Weighted Sobolev Space Estimates for a Class of Singular Integral Operators

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In celebration of the distinguished career
of our esteemed friend, V.G. Maz'ya

Abstract The aim of this paper is to prove the boundedness of a category of integral operators mapping functions from Besov spaces on the boundary of a Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ into functions belonging to weighted Sobolev spaces in Ω. The model we have in mind is the Poisson integral operator

$$(\mathrm{PI}f)(x) := -\int_{\partial\Omega} \partial_{\nu(y)} G(x,y) f(y)\, d\sigma(y), \quad x \in \Omega,$$

where $G(\cdot,\cdot)$ is the Green function for the Dirichlet Laplacian in Ω, ∂_ν is the normal derivative, and σ is the surface area on $\partial\Omega$, in the case where $\Omega \subseteq \mathbb{R}^n$ is a bounded Lipschitz domain satisfying a uniform exterior ball condition.

1 Introduction

The main result of this paper is the following theorem.

Theorem 1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 2$. Denote by σ the surface measure on $\partial\Omega$, and set*

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$$\delta(x) := \operatorname{dist}(x, \partial\Omega), \quad x \in \mathbb{R}^n. \tag{1.1}$$

Consider the integral operator

$$\mathcal{Q}f(x) := \int_{\partial\Omega} q(x,y) f(y)\, d\sigma(y), \quad x \in \Omega, \tag{1.2}$$

satisfying the following conditions:

(1) $\mathcal{Q}1 =$*constant in* Ω,

(2) *there exists* $N \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ *and* $\varepsilon \in [0,1)$ *such that for each* $k \in \{0,1,\dots,N\}$

$$|\nabla_x^{k+1} q(x,y)| \leqslant c_o\, \delta(x)^{-k-\varepsilon} |x-y|^{-n+\varepsilon} \tag{1.3}$$

for all $x \in \Omega$ *and almost every* $y \in \partial\Omega$*, for some constant* $c_o = c_o(\Omega, k) > 0$.

Assume that

$$\frac{n-1}{n-\varepsilon} < p \leqslant \infty, \quad (n-1)\Big(\frac{1}{p} - 1\Big)_+ < s < 1 - \varepsilon. \tag{1.4}$$

Then for each $k \in \{0,1,2,\dots,N\}$ *there exists* $C = C(\Omega, p, s, k) > 0$ *such that*

$$\|\delta^{k+1-\frac{1}{p}-s} |\nabla^{k+1} \mathcal{Q}f| \|_{L^p(\Omega)} + \sum_{j=0}^{k} \|\nabla^j \mathcal{Q}f\|_{L^p(\Omega)} \leqslant C \|f\|_{B_s^{p,p}(\partial\Omega)} \tag{1.5}$$

for every $f \in B_s^{p,p}(\partial\Omega)$.

Above, $|\nabla^k u| := \sum_{|\beta| \leqslant k} |\partial^\beta u|$ and $(a)_+ := \max\{a, 0\}$. Also, $B_s^{p,p}(\partial\Omega)$ denotes the (diagonal) Besov scale on $\partial\Omega$ (cf. Section 2 for more details).

The primary motivation for considering this type of result comes from the study of the Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad u \in B_{s+1/p}^{p,q}(\Omega), \quad \operatorname{Tr} u = f \in B_s^{p,q}(\partial\Omega), \tag{1.6}$$

where $B_\alpha^{p,q}(\Omega)$ denotes the scale of Besov spaces in Ω and Tr is the boundary trace operator, via the potential theoretic representation

$$u(x) = -\int_{\partial\Omega} \partial_{\nu(y)} G(x,y) f(y)\, d\sigma(y), \quad x \in \Omega, \tag{1.7}$$

where ν is the outward unit normal to $\partial\Omega$ and $G(x,y)$ is the Green function for the Dirichlet Laplacian in Ω. In this scenario, the *a priori* estimate

$$\|u\|_{B_{s+1/p}^{p,q}(\Omega)} \leqslant C \|f\|_{B_s^{p,q}(\partial\Omega)} \tag{1.8}$$

is equivalent to the boundedness of the Poisson integral operator

$$\mathrm{PI}: B^{p,q}_s(\partial\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega) \tag{1.9}$$

defined as

$$(\mathrm{PI}f)(x) := -\int_{\partial\Omega} \partial_{\nu(y)} G(x,y) f(y)\, d\sigma(y), \quad x \in \Omega. \tag{1.10}$$

In the case where $\Omega \subseteq \mathbb{R}^n$ is a smooth bounded domain, it is well known that

$$|\nabla_y G(x,y)| \leqslant C\, |x-y|^{1-n}, \quad x, y \in \Omega, \tag{1.11}$$

from which it can be deduced that, for every $k \in \mathbb{N}_0$,

$$|\nabla_x^{k+1} \nabla_y G(x,y)| \leqslant C\, \frac{|x-y|^{-n}}{\min\{|x-y|,\, \delta(x)\}^k} \quad \forall\, x \in \Omega,\ \forall\, y \in \overline{\Omega}. \tag{1.12}$$

As a consequence, in the case $\partial\Omega \in C^\infty$, the integral operator (1.10) has kernel $q(x,y) := -\partial_{\nu(y)} G(x,y)$ which satisfies conditions (1)–(2) of Theorem 1.1 and hence

$$\|\delta^{k+1-\frac{1}{p}-s} |\nabla^{k+1} \mathrm{PI}\, f|\|_{L^p(\Omega)} + \sum_{j=0}^{k} \|\nabla^j \mathrm{PI}\, f\|_{L^p(\Omega)} \leqslant C \|f\|_{B^{p,p}_s(\partial\Omega)} \quad \forall\, f \in B^{p,p}_s(\partial\Omega). \tag{1.13}$$

In order to pass from the weighted Sobolev space estimate (1.13) to the Besov estimate implicit in (1.9), we need an auxiliary regularity result which we now describe. Let L be a homogeneous, elliptic differential operator of even order with (possibly matrix-valued) constant coefficients. Fix a Lipschitz domain $\Omega \subset \mathbb{R}^n$. Denote by $\operatorname{Ker} L$ the space of functions u satisfying $Lu = 0$ in Ω. Then for $0 < p \leqslant \infty$ and $s \in \mathbb{R}$ denote by $\mathbb{H}^p_s(\Omega; L)$ the space of functions $u \in \operatorname{Ker} L$ subject to the size/smoothness condition

$$\|u\|_{\mathbb{H}^p_s(\Omega;L)} := \|\delta^{\langle s\rangle - s} |\nabla^{\langle s\rangle} u|\|_{L^p(\Omega)} + \sum_{j=0}^{\langle s\rangle -1} \|\nabla^j u\|_{L^p(\Omega)} < \infty. \tag{1.14}$$

Hereinafter, for a given $s \in \mathbb{R}$ we set

$$\langle s\rangle := \begin{cases} s, & s \in \mathbb{N}_0, \\ [s]+1, & s > 0,\ s \notin \mathbb{N}, \\ 0, & s < 0, \end{cases} \tag{1.15}$$

where $[\cdot]$ is the integer-part function, i.e., $\langle s\rangle$ is the smallest nonnegative integer greater than or equal to s. Let $F^{p,q}_\alpha(\Omega)$ denote the scale of Triebel–Lizorkin spaces in Ω (again, cf. Section 2 for definitions).

