s,x,y)$ with uniform Gaussian upper bounds, or more precisely,*

$$e^{-sA(t)}f(x) = \int_\Omega p_t(s,x,y)f(y)\,dy, \quad x \in \Omega,\ t \in [0,T],\ s \geq 0,$$

and there exist constants $c, b > 0$ (independent of t) such that

$$|p_t(s,x,y)| \leq cg(bs,x,y), \quad x,y \in \Omega,\ t \in [0,T],\ s \geq 0,$$

where g was defined in Theorem 3.3. Then $\mathscr{A}$ enjoys the maximal L^p-regularity property in $L^q(\Omega)$ for all $p, q \in]1,\infty[$.

Reference for the proof. This result is due to M. Hieber and S. Monniaux and its proof, though in a slightly more general setting, can be found in [19], Theorem 1. Remark that in particular, the Gaussian bound for p_t implies by Theorem 3.3 that every operator $A(t)$ has the property of maximal L^p-regularity. □

5.3 Domains constant with time

The case where the domains $D(A(t))$ of the operators $A(t)$, $t \in [0,T]$ do not depend on t, *i.e.* $D(A(t)) = D \subset X$, was investigated by J. Prüss and R. Schnaubelt in [38], H. Amann in [2], in view of applications to quasi-linear evolution equations in [3], and by W. Arendt, R. Chill, S. Fornaro and C. Poupaud in [5].

5.3.1 The abstract result

To state the result in its full generality, we need to define the notion of relative continuity.

Definition 5.9. A function $A : [0,T] \to \mathscr{L}(D,X)$ is called relatively continuous if for each $t \in [0,T]$ and all $\varepsilon > 0$ there exist $\delta > 0$ and $\eta \geq 0$ such that for all $x \in D$,

$$\|A(t)x - A(s)x\|_X \leq \varepsilon\|x\|_D + \eta\|x\|_X$$

holds for all $s \in [0,T]$ with $|t-s| \leq \delta$.

Theorem 5.10. *Let $A : [0,T] \to \mathscr{L}(D,X)$ be strongly measurable and relatively continuous. Assume that $A(t)$ has the maximal L^p-regularity property for all $t \in [0,T]$. Then for each $x \in (X,D)_{\frac{1}{p'},p}$ and $f \in L^p(0,T;X)$ there exists a unique solution u of* (5.1) *satisfying $u \in L^p(0,T;D) \cap W^{1,p}(0,T;X)$.*

References for the proof. This theorem was first proved by J. Prüss and R. Schnaubelt in the case where $A : [0,T] \to \mathscr{L}(D,X)$ is continuous (Theorem 2.5 of [38] ; see also Theorem 7.1 of [2]). They proved in particular that the hypotheses of the theorem imply the existence of an evolution family $\{U(t,s), 0 \leq s \leq t \leq T\}$. The condition $x \in (X,D)_{\frac{1}{p'},p}$ is to compare with Remark 2.3. The theorem, in its full generality as stated above, is due to W. Arendt, R. Chill,

S. Fornaro and C. Poupaud (Theorem 2.7 of [5]). They also proved the existence of an evolution family $\{U(t,s), 0 \leq s \leq t \leq T\}$ and the solution u of (5.1) is given by

$$u(t) = U(t,s)u_s + \int_s^t U(t,r)f(r)\,dr,$$

for $t \in [s,T]$. □

5.3.2 Application to quasi-linear evolution equations

Non-autonomous maximal regularity results seem to be a good starting point to study quasi-linear evolution equations. This has been thoroughly studied by H. Amann (see *e.g.* [3]). The problem is the following : find a solution u of

$$u' + A(u)u = F(u) \text{ on } [0,T], \quad u(0) = u_0, \tag{5.3}$$

with "reasonable" conditions on $u \mapsto A(u)$ and $u \mapsto F(u)$. The idea to treat this problem is to apply a fixed point theorem on the map

$$v \mapsto u, \qquad \text{where } u'(t) + A(v(t))u(t) = F(v(t)),\ t \in [0,T],\ u(0) = u_0. \tag{5.4}$$

The problem is of course to find a suitable Banach space in which one can apply the fixed point theorem, *i.e.* for which the solution of (5.4) has the best possible regularity properties, such as maximal regularity. The situation studied in [3] is the following. Let

$$\mathscr{E}_p = L^p(0,T;D) \cap W^{1,p}(0,T;X)$$

be the space associated to maximal L^p-regularity for (5.4). The operators $A(u)$ and the function F satisfy, for all $u \in \mathscr{E}_p$,

$$A(u) \in L^\infty(0,T;\mathscr{L}(D,X)) \quad \text{ and } \quad F(u) \in L^p(0,T;X).$$

It is also assumed that the restriction of $(A(u), F(u))$ on a subinterval J depends only on the restriction of u on the interval J (*i.e.* A and F are Volterra maps).

