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THE NONLINEAR HODGE-NAVIER-STOKES EQUATIONS IN LIPSCHITZ DOMAINS

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Abstract. We investigate the Navier-Stokes equations in a suitable functional setting, in a three-dimensional bounded Lipschitz domain Ω, equipped with "free boundary" conditions. In this context, we employ the Fujita-Kato method and prove the existence of a local mild solution. Our approach makes essential use of the properties of the Hodge-Laplacian in Lipschitz domains.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. That is, $\partial\Omega$ can be locally described by means of graphs of real-valued Lipschitz functions in $\mathbb{R}^2$, suitably rotated and translated. The Navier-Stokes system with Dirichlet boundary conditions for an incompressible fluid occupying the domain Ω reads (in the absence of body forces) as follows:

$$\begin{array}{rcll} \frac{\partial u}{\partial t} - \Delta u + \nabla p + (u\cdot\nabla)u &=& 0 & \text{in} \quad (0,T)\times\Omega \\ \operatorname{div} u &=& 0 & \text{in} \quad (0,T)\times\Omega \\ u &=& 0 & \text{on} \quad (0,T)\times\partial\Omega \\ u(0,\cdot) &=& u_0 & \text{in} \quad \Omega, \end{array} \tag{1.1}$$

where u denotes the velocity of the fluid, p stands for its pressure, and u_0 is the initial velocity (assumed to be divergence-free and with vanishing normal

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component on $\partial\Omega$). We denote by A the Stokes operator (see Definition 2.4 in [23]).

The space $D(A^{\frac{1}{4}})$ is critical for the problem (1.1) in the Hilbert space setting. For the initial value $u_0 \in D(A^{\frac{1}{4}})$, it has been shown in [23] (see also [22]) that (1.1) admits a solution $u \in \mathcal{C}(0,T;D(A^{\frac{1}{4}}))$ (T depending on the size of u_0) in the case where $\Omega \subset \mathbb{R}^3$ is any domain bounded or unbounded, smooth or not smooth. In the L^p-space setting, the critical space $D(A^{\frac{1}{4}})$ corresponds to $L^3(\Omega;\mathbb{R}^3)$ by Sobolev embeddings. This case is more subtle since nothing is known about the behavior of the Stokes operator in L^p-spaces if the domain Ω is not smooth enough. Taylor conjectured in [28] that the Stokes operator generates an analytic semi-group in L^p for certain values of p, which is a key tool to prove existence of mild solutions, but this remains unproved.

Besides Dirichlet, another natural set of boundary conditions which have received a substantial amount of attention in the literature (cf. [1], [2], [3], [4], [5], [6], [11], [12], [16], [17], [25], [27], [30], [31], and the references therein) is provided by the following "free boundary" conditions (in the terminology employed on page 503 of [30]):

$$\left\{\begin{array}{rcll} \nu\cdot u & = & 0 & \text{on} \quad (0,T)\times\partial\Omega \\ \nu\times \operatorname{curl} u & = & 0 & \text{on} \quad (0,T)\times\partial\Omega, \end{array}\right. \tag{1.2}$$

where ν denotes the outward unit normal to Ω. The first equation above is a "no-penetration" condition, whereas the second one indicates that the vorticity is normal to the boundary. It is of interest to compare (1.2) with the more traditionally used Navier's slip boundary conditions to the effect that

$$\left\{\begin{array}{rcll} \nu\cdot u & = & 0 & \text{on} \quad (0,T)\times\partial\Omega \\ [(\nabla u+\nabla u^\top)\nu]_{tan} & = & 0 & \text{on} \quad (0,T)\times\partial\Omega. \end{array}\right. \tag{1.3}$$

We do so in Section 2 where we show that, if $\partial\Omega \in \mathcal{C}^2$, then (1.2) differs from (1.3) only by a zero-order term (which actually vanishes on the flat portions of $\partial\Omega$). Incidentally, this clarifies a somewhat obscure point in the literature (cf. page 341 in [29] where apparently the incorrect claim is made that (1.2) and (1.3) are identical).

Granted that u is a sufficiently smooth vector field, we may write

$$(u\cdot\nabla)u = \tfrac{1}{2}\nabla\,|u|^2 + u\times \operatorname{curl} u.$$

Therefore, replacing p in (1.1) with the so-called *dynamical pressure* (cf., e.g., [16])

$$\pi := \tfrac{1}{2}|u|^2 + p$$

and adopting (1.2) as boundary conditions, we arrive at the following initial boundary-value problem

$$\begin{array}{rcll} \frac{\partial u}{\partial t} - \Delta u + \nabla\pi + u \times \operatorname{curl} u & = & 0 & \text{in} \quad (0,T)\times\Omega \\ \operatorname{div} u & = & 0 & \text{in} \quad (0,T)\times\Omega \\ \nu\cdot u & = & 0 & \text{on} \quad (0,T)\times\partial\Omega \\ \nu\times\operatorname{curl} u & = & 0 & \text{on} \quad (0,T)\times\partial\Omega \\ u(0,\cdot) & = & u_0 & \text{in} \quad \Omega. \end{array} \tag{1.4}$$

The boundary conditions (1.2) are natural for the Hodge-Laplacian (i.e., the Laplacian acting on vector fields), in which context they are known as relative boundary conditions (cf. [27]). For this reason, we shall refer to (1.4) as the *Hodge-Navier-Stokes system*.

The existence of mild solutions for (1.1) with initial data $u_0 \in L^3(\Omega;\mathbb{R}^3)$ has been studied in [14] in the case where $\Omega = \mathbb{R}^3$, and in [15] in the case when Ω is a bounded domain with a ($\mathcal{C}^\infty$) smooth boundary. In both instances, the major tool in the proof of the existence of a mild solution was the fact that

$$\text{the Stokes semi-group is analytic in } L^p \tag{1.5}$$

for every $1 < p < \infty$. Of course, for arbitrary bounded Lipschitz domains, simple functional analysis gives that (1.5) is always valid when $p = 2$. On the other hand, Deuring has proved in [8] that there exist three-dimensional, bounded, cone-like domains (hence, in particular, Lipschitz) such that (1.5) fails for certain values of p. This spectrum of results raises the issue of determining the optimal range of p's for which (1.5) holds. In [28], Taylor has conjectured that, for bounded Lipschitz domains in $\mathbb{R}^3$, (1.5) holds for all p's in an open interval containing $[\frac{3}{2}, 3]$.

While, in its original formulation, this question remains open at the present time, progress in a related direction has recently been registered in [21], where the authors have proved that the Hodge-Stokes operator (more on this below) generates an analytic semi-group in L^p for all p's in a certain open interval containing $[\frac{3}{2}, 3]$. This is the natural analogue of Taylor's conjecture for the system (1.1). In turn, this result suggests the consideration of the nonlinear problem (1.4), a close relative of (1.1), from the perspective of the classical Fujita-Kato approach.