Theorem 1.2. *Let L be as above, and let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then for any $s \in \mathbb{R}$ and $p, q \in (0, \infty)$,*

$$\mathbb{H}^p_s(\Omega; L) = F^{p,q}_s(\Omega) \cap \operatorname{Ker} L. \tag{1.16}$$

Consequently,

$$F^{p,q}_s(\Omega) \cap \operatorname{Ker} L = B^{p,p}_s(\Omega) \cap \operatorname{Ker} L \tag{1.17}$$

for $s \in \mathbb{R}$ and $p, q \in (0, \infty)$. Finally, for $p = \infty$

$$\mathbb{H}^\infty_{k+s}(\Omega; L) = B^{\infty,\infty}_{k+s}(\Omega) \cap \operatorname{Ker} L \tag{1.18}$$

for any $k \in \mathbb{N}_0$ and $s \in (0, 1)$.

For $1 < p, q < \infty$, $s > 0$, this theorem was proved in [9] for $L = \Delta$ and in [1] for $L = \Delta^2$. The present formulation was stated in [14, 10]. The boundedness of the operator (1.9) directly follows from Theorem 1.2 and (1.13).

Consider the case of an irregular $\partial\Omega$. In this case, the estimate (1.11) is not necessarily satisfied. Indeed, if (1.11) holds, then the Green operator

$$\mathbb{G}v(x) := \int_\Omega G(x, y)v(y)\, dy, \quad x \in \Omega, \tag{1.19}$$

behaves itself like a fractional integral operator of order one. Thus, in particular,

$$\mathbb{G} : L^p(\Omega) \longrightarrow L^{p^*}(\Omega) \tag{1.20}$$

would be bounded whenever

$$1 < p < n \quad \text{and} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \tag{1.21}$$

by the Hardy–Littlewood–Sobolev fractional integration theorem (cf., for example, [22]). However, Dahlberg [4] showed that for a Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ the operator (1.20) is bounded only if $1 < p < p_n + \varepsilon$ (with $\varepsilon = \varepsilon(\Omega) > 0$), where

$$p_n := \frac{3n}{n+3} \quad \text{for} \quad n \geqslant 3 \quad \text{and} \quad p_2 := \frac{4}{3}. \tag{1.22}$$

By means of counterexamples, Dahlberg also showed that this result is sharp. Thus, as a consequence, the estimate (1.11) cannot hold in a general Lipschitz domain. Hence extra regularity properties need to be imposed.

Recall that $\Omega \subseteq \mathbb{R}^n$ satisfies the *uniform exterior ball condition* (henceforth abbreviated as UEBC) if, outside Ω, one call "roll" a ball of a fixed size along the boundary. It is easy to show that any convex domain satisfies

UEBC. Parenthetically, we also note that a bounded open set Ω has $C^{1,1}$ boundary if and only if Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ satisfy UEBC. However, UEBC alone does allow the boundary to develop irregularities which are "outwardly directed." Grüter and Widman [7] showed that if $\Omega \subset \mathbb{R}^n$ is a bounded open domain satisfying UEBC, then there exists $C = C(\Omega) > 0$ such that the Green function for the Dirichlet Laplacian satisfies the following estimates for all $x, y \in \Omega$:

(i) $G(x,y) \leqslant C \operatorname{dist}(x, \partial\Omega)|x-y|^{1-n}$;

(ii) $G(x,y) \leqslant C \operatorname{dist}(x, \partial\Omega) \operatorname{dist}(y, \partial\Omega)|x-y|^{-n}$;

(iii) $|\nabla_x G(x,y)| \leqslant C|x-y|^{1-n}$;

(iv) $|\nabla_x G(x,y)| \leqslant C \operatorname{dist}(y, \partial\Omega)|x-y|^{-n}$;

(v) $|\nabla_x \nabla_y G(x,y)| \leqslant C|x-y|^{-n}$.

Thus, it is possible to run the above program (based on Theorems 1.1 and 1.2) in order to conclude that for a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ satisfying UEBC the Poisson integral operator (1.9) is bounded whenever

$$0 < q \leqslant \infty, \quad \frac{n-1}{n} < p \leqslant \infty \text{ and } (n-1)\Big(\frac{1}{p} - 1\Big)_+ < s < 1. \tag{1.23}$$

In addition, a similar result is valid for

$$\mathrm{PI} : B^{p,p}_s(\partial\Omega) \longrightarrow F^{p,q}_{s+\frac{1}{p}}(\Omega) \tag{1.24}$$

provided that $p, q < \infty$ (cf. Theorem 3.4).

The layout of the paper is as follows. Section 2 contains a background material pertaining to Lipschitz domains; smoothness spaces defined first in the whole Euclidean space $\mathbb{R}^n$, then in open subsets of $\mathbb{R}^n$ and, finally, on Lipschitz surfaces of codimension one in $\mathbb{R}^n$, as well as basic interpolation results and Green function estimates. In Section 3, we deduce a number of estimates depending on the geometric properties of a domain, which are then used to prove the main result, Theorem 1.1. We mention that a result similar to Theorem 1.1 holds for matrix-valued kernels $q(\cdot,\cdot)$ and vector-valued functions f (in this case, condition (1) should read: $\mathcal{Q}$ maps constant vectors defined on $\partial\Omega$ into constant vectors in Ω). A result similar to Theorem 1.1, but for a more restrictive class of operators was proved in [19, 14].

2 Preliminaries

Recall that an open, bounded set Ω in $\mathbb{R}^n$ is called a bounded Lipschitz domain if there exists a finite open covering $\{\mathcal{O}_j\}_{1\leqslant j\leqslant N}$ of $\partial\Omega$ with the property that, for every $j \in \{1, ..., N\}$, $\mathcal{O}_j \cap \Omega$ coincides with the portion of $\mathcal{O}_j$ lying above the graph of a Lipschitz function $\varphi_j : \mathbb{R}^{n-1} \to \mathbb{R}$ (where $\mathbb{R}^{n-1} \times \mathbb{R}$ is a new system of coordinates obtained from the original one via a rigid motion). As is known, for a Lipschitz domain Ω (bounded or unbounded), the surface measure $d\sigma$ is well defined on $\partial\Omega$ and there exists an outward pointing normal vector $\nu = (\nu_1, \cdots, \nu_n)$ at almost every point on $\partial\Omega$. In particular, this allows one to define the Lebesgue scale in the usual fashion, i.e., for $0 < p \leqslant \infty$

$$L^p(\partial\Omega) := \Big\{ f : \partial\Omega \to \mathbb{R} : \ f \text{ measurable, and} \\ \|f\|_{L^p(\partial\Omega)} := \Big(\int_{\partial\Omega} |f|^p \, d\sigma \Big)^{1/p} < \infty \Big\}.$$

The Besov and Triebel–Lizorkin scales for a Lipschitz domain Ω are defined by restrictions of the corresponding Besov and Triebel–Lizorkin spaces on $\mathbb{R}^n$, so we start by briefly reviewing the latter. One convenient point of view is offered by the classical Littlewood–Paley theory (cf., for example, [20, 23, 24]). More specifically, let Ξ be the collection of all systems $\{\zeta_j\}_{j=0}^{\infty}$ of Schwartz functions with the following properties:

(i) there exist positive constants A, B, C such that

$$\begin{aligned} &\operatorname{supp}(\zeta_0) \subset \{x : \ |x| \leqslant A\}; \\ &\operatorname{supp}(\zeta_j) \subset \{x : \ B2^{j-1} \leqslant |x| \leqslant C2^{j+1}\} \quad \text{if } j \in \mathbb{N}; \end{aligned} \tag{2.1}$$

(ii) for every multiindex α there is a positive finite constant C_α such that

$$\sup_{x\in\mathbb{R}^n} \sup_{j\in\mathbb{N}} 2^{j|\alpha|} |\partial^\alpha \zeta_j(x)| \leqslant C_\alpha; \tag{2.2}$$

(iii)

$$\sum_{j=0}^{\infty} \zeta_j(x) = 1 \ \text{ for every } \ x \in \mathbb{R}^n. \tag{2.3}$$

Fix a family $\{\zeta_j\}_{j=0}^{\infty} \in \Xi$. Also, let $\mathcal{F}$ and $S'(\mathbb{R}^n)$ denote the Fourier transform and the class of tempered distributions in $\mathbb{R}^n$ respectively.. Then the Triebel–Lizorkin space $F_s^{p,q}(\mathbb{R}^n)$ is defined for $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leqslant \infty$ as

$$F_s^{p,q}(\mathbb{R}^n) := \Big\{ f \in S'(\mathbb{R}^n) : \\ \|f\|_{F_s^{p,q}(\mathbb{R}^n)} := \Big\| \Big(\sum_{j=0}^{\infty} |2^{sj} \mathcal{F}^{-1}(\zeta_j \mathcal{F} f)|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} < \infty \Big\} \tag{2.4}$$

(with a natural interpretation when $q = \infty$). The case $p = \infty$ is somewhat special, in that a suitable version of (2.4) needs to be used (cf., for example, [20, p. 9]).

If $s \in \mathbb{R}$ and $0 < p, q \leqslant \infty$, then the Besov space $B^{p,q}_s(\mathbb{R}^n)$ can be defined as

$$B^{p,q}_s(\mathbb{R}^n) := \Big\{ f \in S'(\mathbb{R}^n) : \\ \|f\|_{B^{p,q}_s(\mathbb{R}^n)} := \Big(\sum_{j=0}^{\infty} \|2^{sj}\mathcal{F}^{-1}(\zeta_j \mathcal{F} f)\|^q_{L^p(\mathbb{R}^n)} \Big)^{1/q} < \infty \Big\}. \tag{2.5}$$

Different choices of the system $\{\zeta_j\}_{j=0}^{\infty} \in \varXi$ yield the same spaces (2.4)–(2.5) equipped with equivalent norms. Furthermore, the class of Schwartz functions in $\mathbb{R}^n$ is dense in both $B^{p,q}_s(\mathbb{R}^n)$ and $F^{p,q}_s(\mathbb{R}^n)$ provided $s \in \mathbb{R}$ and $0 < p, q < \infty$.

Next, we discuss the adaptation of certain smoothness classes to the situation where the Euclidean space is replaced with the boundary of a Lipschitz domain $\varOmega$. Consider three parameters p, q, s such that

$$0 < p, q \leqslant \infty, \quad (n-1)\Big(\frac{1}{p} - 1\Big)_+ < s < 1 \tag{2.6}$$

and assume that $\varOmega \subset \mathbb{R}^n$ is the upper-graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. We then define $B^{p,q}_s(\partial\varOmega)$ as the space of locally integrable functions f on $\partial\varOmega$ for which the assignment $\mathbb{R}^{n-1} \ni x \mapsto f(x, \varphi(x))$ belongs to $B^{p,q}_s(\mathbb{R}^{n-1})$, the classical Besov space in $\mathbb{R}^{n-1}$. We equip this space with the (quasi-) norm

$$\|f\|_{B^{p,q}_s(\partial\varOmega)} := \|f(\cdot, \varphi(\cdot))\|_{B^{p,q}_s(\mathbb{R}^{n-1})}. \tag{2.7}$$

As far as Besov spaces with a negative amount of smoothness are concerned, in the same context as above, we set

$$f \in B^{p,q}_{s-1}(\partial\varOmega) \iff f(\cdot, \varphi(\cdot))\sqrt{1 + |\nabla\varphi(\cdot)|^2} \in B^{p,q}_{s-1}(\mathbb{R}^{n-1}), \tag{2.8}$$

$$\|f\|_{B^{p,q}_{s-1}(\partial\varOmega)} := \|f(\cdot, \varphi(\cdot))\sqrt{1 + |\nabla\varphi(\cdot)|^2}\|_{B^{p,q}_{s-1}(\mathbb{R}^{n-1})}. \tag{2.9}$$

As is known, the case $p = q = \infty$ corresponds to the usual (inhomogeneous) Hölder spaces $C^s(\partial\varOmega)$ defined by the requirement

$$\|f\|_{C^s(\partial\varOmega)} := \|f\|_{L^\infty(\partial\varOmega)} + \sup_{\substack{x \neq y \\ x, y \in \partial\varOmega}} \frac{|f(x) - f(y)|}{|x - y|^s} < +\infty, \tag{2.10}$$

i.e.,

$$B^{\infty,\infty}_s(\partial\varOmega) = C^s(\partial\varOmega), \quad s \in (0, 1). \tag{2.11}$$

All the definitions then readily extend to the case of (bounded) Lipschitz domains in $\mathbb{R}^n$ via a standard partition of unity argument. These Besov spaces have been defined in such a way that a number of basic properties from the Euclidean setting carry over to spaces defined on $\partial\Omega$ in a rather direct fashion. We continue by recording an interpolation result which is going to be useful for us here (for a proof see [14, 10]). To state it, recall that $(\cdot,\cdot)_{\theta,q}$ and $[\cdot,\cdot]_\theta$ stand for the real and complex interpolation brackets.

Proposition 2.1. *Suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Assume that $0 < p, q, q_0, q_1 \leqslant \infty$ and*

$$\begin{aligned} &\textit{either} \quad , (n-1)\Big(\frac{1}{p}-1\Big)_+ < s_0 \neq s_1 < 1, \\ &\textit{or} \quad -1 + (n-1)\Big(\frac{1}{p}-1\Big)_+ < s_0 \neq s_1 < 0. \end{aligned} \tag{2.12}$$

Then with $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$

$$(B^{p,q_0}_{s_0}(\partial\Omega), B^{p,q_1}_{s_1}(\partial\Omega))_{\theta,q} = B^{p,q}_s(\partial\Omega). \tag{2.13}$$

Furthermore, if $s_0 \neq s_1$ and $0 < p_i, q_i \leqslant \infty$, $i = 0,1$, satisfy $\min\{q_0, q_1\} < \infty$ *as well as either of the following two conditions:*

$$\begin{aligned} &\textit{either} \ (n-1)\Big(\tfrac{1}{p_i}-1\Big)_+ < s_i < 1, \ i = 0,1, \\ &\textit{or} \ -1 + (n-1)\Big(\tfrac{1}{p_i}-1\Big)_+ < s_i < 0, \ i = 0,1, \end{aligned} \tag{2.14}$$

then

$$[B^{p_0,q_0}_{s_0}(\partial\Omega), B^{p_1,q_1}_{s_1}(\partial\Omega)]_\theta = B^{p,q}_s(\partial\Omega), \tag{2.15}$$

where $0 < \theta < 1$, $s := (1-\theta)s_0 + \theta s_1$, $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