Theorem 5.11. *Under the above assumptions, the problem* (5.3) *has a unique maximal solution.*

Proof. Reference for the proof This result is due to H. Amann, Section 2, [3]. □

6 Appendix

We collect here some of the results used in the previous sections, mostly without proofs, but with references where they can be found.

6.1 Interpolation of operators

Theorem 6.1 (Riesz-Thorin theorem)**.** *Let* $(X, \mathcal{X}, \mu)$ *and* $(Y, \mathcal{Y}, \nu)$ *be two fixed measure spaces. Let* $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ *and* $\theta \in]0, 1[$*. Let*

$$T : L^{p_0}(X) + L^{p_1}(X) \to L^{q_0}(Y) + L^{q_1}(Y)$$

be a linear operator such that there exist two constants $c_0, c_1 > 0$ *with*

$$\|Tf\|_{L^{q_i}(Y)} \leq c_i \|f\|_{L^{p_i}(X)}, \quad \text{for all } f \in L^{p_i}(X),\ i = 0, 1.$$

Then for all $f \in L^{p_\theta}(X)$*, we have*

$$\|Tf\|_{L^{q_\theta}(Y)} \leq c_\theta \|f\|_{L^{p_\theta}(X)},$$

where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ *and* $c_\theta = c_0^{1-\theta} c_1^{\theta}$.

References for the proof. A proof for this theorem can be found in [8], Theorem 1.1.1. Another nice reference is Terence Tao's lecture on this subject [39], Theorem 3. □

Definition 6.2. A sublinear operator T from a measure space $(X, \mathcal{X}, \mu)$ to a measure space $(Y, \mathcal{Y}, \nu)$ maps simple functions $f : X \to \mathbb{C}$ of finite measure support in X to nonnegative-valued functions on Y (modulo almost everywhere equivalence) obeying the homogeneity relationship

$$T(cf) = |c|\, Tf \quad \text{for all } c \in \mathbb{C}$$

and the pointwise bound

$$|T(f+g)| \leq |Tf| + |Tg|,$$

for all simple functions f, g of finite measure support.

Remark 6.3. If S is a linear operator, then $T = |S|$ is sublinear.

Definition 6.4.
1. A linear or sublinear operator from X to Y is said to be of strong type (p, q) if there exists a constant $c > 0$ such that

$$\|Tf\|_{L^q(Y)} \leq c \|f\|_{L^p(X)}, \quad \forall f \in L^p(X).$$

2. A linear or sublinear operator from X to Y is said to be of weak type (p, q) if there exists a constant $c > 0$ such that

$$\sup_{t \geq 0} \left[t \nu\Big(\{ y \in Y; |Tf(y)| \geq t \} \Big)^{\frac{1}{q}} \right] \leq c \|f\|_{L^p(X)}, \quad \forall f \in L^p(X).$$

Theorem 6.5 (Marcinkiewicz interpolation theorem)**.** *Let* $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ *and* $\theta \in]0, 1[$ *such that* $q_0 \neq q_1$ *and* $p_i \leq q_i$ *for* $i = 0, 1$*. Let* T *be a sublinear operator of weak type* (p_0, q_0) *and of weak type* (p_1, q_1)*. Then* T *is of strong type* (p_θ, q_θ) *where* $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

References for the proof. A proof for this theorem can be found in [8], Theorem 1.3.1. Another nice reference is Terence Tao's lecture on this subject [39], Theorem 4. $\square$

Theorem 6.6 (Mihlin multiplier theorem)**.** *In the case where X and Y are both Hilbert spaces, if $M : \mathbb{R} \setminus \{0\} \to \mathscr{L}(X,Y)$ satisfies for some constant $C > 0$,*

$$\|M(t)\|_{\mathscr{L}(X,Y)} \leq C \quad \text{and} \quad \|tM'(t)\|_{\mathscr{L}(X,Y)} \leq C \quad \text{for all } t \in \mathbb{R} \setminus \{0\}$$

then M is a Fourier multiplier (see Definition 2.11) in $L^p(\mathbb{R}; X)$ for all $p \in]1, \infty[$.