Concerning the linear part of (1.4), we summarize the results of [21] in Theorem 1.1 below. Let A be the operator associated with the linear part of (1.4), which we shall refer to as the *Hodge-Stokes operator*. In the context of L^p−spaces ($1 < p < \infty$), $A = A_p$ is defined as follows:

$$\begin{aligned} D(A_p) &:= \{u \in L^p(\Omega;\mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } \Omega,\ \operatorname{curl} u \in L^p(\Omega;\mathbb{R}^3) \\ &\qquad \Delta u \in L^p(\Omega;\mathbb{R}^3) \text{ and } \nu\cdot u = 0,\ \nu\times\operatorname{curl} u = 0 \text{ on } \partial\Omega\} \\ A_p u &:= -\Delta u = \operatorname{curl}\operatorname{curl} u, \qquad \forall\, u \in D(A_p). \end{aligned} \tag{1.6}$$

The operator A_p acts as an unbounded operator in

$$X_p := \{u \in L^p(\Omega;\mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } \Omega \text{ and } \nu\cdot u = 0 \text{ on } \partial\Omega\}.$$

The orthogonal projection $\mathbb{P} : L^2(\Omega;\mathbb{R}^3) \to X_2$, known as the Helmholtz projection extends to a bounded operator $\mathbb{P}_p : L^p(\Omega;\mathbb{R}^3) \to X_p$ whenever p belongs to an interval (p_Ω, q_Ω) whose endpoints satisfy $1 \le p_\Omega < \frac{3}{2} < 3 < q_\Omega \le \infty$ and $1/p_\Omega + 1/q_\Omega = 1$; in particular, (p_Ω, q_Ω) contains $[\frac{3}{2}, 3]$. See Theorem 11.1 of [13] where it has also been pointed out that if Ω is of class $\mathcal{C}^1$ then one can take $p_\Omega = 1$ and $q_\Omega = \infty$. This implies, in particular, that X_p is a closed subspace of $L^p(\Omega;\mathbb{R}^3)$ for $p \in (p_\Omega, q_\Omega)$ and, when equipped with the L^p norm, a Banach space. The following result is a combination of Lemma 3.9, Theorem 6.1 and Theorem 7.3 of [21].

Theorem 1.1. *For each $p \in (p_\Omega, q_\Omega)$, the operator $-A_p$ generates an analytic semi-group $(e^{-tA_p})_{t\ge 0}$ in X_p, referred to in the sequel as the Hodge-Stokes semi-group, satisfying*

$$\sup_{t\ge 0}\Big(\|e^{-tA_p}\|_{X_p\to X_p} + \|\sqrt{t}\,\operatorname{curl} e^{-tA_p}\|_{X_p\to L^p} + \|t\operatorname{curl}\operatorname{curl} e^{-tA_p}\|_{X_p\to L^p}\Big) < \infty.$$

The nonlinear problem (1.4) can be now rewritten in the following form:

$$\begin{aligned} u(t) &\in D(A_p), & t &\in (0,T] \\ u'(t) + A_p u(t) + \mathbb{P}_p\big(u(t)\times\operatorname{curl} u(t)\big) &= 0, & t &\in (0,T] \\ u(0) &= u_0 \in X_p, \end{aligned} \tag{1.7}$$

by formally applying the projection $\mathbb{P}_p$ to the first equation listed in (1.4). Our goal is to show the existence of a solution for the problem (1.7) with small initial data $u_0 \in X_3$, when Ω is a bounded Lipschitz domain. See Theorem 5.4 for a precise formulation.

2. The relationship between free-surface and slip boundary conditions

In the three-dimensional context, one has the readily verified identity

$$\big(\nu \times \operatorname{curl} u\big)_j = \nu \cdot \partial_j u - \partial_\nu u_j, \qquad 1 \leq j \leq 3. \tag{2.1}$$

Consider now the case when Ω is a domain in $\mathbb{R}^n$ (where $n \geq 2$) and $u = (u_1, ..., u_n)$ is a vector field (as before, playing the role of the velocity field of a fluid). In this context, the analogue of our boundary conditions (1.2) is

$$\nu \cdot u = 0 \text{ and } \nu \cdot \partial_j u - \partial_\nu u_j = 0 \text{ on } (0,T) \times \partial\Omega, \text{ for every } j \in \{1, .., n\}. \tag{2.2}$$

We wish to contrast these so-called free-boundary conditions with Navier's slip boundary conditions, which we now proceed to recall. To get started, introduce the deformation tensor of u as

$$\operatorname{Def}(u) := \tfrac{1}{2}\big(\partial_j u_k + \partial_k u_j\big)_{1 \leq j,k \leq n} = \tfrac{1}{2}(\nabla u + \nabla u^\top), \tag{2.3}$$

and, with $I_{n\times n}$ denoting the $n \times n$ identity matrix, recall Cauchy's stress tensor

$$T(u, \pi) := 2\operatorname{Def}(u) - \pi I_{n\times n}. \tag{2.4}$$

Also, set

$$\begin{aligned} B(u) &:= [T(u,\pi)\nu]_{tan} = T(u,\pi)\nu - \big\langle T(u,\pi)\nu, \nu\big\rangle \nu \\ &= (\nabla u + \nabla u^\top)\nu - 2\big\langle (\nabla u)\nu, \nu\big\rangle \nu. \end{aligned} \tag{2.5}$$

Then Navier's slip boundary conditions read

$$\nu \cdot u = 0 \ \text{ and } \ B(u) = 0 \text{ on } (0,T) \times \partial\Omega. \tag{2.6}$$

To compare the two sets of boundary conditions (2.2) and (2.6), we wish to compare $B(u)$ with the vector $\big(\nu \cdot \partial_j u - \partial_\nu u_j\big)_{1 \leq j \leq n}$. We shall do so under the assumption that the boundary of the underlying domain $\Omega \subset \mathbb{R}^n$ is of class $\mathcal{C}^2$. The reader is referred to the Appendix (Section 6) of this paper for definitions and properties of a number of geometrical entities associated with the $\mathcal{C}^2$ surface $\mathcal{S} := \partial\Omega$. Then, if $\mathcal{U}$, ν are as in Proposition 6.1, for a reasonably well-behaved vector field u in $\mathcal{U}$ we may write

$$\begin{aligned} B(u) &\overset{(1)}{=} (\nabla u + \nabla u^\top)\nu - 2\big\langle (\nabla u)\nu, \nu\big\rangle \nu \\ &\overset{(2)}{=} \Big(\nu \cdot \partial_j u + \partial_\nu u_j - 2(\nu \cdot \partial_\nu u)\nu_j\Big)_{1 \leq j \leq n} \end{aligned}$$

$$\stackrel{(3)}{=} \; -\Big(\nu\cdot\partial_j u - \partial_\nu u_j\Big)_{1\le j\le n} + 2\Big(\nu\cdot\partial_j u - (\nu\cdot\partial_\nu u)\nu_j\Big)_{1\le j\le n}.$$

Above, the first equality is just the definition of $B(u)$, the second is equality obtained by expanding $(\nabla u + \nabla u^\top)\nu$, while the third equality is a matter of trivial algebra.