We next discuss atomic decompositions of the diagonal Besov scale on $\partial\Omega$. We call $S = S_r = S_r(x)$ a *surface ball* provided that $x \in \partial\Omega$, $0 < r \leqslant \operatorname{diam}(\Omega)$, and $S_r = B(x,r) \cap \partial\Omega$. Also, for $\varkappa > 0$ and $S_r(x)$ surface ball we write $\varkappa S := B(x, \varkappa r) \cap \partial\Omega$. Recall that the tangential gradient is defined by $\nabla_{\tan} u := \nabla u - (\partial_\nu u)\nu$. A function $a_S \in \operatorname{Lip}(\partial\Omega)$ is called an *atom* for $B^{p,p}_s(\partial\Omega)$, $(n-1)/n < p \leqslant 1$, $(n-1)(\frac{1}{p}-1) < s < 1$, if

$$(1)\ \exists\, S = S_r, \ \text{ surface ball, such that } \operatorname{supp}(a_S) \subseteq S, \tag{2.16}$$

$$(2)\ \|\nabla_{\tan} a_S\|_{L^\infty(\partial\Omega)} \leqslant r^{s - \frac{n-1}{p} - 1}. \tag{2.17}$$

It is useful to observe that, by Fundamental Theorem of Calculus, (1) & (2) above also entail

$$\|a_S\|_{L^\infty(\partial\Omega)} \leqslant C r^{s - \frac{n-1}{p}}, \tag{2.18}$$

where C depends exclusively on the Lipschitz character of Ω. The following proposition, extending well-known results from the Euclidean setting, appeared in [14].

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Fix $(n-1)/n < p \leqslant 1$ and $(n-1)(\frac{1}{p}-1) < s < 1$. Then*

$$\|f\|_{B^{p,p}_s(\partial\Omega)} \approx \inf\Big\{ \Big(\sum_S |\lambda_S|^p\Big)^{1/p} :\ f = \sum_S \lambda_S a_S,$$
$$a_S \ \textit{are}\ B^{p,p}_s(\partial\Omega)\ \textit{atoms},\ \{\lambda_S\}_S \in \ell^p \Big\}, \tag{2.19}$$

uniformly for $f \in B^{p,p}_s(\partial\Omega)$.

In (2.19), the infimum is taken over all possible representations of f as $\sum_S \lambda_S a_S$, for countable families of surface balls, and the series is assumed to converge absolutely in $L^1_{\rm loc}(\partial\Omega)$.

Given an arbitrary open subset Ω of $\mathbb{R}^n$, we denote by $f|_\Omega$ the restriction of a distribution f in $\mathbb{R}^n$ to Ω. For $0 < p, q \leqslant \infty$ and $s \in \mathbb{R}$, both $B^{p,q}_s(\mathbb{R}^n)$ and $F^{p,q}_s(\mathbb{R}^n)$ are spaces of (tempered) distributions, hence it is meaningful to define

$$\begin{aligned} &A^{p,q}_s(\Omega) := \{f \text{ distribution in } \Omega :\ \exists\, g \in A^{p,q}_s(\mathbb{R}^n) \ \text{ such that } \ g|_\Omega = f\},\\ &\|f\|_{A^{p,q}_s(\Omega)} := \inf\{\|g\|_{A^{p,q}_s(\mathbb{R}^n)} :\ g \in A^{p,q}_s(\mathbb{R}^n),\ g|_\Omega = f\}, \quad f \in A^{p,q}_s(\Omega), \end{aligned} \tag{2.20}$$

where $A = B$ or $A = F$.

The existence of a universal extension operator for Besov and Triebel–Lizorkin spaces in an arbitrary Lipschitz domain $\Omega \subset \mathbb{R}^n$ was established by Rychkov [21]. This allows us transferring a number of properties of the Besov–Triebel–Lizorkin spaces in the Euclidean space $\mathbb{R}^n$ to the setting of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. If k is a nonnegative integer and $1 < p < \infty$, then

$$F^{p,2}_k(\Omega) = W^{k,p}(\Omega) := \{f \in L^p(\Omega) :\ \partial^\alpha f \in L^p(\Omega),\ |\alpha| \leqslant k\}, \tag{2.21}$$

the classical Sobolev spaces in Ω.

A proof of the following proposition can be found in [10].

Proposition 2.3. *Suppose Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Let $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \neq \alpha_1$, $0 < q_0, q_1, q \leqslant \infty$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. Then*

$$(F^{p,q_0}_{\alpha_0}(\Omega), F^{p,q_1}_{\alpha_1}(\Omega))_{\theta,q} = B^{p,q}_\alpha(\Omega), \quad 0 < p < \infty, \tag{2.22}$$
$$(B^{p,q_0}_{\alpha_0}(\Omega), B^{p,q_1}_{\alpha_1}(\Omega))_{\theta,q} = B^{p,q}_\alpha(\Omega), \quad 0 < p \leqslant \infty. \tag{2.23}$$

Furthermore, if $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1 \leqslant \infty$, and $0 < q_0, q_1 \leqslant \infty$ are such that

$$\textit{either} \quad \max\{p_0, q_0\} < \infty, \quad \textit{or} \quad \max\{p_1, q_1\} < \infty, \tag{2.24}$$

then

$$[F^{p_0,q_0}_{\alpha_0}(\Omega), F^{p_1,q_1}_{\alpha_1}(\Omega)]_\theta = F^{p,q}_\alpha(\Omega), \tag{2.25}$$

where $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

On the other hand, if $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1, q_0, q_1 \leqslant \infty$ *are such that*

$$\min\{q_0, q_1\} < \infty, \tag{2.26}$$

then

$$[B^{p_0,q_0}_{\alpha_0}(\Omega), B^{p_1,q_1}_{\alpha_1}(\Omega)]_\theta = B^{p,q}_{\alpha}(\Omega), \tag{2.27}$$

where θ, α, p, q *are as above.*

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. The Green function for the Laplacian in Ω is a unique function $G : \Omega \times \Omega \to [0, +\infty]$ satisfying

$$G(\cdot, y) \in W^{1,2}(\Omega \setminus B_r(y)) \cap \mathring{W}^{1,1}(\Omega) \quad \forall\, y \in \Omega,\ \ \forall\, r > 0, \tag{2.28}$$

($\mathring{W}^{1,1}(\Omega)$ denotes the closure in $W^{1,1}(\Omega)$ of smooth compactly supported functions in Ω), and

$$\int_\Omega \langle \nabla_x G(x, y), \nabla \varphi(x)\rangle\, dx = \varphi(y) \quad \forall\, \varphi \in C^\infty_c(\Omega). \tag{2.29}$$

Thus,

$$\begin{aligned} G(x, y)\Big|_{x\in\partial\Omega} &= 0 \quad \text{for every } y \in \Omega, \\ -\Delta G(\cdot, y) &= \delta_y \quad \text{for each fixed } y \in \Omega, \end{aligned} \tag{2.30}$$

where the restriction to the boundary is taken in the sense of Sobolev trace theory and δ_y is the Dirac distribution in Ω with mass at y (cf., for example, [7, 11]). As is well known, the Green function is symmetric, i.e.,

$$G(x, y) = G(y, x) \quad \forall\, x, y \in \Omega, \tag{2.31}$$

so that, by the second line in (2.30),

$$-\Delta G(x, \cdot) = \delta_x \quad \text{for each fixed } x \in \Omega. \tag{2.32}$$

Definition 2.4. An open set $\Omega \subset \mathbb{R}^n$ satisfies a *uniform exterior ball condition* (UEBC) if there exists $r > 0$ with the following property: For every $x \in \partial\Omega$ there exists a point $y = y(x) \in \mathbb{R}^n$ such that

$$\overline{B_r(y)} \setminus \{x\} \subseteq \mathbb{R}^n \setminus \Omega \quad \text{and} \quad x \in \partial B_r(y). \tag{2.33}$$

The largest radius r satisfying the above property will be referred to as the UEBC *constant* of Ω.