References for the proof. Section 6.1 of [8]. $\square$

6.2 Calderón-Zygmund theory

Theorem 6.7 (Calderón-Zygmund decomposition)**.** *Let $f \in L^1(\mathbb{R}^n)$ and fix $\lambda > 0$. Then the following decomposition for f holds. We can write $f = g + \sum b_k$ where*

(i) $|g| \leq 2^n \lambda$ *almost everywhere ;*

(ii) $\|g\|_1 + \sum_k \|b_k\|_1 \leq 3\|f\|_1$ *;*

(iii) there exists a family of disjoint cubes $(Q_k)_{k \in \mathbb{N}}$ of $\mathbb{R}^n$ such that

$$\operatorname{supp} b_k \subset Q_k, \quad \int b_k = 0 \text{ and } \sum_k |Q_k| \leq \frac{1}{\lambda}\|f\|_1.$$

Proof. Let $f \in L^1(\mathbb{R}^n)$ and fix $\lambda > 0$. We may assume $\|f\|_1 = 1$. We decompose $\mathbb{R}^n$ into cubes of measure $\frac{1}{\lambda}$: $\mathbb{R}^n = \bigcup_m \tilde{Q}_{0,m}$. Then we have for all m

$$\frac{1}{|\tilde{Q}_{0,m}|} \int_{\tilde{Q}_{0,m}} |f| \, dx \leq \lambda \|f\|_1 = \lambda.$$

We then decompose each cube $\tilde{Q}_{0,m}$ into cubes of measure $\frac{1}{2^n \lambda}$. We denote by $(Q_{1,m})_m$ all cubes for which

$$\frac{1}{|Q_{1,m}|} \int_{Q_{1,m}} |f| \, dx > \lambda.$$

The other cubes are denoted by $\tilde{Q}_{1,m}$. We repeat this operation with these cubes $\tilde{Q}_{1,m}$ and obtain cubes $(Q_{2,m})_m$ of measure $\frac{1}{4^n \lambda}$ for which

$$\frac{1}{|Q_{2,m}|} \int_{Q_{2,m}} |f| \, dx > \lambda,$$

the remaining cubes beeing denoted by $\tilde{Q}_{2,m}$. After a countable number of steps, we obtain a family of cubes $(Q_{j,m})_{j,m \in \mathbb{N}}$ renamed as $(Q_k)_{k \in \mathbb{N}}$. We now define b_k as $b_k = (f - m_{Q_k}(f))\chi_{Q_k}$ with the notation

$$m_Q(f) = \frac{1}{|Q|} \int_Q f \, dx.$$

We denote by g the quantity

$$g = f - \sum_k b_k.$$

It remains to show that g and the b_k's satisfy the conditions (i), (ii) and (iii) of the theorem. First, if $x \in \mathbb{R}^n \setminus \bigcup_k Q_k$, then for all $j \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $x \in \tilde{Q}_{j,m}$. By construction, we have

$$\lambda \geq \frac{1}{|\tilde{Q}_{j,m}|} \int_{\tilde{Q}_{j,m}} |f| \xrightarrow[j \to \infty]{} |f(x)|$$

if x is a Lebesgue point of f. If $x \in Q_k$ for one $k \in \mathbb{N}$, then

$$|g(x)| = |m_{Q_k}(f)| \leq \frac{|\tilde{Q}_k|}{|Q_k|} \lambda = 2^n \lambda.$$

All together, this gives (i) since the set of non Lebesgue points of f is of measure zero. Again by construction, we have

$$\operatorname{supp} b_k \subset Q_k \quad \text{and} \quad \int_{Q_k} b_k = 0.$$

Moreover, we have $\frac{1}{|Q_k|} \int_{Q_k} |f| > \lambda$. Therefore $|Q_k| < \frac{1}{\lambda} \int_{Q_k} |f|$. Since the cubes Q_k are disjoint, this gives

$$\sum_k |Q_k| \leq \sum_k \frac{1}{\lambda} \int_{Q_k} |f| \leq \frac{1}{\lambda} \int_{\mathbb{R}^n} |f| = \frac{1}{\lambda} \|f\|_1.$$