Let now $j\in\{1,...,n\}$ be fixed. By decomposing the vector e_j into its tangential part $(e_j)_{tan}$ and its normal part $(e_j\cdot\nu)\nu = \nu_j\nu$, the quantity $\nu\cdot\partial_j u$ becomes

$$\begin{aligned}
\nu\cdot\partial_j u &\stackrel{(1)}{=} \nu\cdot\Big((e_j\cdot\nabla)u\Big) \stackrel{(2)}{=} \nu\cdot\Big(([(e_j)_{tan}+\nu_j\nu]\cdot\nabla)u\Big)\\
&\stackrel{(3)}{=} \nu\cdot\Big(\nabla_{(e_j)_{tan}}u + \nu_j\partial_\nu u\Big)\\
&\stackrel{(4)}{=} \nabla_{(e_j)_{tan}}(\nu\cdot u) - [\nabla_{(e_j)_{tan}}\nu]\cdot u + (\nu\cdot\partial_\nu u)\nu_j.
\end{aligned}$$

The first equality is a consequence of $\partial_j = e_j\cdot\nabla$, the second equality is due to the decomposition of e_j into its tangential and its normal parts, while the third equality is based on the fact that $\nu\cdot\nabla = \partial_\nu$ and $(e_j)_{tan}\cdot\nabla = \nabla_{(e_j)_{tan}}$. It is relevant to note that $\nabla_{(e_j)_{tan}}$ is a tangential derivation operator along $\partial\Omega$. Plugging this back into the expression of $B(u)$ we obtain

$$\begin{aligned}
B(u) &= -\Big(\nu\cdot\partial_j u - \partial_\nu u_j\Big)_{1\le j\le n} + 2\Big(\nu\cdot\partial_j u - (\nu\cdot\partial_\nu u)\nu_j\Big)_{1\le j\le n}\\
&= -\Big(\nu\cdot\partial_j u - \partial_\nu u_j\Big)_{1\le j\le n} + 2\Big(\nabla_{(e_j)_{tan}}(\nu\cdot u) - [\nabla_{(e_j)_{tan}}\nu]\cdot u\Big)_{1\le j\le n}.
\end{aligned}$$

The last step is now to identify the quantity $\nabla_{(e_j)_{tan}}\nu$. One has

$$\nabla_{(e_j)_{tan}}\nu = \Big(\nabla_{e_j}\nu_k - \nabla_{(\nu_j\nu)}\nu_k\Big)_{1\le k\le n} = \Big(\partial_j\nu_k - \nu_j\partial_\nu\nu_k\Big)_{1\le k\le n},$$

which shows that, in the neighborhood $\mathcal{U}$ of $\partial\Omega$, we have

$$B(u) = -\Big(\nu\cdot\partial_j u - \partial_\nu u_j\Big)_{1\le j\le n} + 2\Big(\nabla_{(e_j)_{tan}}(\nu\cdot u)\Big)_{1\le j\le n} - 2Ru + 2[(\partial_\nu\nu)\cdot u]\nu,$$

where R is the matrix defined by (6.7). Restricting both sides to $\partial\Omega$ yields, thanks to Proposition 6.1(iii) and (6.8),

$$B(u) = -\Big(\nu\cdot\partial_j u - \partial_\nu u_j\Big)_{1\le j\le n} - 2Ru + 2\Big(\nabla_{(e_j)_{tan}}(\nu\cdot u)\Big)_{1\le j\le n}. \tag{2.7}$$

Since $\nabla_{(e_j)_{tan}}$ is a tangential derivation operator along $\partial\Omega$, the extra assumption that $\nu\cdot u = 0$ on $\partial\Omega$ guarantees both that the last term in (2.7) vanishes, and that $u|_{\partial\Omega}$ is a tangential field. Consequently, $Ru = -\mathcal{W}u$ on

$\partial\Omega$, where $\mathcal{W}$ is the Weingarten map on $\partial\Omega$ (cf. Section 6). In summary, the above shows that an equivalent way of expressing the boundary conditions (2.6) is

$$\nu \cdot u = 0 \quad \text{and} \quad -\Big(\nu \cdot \partial_j u - \partial_\nu u_j\Big)_{1\leq j\leq n} + 2\mathcal{W}u = 0 \quad \text{on} \quad (0,T)\times\partial\Omega. \tag{2.8}$$

In particular, if $n = 3$, then the above Navier's slip boundary conditions become

$$\nu \cdot u = 0 \quad \text{and} \quad -\nu \times \operatorname{curl} u + 2\mathcal{W}u = 0 \qquad \text{on} \quad (0,T)\times\partial\Omega. \tag{2.9}$$

A few comments are in order here. The quantity $\mathcal{W}u$ appearing in (2.8) and (2.9) is a zero-order term. It has a clear geometrical significance vis-a-vis the surface $\partial\Omega$ (cf. Section 6 for a discussion) and it depends linearly on the velocity field u. Furthermore, on the flat portions of $\partial\Omega$ we have that $\mathcal{W} = 0$ (since ν is constant). Hence, there is genuine agreement between the boundary conditions (2.6) and (2.2) on the flat patches of $\partial\Omega$. Finally, we wish to point out that using the language of differential forms on $\mathbb{R}^n$ (so that d denotes the exterior derivative operator, $*$ stands for the Hodge-star operator, and $\wedge$, $\vee$ are the exterior and interior product, respectively) and canonically identifying vector fields with 1-forms, we have

$$-\nu \times \operatorname{curl} u = \nu \vee du, \tag{2.10}$$

in the three-dimensional setting. Indeed, for any 1-form u in $\mathbb{R}^n$ we have

$$\begin{aligned}
\nu \vee du &= \sum_{i=1}^{n} \nu_i\Big(\sum_{j,k=1}^{n} \partial_j u_k[dx_i \vee (dx_j \wedge dx_k)]\Big)\\
&= \sum_{1\leq j<k\leq n} \nu_j(\partial_j u_k - \partial_k u_j)dx_k - \sum_{1\leq j<k\leq n} \nu_k(\partial_j u_k - \partial_k u_j)dx_j\\
&= \sum_{j=1}^{n}\Big(\sum_{k=1}^{n} \nu_k(\partial_k u_j - \partial_j u_k)\Big)dx_j = \sum_{j=1}^{n}(\partial_\nu u_j - \nu\cdot\partial_j u)dx_j,
\end{aligned}$$

so (2.10) follows from (2.1). Hence, the correct substitute for $\nu \times \operatorname{curl} u = 0$ in the n-dimensional setting is $\nu \vee du = 0$. In particular, when $n = 2$ this takes the simpler form $du = 0$, or $\partial_2 u_1 - \partial_1 u_2 = 0$. Indeed, in general we have $du = \nu \wedge (\nu \vee du) + \nu \vee (\nu \wedge du)$ and, in the two-dimensional setting, the 3-form $\nu \wedge du$ necessarily vanishes.