The relevance of the above concept is apparent from the following result of Grüter and Widman (which is contained in [7, Theorem 3.3]).

Theorem 2.5. *Let* $\Omega \subset \mathbb{R}^n$ *be open and satisfy a* UEBC. *Then there exists* $C = C(\Omega) > 0$ *such that the Green function for the Laplacian satisfies*

$$|\nabla_x \nabla_y G(x,y)| \leqslant C|x-y|^{-n} \quad \textit{for all } x, y \in \Omega. \tag{2.34}$$

We record here a version of the interpolation theorem of E. Stein for analytic families of operators which will be needed in the sequel. This particular variant appeared in [2].

Theorem 2.6. *Let (A_0, A_1) be an interpolation pair of complex Banach spaces. We set $\mathcal{X} = A_0 \bigcap A_1$ and $\mathcal{X}_\theta = [A_0, A_1]_\theta$ for $0 \leqslant \theta \leqslant 1$. Analogously, let (B_0, B_1) be another interpolation pair of complex Banach spaces. We set $\mathcal{Y} = B_0 \bigcap B_1$ and $\mathcal{Y}_\theta = [B_0, B_1]_\theta$ for $0 \leqslant \theta \leqslant 1$.*

Let L_z be a family of linear operators defined in $\mathcal{X}$, with values in $\mathcal{Y}$, indexed by a complex parameter z, with $0 \leqslant \Re e z \leqslant 1$. Assume that $l(L_z f)$ is continuous and bounded in $0 \leqslant \Re e z \leqslant 1$, and analytic in $0 < \Re e z < 1$ for every $f \in \mathcal{X}$ and every continuous linear functional l on $\mathcal{Y}$. Assume further that for $\Re e z = 0$ and $f \in \mathcal{X}$

$$\|L_z f\|_{\mathcal{Y}_0} \leqslant c_0 \|f\|_{\mathcal{X}_0} \tag{2.35}$$

and for $\Re e z = 1$ and $f \in \mathcal{X}$

$$\|L_z f\|_{\mathcal{Y}_1} \leqslant c_1 \|f\|_{\mathcal{X}_1}. \tag{2.36}$$

Then for $0 < \Re e z = \theta < 1$ there exists $c = c(s, q_0, q_1, c_0, c_1)$ such that

$$\|L_z f\|_{\mathcal{Y}_\theta} \leqslant c \|f\|_{\mathcal{X}_\theta} \tag{2.37}$$

uniformly for $f \in \mathcal{X}$.

3 Geometric Estimates and the Proof of the Main Result

In the proof of Theorem 1.1, we use a couple of geometric lemmas which we discuss below (recall the notation in (1.1)).

Lemma 3.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain. Then for each point $y \in \partial\Omega$ and parameters $\alpha < 1$, $N < n - \alpha$ there exists a finite constant $C = C(\Omega, N, \alpha) > 0$ such that*

$$\int_{\Omega \cap B(y,r)} \frac{\delta(x)^{-\alpha}}{|x-y|^N}\, dx \leqslant C r^{n-\alpha-N} \quad \forall\, r > 0. \tag{3.1}$$

Furthermore, if $N > n-1$ and $1 > \alpha > n - N$, then

$$\int_{\Omega \setminus B(y,r)} \frac{\delta(x)^{-\alpha}}{|x-y|^N}\, dx \leqslant C r^{n-\alpha-N} \quad \forall\, r > 0, \tag{3.2}$$

for some $C = C(\Omega, N, \alpha) > 0$.

This is the basic geometric result on which our entire subsequent analysis is based. The proof is straightforward if $\Omega = \mathbb{R}^n_+$ and is reduced to this special case in the case of a general Lipschitz domain by localizing and flattening the boundary via a bi-Lipschitz map (which does not distort distances by more than a fixed factor). We omit the details, but parenthetically mention that Lemma 3.1 continues to hold for a more general class of domains (more specifically, for domains satisfying an interior corkscrew condition and such that $\partial\Omega$ is Ahlfors regular of dimension $n-1$; for definitions, background, and pertinent references the interested reader is referred to [8, 11]).

Lemma 3.2. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a Lipschitz domain. Then for any points* $y, z \in \partial\Omega$ *and parameters* $c > 1$, $\beta < n$, $M > n - \beta$ *there exists a finite constant* $C = C(\Omega, c, N, \alpha) > 0$ *such that*

$$\int_{\Gamma(z)} \frac{\delta(x)^{-\beta}}{|x-y|^M}\, dx \leqslant C|y-z|^{n-\beta-M}, \tag{3.3}$$

where

$$\Gamma(z) := \{x \in \Omega :\ |x-z| < c\,\delta(x)\}. \tag{3.4}$$

Proof. Once again, it is possible to show that the conclusion in Lemma 3.2 remains valid if $\Omega \subseteq \mathbb{R}^n$ is a domain satisfying an interior corkscrew condition and such that $\partial\Omega$ is Ahlfors regular of dimension $n-1$. We shall, however, not pursue this avenue here.

To start the proof in earnest, fix $c > 1$, $z, y \in \partial\Omega$ and set $r := |z-y|$. For $j \in \mathbb{N}$ introduce

$$\Gamma_j(z) := \{x \in \Gamma(z) :\ 2^{j-1}r < |x-z| < 2^j r\} \tag{3.5}$$

and define

$$I_j := \int_{\Gamma_j(z)} \frac{\delta(x)^{-\beta}}{|x-y|^M}\, dx. \tag{3.6}$$

Note that for $x \in \Gamma_j(z)$ we have $|x-y| \leqslant |x-z| + |z-y| \leqslant (2^j+1)r \leqslant 2^{j+1}r$ and $\delta(x) \approx |x-z| \approx 2^j r$, where the notation $a \approx b$ means that there exist c_1, $c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. Keeping these in mind, we can write for each $\alpha \in \mathbb{R}$

$$I_j \leqslant C(2^j r)^{\alpha-\beta} \int_{B(y,2^{j+1}r)\cap\Omega} \frac{\delta(x)^{-\alpha}}{|x-y|^M}\, dx \quad \forall\, j \in \mathbb{N}. \tag{3.7}$$

Now, we choose $\alpha < \min\{1, n-M\}$ and apply Lemma 3.1 to the integral in (3.7) to further obtain

$$I_j \leqslant C(2^j r)^{\alpha-\beta}(2^j r)^{n-\alpha-M} = 2^{j(n-M-\beta)} r^{n-M-\beta} \quad \forall\, j \in \mathbb{N}. \tag{3.8}$$

Next, we note that from our hypothesis $n-M-\beta < 0$, so $\sum\limits_{j=1}^{\infty} 2^{j(n-M-\beta)} < \infty$ which, in the combination with (3.8), gives that there exists $C > 0$ such that

$$\int_{\Gamma(z)\setminus B(z,r/2)} \frac{\delta(x)^{-\beta}}{|x-y|^M}\, dx \leqslant C|y-z|^{n-\beta-M}. \tag{3.9}$$