This proves (iii). Finally, we have $\|g\|_1 \leq \|f\|_1$ and $\|b_k\|_1 \leq 2 \int_{Q_k} |f|$ for all $k \in \mathbb{N}$, and this gives (ii). $\square$

Remark 6.8. The Calderón-Zygmund decomposition works also in a measurable space (E, μ, d) of homogeneous type where μ is a $\sigma-$finite measure and d is a quasi-metric, *i.e.* there exists a constant $c > 0$ such that for any ball $B = \{y; d(x,y) < r\}$, if we write $2B$ the ball with the same center and 2 times the radius of B, it holds $\mu(2B) \leq c\mu(B)$.

6.3 Bounded imaginary powers

Definition 6.9. A (linear) operator A on a Banach space X is sectorial if it is closed, densely defined, has empty kernel, dense range $R(A)$ and verifies

$$\sup_{t>0} \|t(tI + A)^{-1}\|_{\mathscr{L}(X)} < \infty.$$

Let $x \in D(A) \cap R(A)$. Then one can define for $z \in \mathbb{C}$, $|\Re e(z)| < 1$

$$\begin{aligned} A^z x &= \frac{\sin \pi z}{\pi} \Big(\frac{x}{z} - \frac{1}{1+z} A^{-1} x + \int_0^1 t^{z+1} (tI + A)^{-1} A^{-1} x \, dt \\ &\quad + \int_1^\infty t^{z-1} (tI + A)^{-1} A x \, dt \Big). \end{aligned}$$

We are now in position to give the definition of operators with bounded imaginary powers.

Definition 6.10. A sectorial operator on a Banach space X has bounded imaginary powers if the closure of the operator $(A^{is}, D(A) \cap R(A))$ defines a bounded operator on X for all $s \in \mathbb{R}$ and if $\sup_{|s| \leq 1} \|A^{is}\|_{\mathscr{L}(X)} < \infty$.

Remark 6.11. If A admits bounded imaginary powers, then $(A^{is})_{s\in\mathbb{R}}$ forms a strongly continuous group on X.

Remark 6.12. Conversely, it has been proved in [32] that a given strongly continuous group with type strictly less than π on a $UMD-$Banach space can be represented as the imaginary powers of a sectorial operator, called its analytic generator.

In the class of $UMD-$spaces, a positive result for operators having bounded imaginary powers has been proved by G. Dore and A. Venni.

Theorem 6.13 (Dore-Venni, 1987)**.** *Let X be a Banach space in the $UMD-$class. Let A be an operator with bounded imaginary powers (see Definition 6.10) for which the type of the group $(A^{is})_{s\in\mathbb{R}}$ is strictly less than $\frac{\pi}{2}$. Then A has the maximal L^p-regularity property.*

Idea of the proof. The idea is to show that $\mathcal{A}+\mathcal{B}$ with domain $D(\mathcal{A})\cap D(\mathcal{B})$ is invertible, where

$$D(\mathcal{A}) = L^2(0,\infty; D(A)),\ (\mathcal{A}u)(t) = Au(t),\ t>0$$

and

$$D(\mathcal{B}) = H_0^1(0,\infty;X),\ \mathcal{B}u = u'.$$

The operator $\mathcal{A}$ has bounded imaginary powers with angle strictly less than $\frac{\pi}{2}$ and the operator $\mathcal{B}$ has bounded imaginary powers with angle $\frac{\pi}{2}$. We define then, for $c\in]0,1[$,

$$S = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty} \frac{\mathcal{A}^{-z}\mathcal{B}^{z-1}}{\sin \pi z}\, dz.$$

The purpose is to show that $\mathcal{B}S$ is bounded and that $S = (\mathcal{A}+\mathcal{B})^{-1}$, and this is done by letting $c\to 0^+$ and taking into account that the Hilbert transform is bounded in $L^2(\mathbb{R};X)$.
Another (shorter) proof uses the result presented in Remark 6.12 by showing that the group $(\mathcal{A}^{-is}\mathcal{B}^{is})_{s\in\mathbb{R}}$ has a sectorial analytic generator. □

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