3. An inverse of the curl, modulo gradient vectors

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. The Bessel potential scale $L^p_s(\Omega)$ is then defined, for $s \in \mathbb{R}$ and $1 < p < \infty$, by

$$L^p_s(\Omega) := \{f|_\Omega : \ f \in (I - \Delta)^{-s/2} L^p(\mathbb{R}^3)\}, \tag{3.1}$$

equipped with the natural infimum norm, which we shall denote by $\|\cdot\|_{s,p}$. As is well known, if k is a nonnegative integer, then

$$L^p_k(\Omega) = \{f \in L^p(\Omega) : \ \textstyle\sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{L^p(\Omega)} < \infty\}, \tag{3.2}$$

the classical L^p-based Sobolev space of order k in Ω.

Assume next that $\Omega \subset \mathbb{R}^3$ is a bounded domain, star-shaped with respect to a ball $B \subset \Omega$. In particular, from the lemma on page 20 in [19], it follows that Ω is a Lipschitz domain. In this setting, we proceed to review an assortment of results from Section 4 of [20], phrased in the context and terminology of the current paper. To set the stage, fix a function $\theta \in \mathcal{C}^\infty_c(B)$ with the property that $\int \theta = 1$. Then there exist three linear operators K_1, K_2 and K_3 such that

$$K_\ell : \Big(\mathcal{C}^\infty_c(\Omega; \Lambda^\ell)\Big)' \longrightarrow \Big(\mathcal{C}^\infty_c(\Omega; \Lambda^{\ell-1})\Big)', \quad 1 \leq \ell \leq 3,$$

where we have set $\Lambda^0 := \mathbb{R}$, $\Lambda^1 := \mathbb{R}^3$, $\Lambda^2 := \mathbb{R}^3$ and $\Lambda^3 := \mathbb{R}$. The operators K_ℓ are regularizing of order one in the sense that

$$K_\ell : L^p(\Omega; \Lambda^\ell) \longrightarrow L^p_1(\Omega; \Lambda^{\ell-1}), \quad p \in (1, \infty), \tag{3.3}$$

and

$$K_\ell : L^p_{-1}(\Omega; \Lambda^\ell) \longrightarrow L^p(\Omega; \Lambda^{\ell-1}), \quad p \in (1, \infty). \tag{3.4}$$

Moreover, for $u : \Omega \to \Lambda^\ell$ sufficiently smooth, the following formulas are valid:

$$\begin{array}{lcll} u & = & K_1(\nabla u) + \int_\Omega (\theta u) & \text{for } \ell = 0, \\ u & = & K_2(\operatorname{curl} u) + \nabla(K_1 u) & \text{for } \ell = 1, \\ u & = & K_3(\operatorname{div} u) + \operatorname{curl}(K_2 u) & \text{for } \ell = 2, \\ u & = & \operatorname{div}(K_3 u) & \text{for } \ell = 3. \end{array} \tag{3.5}$$

When acting on a sufficiently smooth function $u : \Omega \to \Lambda^\ell$, the operators K_ℓ take the following form:

$$K_\ell u(x) = \int_B \int_0^1 t^{\ell-1} \theta(y)(x-y) \times_\ell u(tx + (1-t)y)\, dt dy, \quad x \in \Omega, \ 1 \leq \ell \leq 3, \tag{3.6}$$

where $\times_\ell$ denotes, respectively, the scalar product between two vectors if $\ell = 1$, the cross product between two vectors if $\ell = 2$, and multiplication of a scalar and a vector if $\ell = 3$. Assume now that $p \in (p_\Omega, q_\Omega)$. By the second equality in (3.5), for $u \in X_p$ such that $\operatorname{curl} u \in L^p(\Omega; \mathbb{R}^3)$, we have

$$u = \mathbb{P}_p u = \mathbb{P}_p(K_2 \operatorname{curl} u). \tag{3.7}$$

At this stage, (3.7) and Theorem 1.1 suggest the following.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain which is star-shaped with respect to a ball, and let $p \in (p_\Omega, q_\Omega)$. Fix $q \in (p, q_\Omega)$ and assume that $\alpha \in (0,1)$ is such that $\frac{1}{p} - \frac{\alpha}{3} = \frac{1}{q}$. Then the Hodge-Stokes semi-group $(e^{-tA_p})_{t \geq 0}$, considered in X_p, satisfies the estimate*

$$\sup_{t \geq 0} \| t^{\frac{\alpha}{2}} e^{-tA_p} \|_{X_p \to L^q} + \sup_{t \geq 0} \| t^{\frac{1+\alpha}{2}} \operatorname{curl} e^{-tA_p} \|_{X_p \to L^q} < \infty. \tag{3.8}$$

Proof. For each $u \in X_p \cap X_q (= X_q)$, which is a dense subspace of X_p, Theorem 1.1 gives that

$$\| e^{-tA_p} u \|_p \leq c \, \| u \|_p, \qquad \text{for } t > 0 \tag{3.9}$$

and

$$\| \sqrt{t} \, \operatorname{curl} e^{-tA_p} u \|_p \leq c \, \| u \|_p, \qquad \text{for } t > 0. \tag{3.10}$$

From (3.4) and (3.9) we then obtain

$$\begin{aligned} \| K_2 \operatorname{curl} e^{-tA_p} u \|_p &\leq c \, \| \operatorname{curl} e^{-tA_p} u \|_{-1,p} \\ &\leq c \, \| e^{-tA_p} u \|_p \leq c \, \| u \|_p, \qquad \text{for } t > 0. \end{aligned} \tag{3.11}$$

Going further, from (3.10) and (3.3), we also have

$$\| \sqrt{t} K_2 \operatorname{curl} e^{-tA_p} u \|_{1,p} \leq c \, \| \sqrt{t} \operatorname{curl} e^{-tA_p} u \|_p \leq c \, \| u \|_p, \quad \text{for } t > 0. \tag{3.12}$$