It remains to estimate

$$\int_{\Gamma(z)\cap B(z,r/2)} \frac{\delta(x)^{-\beta}}{|x-y|^M}\, dx.$$

Observe that if $x \in \Gamma(z) \cap B(z,r/2)$, then $|x-y| \approx |z-y| = r$. Thus, it suffices to prove that

$$\int_{\Gamma(z)\cap B(z,r/2)} \delta(x)^{-\beta}\, dx \leqslant C r^{n-\beta}. \tag{3.10}$$

For each $j = 0, 1, 2, \ldots$ we consider

$$\Gamma^j(z) := \{x \in \Gamma(z) :\, 2^{-j-1} r \leqslant |x-z| \leqslant 2^{-j} r\}. \tag{3.11}$$

If $x \in \Gamma^j(z)$, we have $\delta(x) \approx |x-z| \approx 2^{-j} r$. Thus,

$$\int_{\Gamma^j(z)} \delta(x)^{-\beta}\, dx \leqslant C(2^{-j}r)^{-\beta}(2^{-j}r)^n \quad \forall\, j = 0, 1, 2, \ldots. \tag{3.12}$$

Furthermore,

$$\begin{aligned}\int_{\Gamma(z)\cap B(z,r/2)} \delta(x)^{-\beta}\, dx &= \sum_{j=0}^{\infty} \int_{\Gamma^j(z)} \delta(x)^{-\beta}\, dx \\ &\leqslant C r^{n-\beta} \sum_{j=0}^{\infty} 2^{j(\beta-n)} \leqslant C r^{n-\beta},\end{aligned} \tag{3.13}$$

as desired, where for the last inequality in (3.13) we used the fact that $\beta - n < 0$. This proves (3.10). The proof of Lemma 3.2 is complete. □

After these preliminaries, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Consider first the case $p = 1$, in which scenario we prove that for each $k \in \{0, 1, \ldots, N\}$ there exists $C = C(\Omega, k) > 0$ such that

$$\|\delta^{k-s} |\nabla^{k+1} \mathcal{Q} f|\,\|_{L^1(\Omega)} \leqslant C \|f\|_{B^{1,1}_s(\partial\Omega)} \quad \forall\, f \in B^{1,1}_s(\partial\Omega). \tag{3.14}$$

Fix $f \in B_s^{1,1}(\partial\Omega)$. By property (1), the operator $\nabla^{k+1}\mathcal{Q}$ annihilates constants. Hence

$$(\nabla^{k+1}\mathcal{Q}f)(x) = \int_{\partial\Omega} \nabla^{k+1} q(x,y)(f(y) - f(z))\, d\sigma(y) \quad \forall x \in \Omega,\ z \in \partial\Omega. \tag{3.15}$$

Combining (3.15) with *(2)*, we obtain

$$|(\nabla^{k+1}\mathcal{Q}f)(x)| \leqslant \int_{\partial\Omega} \delta(x)^{-k-\varepsilon}|x-y|^{-n+\varepsilon}|f(y)-f(z)|\, d\sigma(y) \tag{3.16}$$

for all $x \in \Omega$ and $z \in \partial\Omega$. Next, fix $c > 1$ and for each $x \in \Omega$ define the set

$$E_x := \{z \in \partial\Omega : \, |x-z| < c\delta(x)\}. \tag{3.17}$$

Now, consider $x^* \in \partial\Omega$ such that $|x - x^*| = \delta(x)$. Then for every $z \in E_x$ we have $|z-x^*| \leqslant |x-z|+|x-x^*| < (c+1)\delta(x)$. Moreover, if $0 < \theta < c-1$, then for every $z \in \partial\Omega$ such that $|z-x^*| < \theta\delta(x)$ we have $|x-z| \leqslant |x-x^*|+|x^*-z| < c\delta(x)$. Thus,

$$B(x^*, \theta\delta(x)) \cap \partial\Omega \subseteq E_x \subseteq B(x^*, (c+1)\delta(x)) \cap \partial\Omega. \tag{3.18}$$

From (3.18) it follows that

$$\sigma(E_x) \approx \delta(x)^{n-1} \tag{3.19}$$

(note that for (3.19) to hold we only need $\partial\Omega$ to be Ahlfors regular). Next, we take the integral average over E_x of (3.16) and use (3.19) to conclude that, for all $x \in \Omega$,

$$|(\nabla^{k+1}\mathcal{Q}f)(x)| \leqslant C\delta(x)^{1-n-k-\varepsilon} \int_{E_x}\int_{\partial\Omega} \frac{|f(y)-f(z)|}{|x-y|^{n-\varepsilon}}\, d\sigma(y)\, d\sigma(z). \tag{3.20}$$

Multiplying the left- and right-hand sides of (3.20) by $\delta(x)^{k-s}$ and then integrating over Ω with respect to x, we obtain

$$\begin{aligned}
&\int_\Omega \delta(x)^{k-s}|\nabla_x^{k+1}\mathcal{Q}f(x)|\, dx \\
&\leqslant C\int_\Omega \delta(x)^{1-n-s-\varepsilon}\int_{E_x}\int_{\partial\Omega} |x-y|^{-n+\varepsilon}|f(y)-f(z)|\, d\sigma(y)\, d\sigma(z)\, dx \\
&= C\int_{\partial\Omega}\int_{\partial\Omega} |f(y)-f(z)|\Big(\int_{\Gamma(z)} \frac{\delta(x)^{1-n-s-\varepsilon}}{|x-y|^{n-\varepsilon}}\, dx\Big)\, d\sigma(y)\, d\sigma(z),
\end{aligned} \tag{3.21}$$

where

$$\Gamma(z) = \{x \in \Omega : \, |x-z| < c\delta(x)\}. \tag{3.22}$$

At this point, we use Lemma 3.2 with $\beta = -1 + n + s + \varepsilon$ and $M = n - \varepsilon$ (note that since $p = 1$, we have $0 < s < 1 - \varepsilon$, so $\beta < n$ and $n - M - \beta < 0$ as needed). By Lemma 3.2, the integral over $\Gamma(z)$ in (3.21) is bounded by $C|x-y|^{-(n-1+s)}$. The latter used back in (3.21) yields (3.14) since

$$\|f\|_{B_s^{1,1}(\partial\Omega)} \approx \|f\|_{L^1(\partial\Omega)} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(x)-f(y)|}{|x-y|^{n-1+s}}\, d\sigma(x)\, d\sigma(y). \tag{3.23}$$

Consider the case $p = \infty$. The goal is to show that

$$\|\delta^{k+1-s}|\nabla^{k+1}\mathcal{Q}f|\|_{L^\infty(\Omega)} \leqslant C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)}, \quad 0 \leqslant k \leqslant N. \tag{3.24}$$

For this purpose, we assume that $x \in \Omega$ is arbitrary and again denote by $x^* \in \partial\Omega$ a point such that $|x - x^*| = \delta(x)$. Then

$$\delta(x)^{k+1-s}|\nabla^{k+1}\mathcal{Q}f(x)| = \delta(x)^{k+1-s}\left|\int_{\partial\Omega} \nabla_x^{k+1} q(x,y)(f(y) - f(x^*))\, d\sigma(y)\right|. \tag{3.25}$$