If α is as in the statement of the theorem, (3.11), (3.12) plus standard interpolation and embedding estimates give that

$$\begin{aligned} \| t^{\frac{\alpha}{2}} \, K_2 \operatorname{curl} e^{-tA_p} u \|_q &\leq c \, \| t^{\frac{\alpha}{2}} \, K_2 \operatorname{curl} e^{-tA_p} u \|_{\alpha,p} \\ &\leq c \, \| K_2 \operatorname{curl} e^{-tA_p} u \|_p^{1-\alpha} \| \sqrt{t} \, K_2 \operatorname{curl} e^{-tA_p} u \|_{1,p}^{\alpha} \\ &\leq c \, \| u \|_p, \qquad \text{for } t > 0. \end{aligned} \tag{3.13}$$

The fact that $u \in X_q$ guarantees that $e^{-tA} u \in X_q$. Also, as already discussed, the projection $\mathbb{P}_q$ is known to be bounded on $L^q(\Omega; \mathbb{R}^3)$. Using (3.7) (with p replaced by q and u replaced by $e^{-tA} u$) and (3.13) we may write

$$\| t^{\frac{\alpha}{2}} e^{-tA_p} u \|_q \;=\; \| \mathbb{P}_q (t^{\frac{\alpha}{2}} e^{-tA_p} u) \|_q = \| \mathbb{P}_q (t^{\frac{\alpha}{2}} \, K_2 \operatorname{curl} e^{-tA_p} u) \|_q$$

$$\leq \quad c\,\|t^{\frac{\alpha}{2}}\,K_2\,\mathrm{curl}\,e^{-tA_p}u\|_q \leq c\,\|u\|_p, \qquad \text{for } t>0. \quad (3.14)$$

This accounts for the first part of (3.8). As for the second part of (3.8), we use the semi-group property, Theorem 1.1, as well as (3.14), in order to write

$$\begin{aligned}\|\mathrm{curl}\,e^{-tA_p}u\|_q &\leq \|\mathrm{curl}\,e^{-\frac{t}{2}A_q}\|_{X_q\to L^q}\|e^{-\frac{t}{2}A_p}u\|_q \\ &\leq c\Big(\frac{t}{2}\Big)^{-\frac{1}{2}}\Big(\frac{t}{2}\Big)^{-\frac{\alpha}{2}}\|u\|_p, \qquad \text{for } t>0. \qquad (3.15)\end{aligned}$$

This completes the proof of the theorem. □

4. The case of arbitrary Lipschitz domains

Let $\Omega\subset\mathbb{R}^3$ be an arbitrary bounded Lipschitz domain (not necessarily star-shaped with respect to a ball, as assumed in most of Section 2). By Lemmas 1-2 on page 22 and the lemma on page 25 of [19], there exist a finite, open cover of $\bar\Omega$ by domains star-shaped with respect to a ball and a smooth partition of unity subordinate to this cover. More specifically, there exists a family of open sets $\mathcal{O}_j$ such that

$$\bar\Omega\subset\bigcup_{j=1}^{N}\mathcal{O}_j \qquad (4.1)$$

and, for $j=1,...,N$, the domain $\Omega_j:=\Omega\cap\mathcal{O}_j$ is star-shaped with respect to a ball B_j, along with a family of functions $\phi_j\in\mathcal{C}^\infty_c(\Omega_j)$, $1\leq j\leq N$, such that

$$\sum_{j=1}^{N}\phi_j^2(x)=1, \qquad \text{for all } x\in\bar\Omega. \qquad (4.2)$$

For each $j=1,...,N$, select $\theta_j\in\mathcal{C}^\infty(\Omega_j)$ with $\mathrm{supp}\,\theta_j\subset B_j$ and $\int_{B_j}\theta_j=1$, and then define K^j_ℓ as in (3.6), relative to the domain Ω_j. These operators satisfy properties similar to (3.5), in each domain Ω_j. Finally, for $\ell=1,2,3$, we introduce

$$\begin{aligned}&K_\ell:(\mathcal{C}^\infty_c(\Omega,\Lambda^\ell))'\longrightarrow(\mathcal{C}^\infty_c(\Omega,\Lambda^{\ell-1}))'\\ &K_\ell u:=\textstyle\sum_{j=1}^{N}\phi_jK^j_\ell(\phi_ju),\end{aligned} \qquad (4.3)$$

and note that the mapping properties (3.3)-(3.4) remain valid for K_ℓ defined above, even if the bounded Lipschitz domain Ω is not necessarily star-shaped with respect to a ball.

Moreover, by (3.5), for each $u \in (\mathcal{C}_c^\infty(\Omega, \mathbb{R}^3))'$ we may write

$$\begin{aligned}
\nabla(K_1 u) &\overset{(1)}{=} \sum_{j=1}^N (\nabla\phi_j) K_1^j(\phi_j u) + \sum_{j=1}^N \phi_j \nabla(K_1^j(\phi_j u)) \\
&\overset{(2)}{=} \sum_{j=1}^N (\nabla\phi_j) K_1^j(\phi_j u) + \sum_{j=1}^N \phi_j \Big(\phi_j u - K_2^j(\mathrm{curl}\,(\phi_j u))\Big) \\
&\overset{(3)}{=} \sum_{j=1}^N (\nabla\phi_j) K_1^j(\phi_j u) + u - K_2\,\mathrm{curl}\,u - \sum_{j=1}^N \phi_j K_2^j(\nabla\phi_j \times u).
\end{aligned}$$

Equality (1) is routine algebra. Equality (2) is a consequence of (3.5) applied to K_ℓ^j. Finally, equality (3) follows from (4.1) and the identity $\mathrm{curl}\,(\phi_j u) = \phi_j \mathrm{curl}\,u + \nabla\phi_j \times u$. Introducing

$$\begin{aligned}
&R : (\mathcal{C}_c^\infty(\Omega, \mathbb{R}^3))' \longrightarrow (\mathcal{C}_c^\infty(\Omega, \mathbb{R}^3))' \\
&Ru := \textstyle\sum_{j=1}^N \Big(\phi_j K_2^j(\nabla\phi_j \times u) - (\nabla\phi_j) K_1^j(\phi_j u)\Big),
\end{aligned} \tag{4.4}$$

allows us to rephrase the identity just derived in the form

$$u = \nabla(K_1 u) + K_2\,\mathrm{curl}\,u + Ru, \qquad \forall\, u \in \big(\mathcal{C}_c^\infty(\Omega, \mathbb{R}^3)\big)'. \tag{4.5}$$

The following is an extension of Theorem 3.1 to arbitrary Lipschitz domains.

Theorem 4.1. *Assume that $\Omega \subset \mathbb{R}^3$ is an arbitrary bounded Lipschitz domain. Fix $p \in (p_\Omega, q_\Omega)$ and $q \in (p, q_\Omega)$ such that $\frac{1}{p} - \frac{\alpha}{3} = \frac{1}{q}$ for some $\alpha \in (0,1)$. Then the Hodge-Stokes semi-group $(e^{-tA_p})_{t\geq 0}$, considered in X_p, satisfies the estimate*

$$\sup_{t\geq 0} \|t^{\frac{\alpha}{2}} e^{-tA_p}\|_{X_p \to L^q} + \sup_{t\geq 0} \|t^{\frac{1+\alpha}{2}}\,\mathrm{curl}\, e^{-tA_p}\|_{X_p \to L^q} < \infty. \tag{4.6}$$

Proof. Up to (and including) (3.13), we follow the same arguments as in the proof of Theorem 3.1, with K_2 defined as in (4.3). It is in (3.14) that the operator R intervenes for the first time, when (4.5) is employed in place of (3.7). To estimate its contribution, we first note that, from (4.4) and the discussion in the first part of Section 2, we have

$$R : L^p(\Omega, \mathbb{R}^3) \longrightarrow L_1^p(\Omega, \mathbb{R}^3) \quad \text{bounded, whenever } 1 < p < \infty. \tag{4.7}$$

Thus, based on (4.7) and well-known properties of analytic semi-groups (cf., e.g., Theorem 6.13 on page 74 in [24]), we may estimate

$$\|t^{\frac{\alpha}{2}} R(e^{-tA_p} u)\|_q \quad\leq\quad c\,\|t^{\frac{\alpha}{2}} R(e^{-tA_p} u)\|_{\alpha,p}$$

$$\begin{aligned}
&\leq c\,\|R(e^{-tA_p}u)\|_p^{1-\alpha}\|\sqrt{t}\,R(e^{-tA_p}u)\|_{1,p}^{\alpha} \\
&\leq c\,\|t^{\frac{\alpha}{2}}e^{-tA_p}u\|_p = c\,\|A^{-\frac{\alpha}{2}}(t^{\frac{\alpha}{2}}A^{\frac{\alpha}{2}}e^{-tA_p}u)\|_p \\
&\leq c\,\|A^{-\frac{\alpha}{2}}\|_{X_p\to X_p}\|t^{\frac{\alpha}{2}}A^{\frac{\alpha}{2}}e^{-tA_p}\|_{X_p\to X_p}\|u\|_p \\
&= c\,\|u\|_p, \qquad \text{for } t>0, \qquad (4.8)
\end{aligned}$$

which is of the right order. With this in hand, we may then conclude much as in the endgame of the proof of Theorem 3.1. □

5. Mild solution to the Hodge-Navier-Stokes system

Throughout this section, Ω denotes a bounded Lipschitz domain in $\mathbb{R}^3$. Let $T>0$ be fixed and assume that $\varepsilon>0$ is such that $3(1+\varepsilon)<q_\Omega$, where q_Ω was defined in Theorem 1.1. Introduce the following Banach space:

$$\mathcal{E}_T := \Big\{u\in\mathcal{C}([0,T];X_3)\cap\mathcal{C}((0,T];L^{3(1+\varepsilon)}(\Omega;\mathbb{R}^3)) : \operatorname{curl} u\in\mathcal{C}((0,T];L^3(\Omega;\mathbb{R}^3)) \text{ with } \sup_{0<s<T}\Big(\|u(s)\|_3+\|s^{\frac{\varepsilon}{2(1+\varepsilon)}}u(s)\|_{3(1+\varepsilon)}+\|\sqrt{s}\operatorname{curl} u(s)\|_3\Big)<\infty\Big\}$$

endowed with the norm

$$\|u\|_{\mathcal{E}_T} := \sup_{0<s<T}\|u(s)\|_3+\sup_{0<s<T}\|s^{\frac{\varepsilon}{2(1+\varepsilon)}}u(s)\|_{3(1+\varepsilon)}+\sup_{0<s<T}\|\sqrt{s}\operatorname{curl} u(s)\|_3.$$

Proposition 5.1. *Let $u_0\in X_3$ be arbitrary and set $a(t):=e^{-tA}u_0$, for all $t\geq 0$. Then $a\in\mathcal{E}_T$ and $\|a\|_{\mathcal{E}_T}\leq c\,\|u\|_{X_3}$.*

Proof. That $a\in\mathcal{C}([0,T];X_3)$ is a consequence of the fact that $(e^{-tA})_{t\geq 0}$ is a $\mathcal{C}_0$ semi-group in X_3. Thanks to Theorem 1.1, we have

$$\operatorname{curl} a\in\mathcal{C}((0,T];L^3(\Omega;\mathbb{R}^3))$$

with $\sup_{0<s<T}\|\sqrt{s}\operatorname{curl} a(s)\|_3\leq\|u_0\|_3$. By (3.8) with $p=3$, we also get that $a\in L^q(\Omega;\mathbb{R}^3)$ for all $q\in(3,q_\Omega)$ and

$$\sup_{0<s<T}\|s^{\frac{\alpha}{2}}a(s)\|_q\leq\|u_0\|_3$$

provided $\frac{1}{q}=\frac{1}{3}-\frac{\alpha}{3}$. In particular, the choice $q=3(1+\varepsilon)$ entails $\alpha=\frac{\varepsilon}{1+\varepsilon}\in(0,1)$. □

Lemma 5.2. *Let $u,v\in\mathcal{E}_T$ be arbitrary. Then*

$$u\times\operatorname{curl} v\in\mathcal{C}((0,T];L^{\frac{3(1+\varepsilon)}{2+\varepsilon}}(\Omega;\mathbb{R}^3))$$

and

$$\sup_{0<s<T} \|s^{\frac{1+2\varepsilon}{2(1+\varepsilon)}} u(s) \times \operatorname{curl} v(s)\|_{\frac{3(1+\varepsilon)}{2+\varepsilon}} \leq \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}. \tag{5.1}$$

Proof. For $u, v \in \mathcal{E}_T$, it is clear that $u \times \operatorname{curl} v \in \mathcal{C}((0,T]; L^q(\Omega, \mathbb{R}^3))$ provided

$$\frac{1}{q} = \frac{1}{3} + \frac{1}{3(1+\varepsilon)} = \frac{2+\varepsilon}{3(1+\varepsilon)}.$$