Since $f \in B_s^{\infty,\infty}(\partial\Omega) = C^s(\partial\Omega)$, we have

$$|f(y) - f(x^*)| \leqslant \|f\|_{B_s^{\infty,\infty}(\partial\Omega)}|y - x^*|^s \quad \forall\, y \in \partial\Omega. \tag{3.26}$$

To proceed, we split the integral in (3.25) into two parts, I_1 and I_2, corresponding to $y \in B(x^*, cr) \cap \partial\Omega$ and $y \in \partial\Omega \setminus B(x^*, cr)$ respectively, where $r := |x - x^*|$ and $c = c(\partial\Omega) > 0$ is a suitable constant. Using (3.26) and (1.3), we obtain

$$\begin{aligned}
|I_1| &\leqslant C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} r^{k+1-s} \int_{B(x^*,cr)\cap\partial\Omega} \frac{|y-x^*|^s}{r^{k+\varepsilon}|x-y|^{n-\varepsilon}}\, d\sigma(y)\\
&\leqslant C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} r^{1-s-\varepsilon} \int_{B(x^*,cr)\cap\partial\Omega} |x-y|^{s-n+\varepsilon}\, d\sigma(y)\\
&\leqslant C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} r^{1-s-\varepsilon} \int_{B(x^*,cr)\cap\partial\Omega} r^{s-n+\varepsilon}\, d\sigma(y)\\
&= C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)}.
\end{aligned} \tag{3.27}$$

If $y \in B(x^*, cr) \cap \partial\Omega$, then $|y - x^*| \leqslant cr \leqslant c|x - y|$ and $|x - y| \geqslant r$, which are used to obtain the second and third inequalities in (3.27) (for the third one we also recall that $s - n + \varepsilon < 0$). Turning our attention to I_2, we observe that if $y \in \partial\Omega \setminus B(x^*, cr)$, then $|y - x^*| \leqslant |x - x^*| + |x - y| \leqslant 2|x - y|$, which yields

$$\begin{aligned}
|I_2| &\leqslant C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} r^{k+1-s} \int_{\partial\Omega\setminus B(x^*,cr)} \frac{|y-x^*|^s}{r^{k+\varepsilon}|x-y|^{n-\varepsilon}}\, d\sigma(y)\\
&\leqslant C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} r^{1-s-\varepsilon} \int_{\partial\Omega\setminus B(x^*,cr)} |y-x^*|^{s-n+\varepsilon}\, d\sigma(y)
\end{aligned}$$

$$\leqslant C\|f\|_{B^{\infty,\infty}_s(\partial\Omega)} r^{1-s-\varepsilon}\int_{cr}^{\infty}\rho^{s-2+\varepsilon}\,d\rho \leqslant C\|f\|_{B^{\infty,\infty}_s(\partial\Omega)}. \tag{3.28}$$

Now, (3.24) follows by combining (3.27) and (3.28). Thus, the case $p=\infty$ is complete.

To treat the case $1<p<\infty$, we use what we have proved so far and Theorem 2.6. More precisely, for $s_0, s_1 \in (0,1)$ we consider the family of operators

$$L_z f := \delta^{k+z-[(1-z)s_0+zs_1]}|\nabla^{k+1}\mathcal{Q}f| \tag{3.29}$$

such that

$$\begin{aligned}\Re e z = 0 &\Rightarrow |L_0 f| = \delta^{k-s_0}|\nabla^{k+1}\mathcal{Q}f|,\\ \Re e z = 1 &\Rightarrow |L_1 f| = \delta^{k+1-s_1}|\nabla^{k+1}\mathcal{Q}f|.\end{aligned}$$

Our results for $p=1$ and $p=\infty$ lead to the conclusion that the operators

$$\begin{aligned}&L_0 : B^{1,1}_{s_0}(\partial\Omega)\to L^1(\Omega),\\ &L_1 : B^{\infty,\infty}_{s_1}(\partial\Omega)\to L^\infty(\Omega)\end{aligned}$$

are well defined and are bounded for any $s_0, s_1\in(0,1)$. Pick $0<s_0<s_1<1-\varepsilon$, otherwise arbitrary, so that (2.15) applies. In this scenario, Theorem 2.6 can be used, and we can conclude that for each $0\leqslant k\leqslant N$ the operator

$$\delta^{k+1-\frac{1}{p}-s}|\nabla^{k+1}\mathcal{Q}f| : B^{p,p}_s(\partial\Omega)\longrightarrow L^p(\Omega) \tag{3.30}$$

is well define, linear and bounded for every $s\in(0,1)$ and $p\in[1,\infty]$. This takes care of the estimate for the "higher order term" on the left-hand side of (1.5). The "lower order terms" $\nabla^j\mathcal{Q}$ in (1.5) can be handle in a simpler, more straightforward fashion, so we omit the argument.

It remains to analyze the case where $\frac{n-1}{n-\varepsilon}<p<1$, in which scenario (by once again focusing only on the higher order term in the left-hand side of (1.5)), it suffices to prove that for each $k\in\{0,1,\dots,N\}$ there exists some finite constant $C>0$ such that

$$\|\delta^{k+1-\frac{1}{p}-s}|\nabla^{k+1}\mathcal{Q}a\|_{L^p(\Omega)}\leqslant C \quad \text{for every } B^{p,p}_s(\partial\Omega)\text{-atom } a. \tag{3.31}$$

For this purpose, we assume that

$$\begin{aligned}&\operatorname{supp} a\subseteq S_r(x_0)\quad \text{for some}\quad x_0\in\partial\Omega \text{ and } r>0,\\ &\|\nabla_{\tan}a\|_{L^\infty(\partial\Omega)}\leqslant r^{s-1-(n-1)\frac{1}{p}}.\end{aligned} \tag{3.32}$$

Next, we proceed with the rescaling $\widetilde{a}(x):=r^\tau a(x)$ for $x\in\partial\Omega$, with $\tau\in\mathbb{R}$ to be specified soon. Since $s<1$, we have $1-p(2-s)<1-p$, so we can pick $\theta\in(1-p(2-s),1-p)$. Fix such a θ and select $\tau:=\frac{n-\theta}{p}-n$. Then $-1+s+\frac{1-\theta}{p}\in(s,1)$ and $\widetilde{a}$ is a $B^{1,1}_{-1+s+\frac{1-\theta}{p}}(\partial\Omega)$-atom. In particular, there exists $C = C(\Omega, p, s, \theta) > 0$ such that

$$\|\widetilde{a}\|_{B^{1,1}_{-1+s+\frac{1-\theta}{p}}(\partial\Omega)} \leqslant C, \tag{3.33}$$

and, based on what we proved for $p = 1$,

$$\|\delta^{k+1-s-\frac{1-\theta}{p}} |\nabla^{k+1} \mathcal{Q}\widetilde{a}\|_{L^1(\Omega)} \leqslant C\|\widetilde{a}\|_{B^{1,1}_{-1+s+\frac{1-\theta}{p}}(\partial\Omega)} \leqslant C. \tag{3.34}$$