For later use, let us point out here that

$$q := \frac{3(1+\varepsilon)}{2+\varepsilon} \in (p_\Omega, q_\Omega), \tag{5.2}$$

since $p_\Omega < (3(1+\varepsilon))' = \frac{3(1+\varepsilon)}{2+3\varepsilon} < q < 3$. Moreover, since $\|s^{\frac{\varepsilon}{1+\varepsilon}} u(s)\|_{3(1+\varepsilon)} \leq \|u\|_{\mathcal{E}_T}$ for all $s \in (0,T)$ and $\|\sqrt{s} \operatorname{curl} u(s)\|_3 \leq \|v\|_{\mathcal{E}_T}$ for all $s \in (0,T)$, we get

$$\|s^\beta u(s) \times \operatorname{curl} v(s)\|_{\frac{3(1+\varepsilon)}{2+\varepsilon}} \leq \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} \quad \text{ for all } s \in (0,T),$$

where $\beta := \frac{1}{2} + \frac{\varepsilon}{2(1+\varepsilon)} = \frac{1+2\varepsilon}{2(1+\varepsilon)}$. This proves (5.1). □

Consider next the mapping Φ defined on $\mathcal{E}_T \times \mathcal{E}_T$ by

$$\begin{aligned}&[0,T] \ni t \mapsto \Phi(u,v)(t)\\ &\qquad = \int_0^t e^{-(t-s)A} (-\tfrac{1}{2} \mathbb{P}_{\frac{3(1+\varepsilon)}{2+\varepsilon}}) \Big(u(s) \times \operatorname{curl} v(s) + v(s) \times \operatorname{curl} u(s) \Big)\, ds.\end{aligned}$$

Proposition 5.3. *The mapping Φ is bilinear, symmetric, and continuous from $\mathcal{E}_T \times \mathcal{E}_T$ into $\mathcal{E}_T$.*

Proof. The fact that Φ is linear and symmetric is clear from its definition. We shall focus on proving that $\Phi(u,v)$ belongs to $\mathcal{E}_T$ whenever $u, v \in \mathcal{E}_T$. The continuity of Φ follows *a posteriori* from the estimates implicit in the justification of this membership. To get started, we note that from (5.1)-(5.2) we have

$$\Big\| (-\tfrac{1}{2} \mathbb{P}_{\frac{3(1+\varepsilon)}{2+\varepsilon}}) \big(u(s) \times \operatorname{curl} v(s) + v(s) \times \operatorname{curl} u(s) \big) \Big\|_{\frac{3(1+\varepsilon)}{2+\varepsilon}} \leq s^{-\frac{1+2\varepsilon}{2(1+\varepsilon)}} \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T},$$

for all $s \in (0,T]$. The special case of (3.8) when $p = \frac{3(1+\varepsilon)}{2+\varepsilon}$, $q = 3$ and $\alpha = \frac{1}{1+\varepsilon}$, yields

$$\begin{aligned}&\Big\| e^{-(t-s)A} (-\tfrac{1}{2} \mathbb{P}_{\frac{3(1+\varepsilon)}{2+\varepsilon}}) \big(u(s) \times \operatorname{curl} v(s) + v(s) \times \operatorname{curl} u(s) \big) \Big\|_3 \\ &\qquad \leq c\, s^{-\frac{1+2\varepsilon}{2(1+\varepsilon)}} (t-s)^{-\frac{1}{2(1+\varepsilon)}} \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T},\end{aligned} \tag{5.3}$$

and

$$\Big\|\operatorname{curl} e^{-(t-s)A}\big(-\tfrac{1}{2}\mathbb{P}_{\frac{3(1+\varepsilon)}{2+\varepsilon}}\big)\big(u(s)\times\operatorname{curl} v(s)+v(s)\times\operatorname{curl} u(s)\big)\Big\|_3$$
$$\leq c\, s^{-\frac{1+2\varepsilon}{2(1+\varepsilon)}}(t-s)^{-\frac{2+\varepsilon}{2(1+\varepsilon)}}\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}, \tag{5.4}$$

for all $0 < s \leq t \leq T$. Applying now (3.8) with $p = \frac{3(1+\varepsilon)}{2+\varepsilon}$, $q = 3(1+\varepsilon)$ and $\alpha = 1$, we obtain

$$\Big\|e^{-(t-s)A}\big(-\tfrac{1}{2}\mathbb{P}_{\frac{3(1+\varepsilon)}{2+\varepsilon}}\big)\big(u(s)\times\operatorname{curl} v(s)+v(s)\times\operatorname{curl} u(s)\big)\Big\|_{3(1+\varepsilon)}$$
$$\leq c\, s^{-\frac{1+2\varepsilon}{2(1+\varepsilon)}}(t-s)^{-\frac{1}{2}}\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}. \tag{5.5}$$

From the estimate (5.3) it follows that, for each $t \in (0, T]$,

$$\begin{aligned}\|\Phi(u,v)(t)\|_3 &\overset{(1)}{\leq} c\Big(\int_0^t s^{-\frac{1+2\varepsilon}{2(1+\varepsilon)}}(t-s)^{-\frac{1}{2(1+\varepsilon)}}\,ds\Big)\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}\\ &\overset{(2)}{\leq} c\Big(\int_0^1 \sigma^{-\frac{1+2\varepsilon}{2(1+\varepsilon)}}(1-\sigma)^{-\frac{1}{2(1+\varepsilon)}}\,d\sigma\Big)\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}.\end{aligned} \tag{5.6}$$

The inequality (1) above is obtained by integrating (5.3) between 0 and t. Passing from (1) to (2) is done by making the change of variables $s = t\sigma$ in the integral just alluded to.

By the same method, and relying on the estimate (5.4), we obtain that for each $t \in (0, T]$

$$\|\operatorname{curl}\Phi(u,v)(t)\|_3 \leq c\Big(\int_0^t s^{-\frac{1+2\varepsilon}{2(1+\varepsilon)}}(t-s)^{-\frac{2+\varepsilon}{2(1+\varepsilon)}}\,ds\Big)\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T} \tag{5.7}$$

$$\leq c\, t^{-\frac{1}{2}}\Big(\int_0^1 \sigma^{-\frac{1+2\varepsilon}{2(1+\varepsilon)}}(1-\sigma)^{-\frac{2+\varepsilon}{2(1+\varepsilon)}}\,d\sigma\Big)\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}.$$

Using now the estimate (5.5) and employing the same method, we see that for all $t \in (0, T]$

$$\|\Phi(u,v)(t)\|_{3(1+\varepsilon)} \leq c\Big(\int_0^t s^{-\frac{1+2\varepsilon}{2(1+\varepsilon)}}(t-s)^{-\frac{1}{2}}\,ds\Big)\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T} \tag{5.8}$$

$$\leq c\, t^{-\frac{\varepsilon}{2(1+\varepsilon)}}\Big(\int_0^1 \sigma^{-\frac{1+2\varepsilon}{2(1+\varepsilon)}}(1-\sigma)^{-\frac{1}{2}}\,d\sigma\Big)\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}.$$

In concert, the estimates (5.6), (5.7) and (5.8) then imply that $\Phi(u,v) \in \mathcal{E}_T$. □

We are now ready to prove the existence of solutions for the functional analytic equation

$$u = a + \Phi(u,u). \tag{5.9}$$

We shall refer to these as *mild solutions of the Hodge-Navier-Stokes system* (1.4).

Theorem 5.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and fix $T > 0$. Then there exists $\delta > 0$ with the property that for each $u_0 \in X_3$ with $\|u_0\|_3 < \delta$ there exists a unique mild solution u of the Hodge-Navier-Stokes system* (1.4) *(i.e., a function $u \in \mathcal{E}_T$ satisfying* (5.9) *on $[0,T]$).*

Proof. Since $\Phi : \mathcal{E}_T \times \mathcal{E}_T \to \mathcal{E}_T$ is bilinear and continuous, the idea is to implement Picard's fixed-point theorem. As in [14], the sequence $(v_n)_{n\in\mathbb{N}}$ of functions in $\mathcal{E}_T$ defined by $v_0 := a$, as the first term, and then, iteratively,

$$v_{n+1} := a + \Phi(v_n, v_n), \quad n \in \mathbb{N}$$

converges to the unique solution $u \in \mathcal{E}_T$ of (5.9) provided $\|u_0\|_{X_3}$ is small enough so that, say, $\|a\|_{\mathcal{E}_T} < \frac{1}{4\|\Phi\|_{\mathcal{L}(\mathcal{E}_T\times\mathcal{E}_T;\mathcal{E}_T)}}$. That this can be ensured is guaranteed by Proposition 5.1. □

6. Appendix

Let M be a $\mathcal{C}^2$ manifold, possibly with boundary, of (real) dimension n. As usual, by TM and T^*M we denote, respectively, the tangent and cotangent bundle on M. We shall also denote by TM global ($\mathcal{C}^1$) sections in TM (i.e., $TM \equiv \mathcal{C}^1(M,TM)$). Similarly, we identify $T^*M \equiv \mathcal{C}^1(M,T^*M)$. We shall assume that M is equipped with a $\mathcal{C}^1$ Riemannian metric tensor $g = \sum_{j,k} g_{jk} dx_j \otimes dx_k$ and denote by ∇ the associated Levi-Civita connection. Among other things, the metric property

$$Z\langle X,Y\rangle = \nabla_Z \langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle, \quad \forall\, X,Y,Z \in TM, \tag{6.1}$$

holds. Consider next $\mathcal{S} \hookrightarrow M$, a $\mathcal{C}^2$, orientable sub-manifold of codimension one in M, and fix some $\nu \in TM$ such that $\nu|_{\mathcal{S}}$ becomes the outward unit normal to $\mathcal{S}$. If $\nabla^{\mathcal{S}}$ is the induced Levi-Civita connection on $\mathcal{S}$ (from the metric inherited from M) it is then well known that

$$\nabla^{\mathcal{S}}_X Y = \pi(\nabla_X Y), \quad \forall\, X,Y \in T\mathcal{S}, \tag{6.2}$$

where $\pi : TM \longrightarrow T\mathcal{S}$ is the canonical orthogonal projection onto $T\mathcal{S}$, the tangent bundle to $\mathcal{S}$. In particular, the second fundamental form of $\mathcal{S}$

becomes

$$II(X,Y) := \nabla_X Y - \nabla_X^{\mathcal{S}} Y = \pi(\nabla_X Y), \qquad \forall\, X, Y \in T\mathcal{S}. \tag{6.3}$$

In this setting, the Weingarten map

$$\mathcal{W} : T\mathcal{S} \longrightarrow T\mathcal{S}, \tag{6.4}$$

originally defined uniquely by the requirement that

$$\langle \mathcal{W}X, Y\rangle = \langle \nu, II(X,Y)\rangle, \qquad \forall\, X, Y \in T\mathcal{S}, \tag{6.5}$$

reduces to

$$\mathcal{W}X = -\nabla_X \nu \quad \text{on } \mathcal{S}, \qquad \forall\, X \in T\mathcal{S}, \tag{6.6}$$

known as the Weingarten formula. An excellent reference for the material in this section is [27]. The following propositions, proved in [10], describes an extension of the unit normal to a hypersurface enjoying a number of useful properties.

Proposition 6.1. *For a $\mathcal{C}^2$ surface $\mathcal{S} \subseteq \mathbb{R}^n$ there exists a neighborhood $\mathcal{U}$ of $\mathcal{S}$ along with a vector field $\nu \in \mathcal{C}^1(\mathcal{U})$ with the following properties:*

(i) $\|\nu\| = 1$ *in* $\mathcal{U}$;
(ii) $\nu|_{\mathcal{S}}$ *coincides with the unit normal to* $\mathcal{S}$;
(iii) $\nabla_\nu \nu = 0$ *on* $\mathcal{S}$; *i.e.,* $\partial_\nu \nu_j = 0$ *on* $\mathcal{S}$ *for* $j = 1, 2, ..., n$;
(iv) $d\nu = 0$ *on* $\mathcal{S}$; *i.e.,* $\partial_k \nu_j - \partial_j \nu_k = 0$ *on* $\mathcal{S}$, *for* $k,\ j = 1,\ 2, \dots, n$;
(v) $\mathrm{div}\nu\Big|_{\mathcal{S}} = (n-1)\mathcal{H}$, *where* $\mathcal{H}$ *stands for the mean curvature of* $\mathcal{S}$.

Moreover, for the $n \times n$ matrix-valued function

$$R(x) := \nabla\nu(x) = (\partial_k \nu_j(x))_{j,k}, \qquad x \in \mathcal{U}, \tag{6.7}$$

the following are true:

(vi) $R\nu = 0$ *in* $\mathcal{U}$;
(vii) $Tr(R)|_{\mathcal{S}} = (n-1)\mathcal{H}$.

In addition, when restricted to the hypersurface $\mathcal{S}$, R has the following additional properties:

(viii) *R depends only on $\mathcal{S}$ and not on the choice of the extended unit ν;*
(ix) $R^\top = R$ *on* $\mathcal{S}$;
(x) *$(Ru)|_{\mathcal{S}}$ is tangent to $\mathcal{S}$ for any vector field $u : \mathcal{S} \to \mathbb{R}^n$. In fact,*

$$R\Big|_{T\mathcal{S}} = -\mathcal{W}, \tag{6.8}$$

the opposite of the Weingarten map of $\mathcal{S}$. In particular, the eigenvalues $\{\kappa_j\}_{1\le j\le n-1}$ of $-R$ (at points on $\mathcal{S}$) as an operator on $T\mathcal{S}$ are *the principal curvatures of $\mathcal{S}$, and its determinant is Gauss's total curvature of $\mathcal{S}$.*

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