Applying the Hölder inequality, we can write

$$\begin{aligned}
&\int_{B(x_0,2r)\cap\Omega} \Big(\delta(x)^{k+1-\frac{1}{p}-s} |\nabla^{k+1}\mathcal{Q}a(x)|\Big)^p \, dx \\
&= r^{-\tau p} \int_{B(x_0,2r)\cap\Omega} \delta(x)^{kp+p-1-sp} |\nabla^{k+1}\mathcal{Q}\widetilde{a}(x)|^p \, dx \\
&\leqslant r^{-\tau p} \left(\int_{B(x_0,2r)\cap\Omega} \delta(x)^{k+1-s-\frac{1}{p}+\frac{\theta}{p}} |\nabla^{k+1}\mathcal{Q}\widetilde{a}(x)| \, dx \right)^p \\
&\qquad \times \left(\int_{B(x_0,2r)\cap\Omega} \delta(x)^{-\frac{\theta}{1-p}} \, dx \right)^{1-p}.
\end{aligned} \tag{3.35}$$

Since $\theta < 1 - p$, it follows that $-\frac{\theta}{1-p} > -1$ so that

$$\int_{B(x_0,2r)\cap\Omega} \delta(x)^{-\frac{\theta}{1-p}} \, dx \leqslant Cr^{n-1} \int_0^{cr} t^{-\frac{\theta}{1-p}} \, dt = Cr^{n-\frac{\theta}{1-p}}. \tag{3.36}$$

Thus, combining (3.35), (3.36), and (3.34), we obtain

$$\begin{aligned}
&\int_{B(x_0,2r)\cap\Omega} \Big(\delta(x)^{k+1-\frac{1}{p}-s} |\nabla^{k+1}\mathcal{Q}a(x)|\Big)^p \, dx \\
&\qquad \leqslant C \left(\int_{B(x_0,2r)\cap\Omega} \delta(x)^{k+1-s-\frac{1}{p}+\frac{\theta}{p}} |\nabla^{k+1}\mathcal{Q}\widetilde{a}(x)| \, dx \right)^p \leqslant C.
\end{aligned} \tag{3.37}$$

Next, we turn our attention to the contribution away from the support of the atom. For notational simplicity, we assume that $x_0 = 0$ (which can always be arranged via a translation). Taking into account $\|a\|_{L^\infty(\partial\Omega)} \leqslant Cr^{s-(n-1)\frac{1}{p}}$ (cf. (2.18)) and recalling *(2)* again, we have

$$\begin{aligned}
|\nabla^{k+1}\mathcal{Q}a(x)| &\leqslant C \int_{B(0,r)\cap\partial\Omega} \frac{|a(y)|}{\delta(x)^{k+\varepsilon}|x-y|^{n-\varepsilon}} \, d\sigma(y) \\
&\leqslant C \, \frac{r^{s+(n-1)(1-\frac{1}{p})}}{\delta(x)^{k+\varepsilon}|x|^{n-\varepsilon}} \quad \text{if } x \in \Omega \setminus B(0,2r).
\end{aligned} \tag{3.38}$$

At this point, we use (3.38) and (3.2) from Lemma 3.1 with $\alpha = 1-p+sp+\varepsilon p$ and $N = np - \varepsilon p$ to conclude that

$$\int_{\Omega\setminus B(0,2r)} \delta(x)^{(k+1-\frac{1}{p}-s)p} |\nabla^{k+1} \mathcal{Q}a(x)|^p \, dx \leqslant C. \tag{3.39}$$

Now, the estimate (3.31) follows from (3.37) and (3.39), completing the proof of the case $\frac{n-1}{n-\varepsilon} < p < 1$. This completes the proof of the estimate (1.5) for the full range of indices s and p. □

We now discuss a setting where the kernel of the Poisson integral operator satisfies the estimate (1.3).

Theorem 3.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain satisfying a* UEBC. *Then for every $k \in \mathbb{N}_0$ there exists a finite constant $C = C(\Omega, k) > 0$ such that $G(\cdot,\cdot)$, the Green function for the Laplacian in Ω, satisfies*

$$|\nabla_x^{k+1} \nabla_y G(x,y)| \leqslant C \, \frac{|x-y|^{-n}}{\min\{|x-y|,\, \delta(x)\}^k} \quad \forall\, x \in \Omega,\ \forall\, y \in \overline{\Omega} \setminus E, \tag{3.40}$$

for some set $E \subseteq \partial\Omega$ with $\sigma(E) = 0$, where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ for all $x \in \Omega$.

Proof. We first claim that it suffices to prove (3.40) for every $x, y \in \Omega$. Indeed, assume that the latter has been proved. Then, keeping $x \in \Omega$, by the Fatou theorem proved in [3] for bounded harmonic functions in Lipschitz domains, it follows that there exists $E_x \subseteq \partial\Omega$ with $\sigma(E_x) = 0$ such that (3.40) holds for every $y \in \overline{\Omega} \setminus E_x$. Fixing a countable dense subset D of Ω and setting $E := \bigcup_{x \in D} E_x$, we see that $\sigma(E) = 0$ and (3.40) holds for every $x \in D$ and $y \in \overline{\Omega} \setminus E$. Keeping now $y \in \overline{\Omega} \setminus E$ fixed, by density then (3.40) holds for every $x \in \Omega$.

The case $k = 0$ is contained in Theorem 2.5. Thus, it remains to prove the estimate in (3.40) for every $x, y \in \Omega$ and $k \geqslant 1$.

Step I. Proof of the statement for $k = 1$. Here, we distinguish two cases: $x, y \in \Omega$ with $\delta(x) \leqslant |x-y|$ and $x, y \in \Omega$ with $\delta(x) > |x-y|$.

Case (a). $x, y \in \Omega$ with $\delta(x) \leqslant |x-y|$. In this scenario, $\min\{|x-y|, \delta(x)\} = \delta(x)$. So, we need to show that

$$|\nabla_x^2 \nabla_y G(x,y)| \leqslant C\delta(x)^{-1} |x-y|^{-n}. \tag{3.41}$$

Consider $D := B(x, \delta(x)/2) \subseteq \Omega$ so that $y \notin D$ and, if we set $d := \operatorname{dist}(y, \partial D)$, then $d = |x-y| - \frac{1}{2}\delta(x) \geqslant \frac{1}{2}|x-y|$. Thus,

$$\frac{1}{2}|x-y| \leqslant d \leqslant |x-y|. \tag{3.42}$$

Note that $\nabla_x \nabla_y G(\cdot, y)$ is harmonic in D and, by Theorem 2.5, we have

$$|\nabla_x \nabla_y G(x,y)| \leqslant C|x-y|^{-n} \quad \text{for all} \quad x, y \in \Omega. \tag{3.43}$$

Hence $|\nabla_x \nabla_y G(x,y)| \leqslant Cd^{-n}$ if x, y are as in case (a). The latter, combined with interior estimates for the harmonic function $\nabla_x \nabla_y G(\cdot, y)$ in D, implies

$$\begin{aligned} |\nabla_x^2 \nabla_y G(x,y)| &\leqslant C\delta(x)^{-1} \sup_{x \in D} |\nabla_x \nabla_y G(x,y)| \\ &\leqslant C\delta(x)^{-1} d^{-n} \leqslant C\delta(x)^{-1}|x-y|^{-n}, \end{aligned} \tag{3.44}$$

where for the last inequality in (3.44) we used (3.42). This completes the proof of (3.41).

Case (b). $x, y \in \Omega$ with $\delta(x) > |x-y|$.
Now, we have $\min\{|x-y|, \delta(x)\} = |x-y|$, and we seek to prove that

$$|\nabla_x^2 \nabla_y G(x,y)| \leqslant C|x-y|^{-n-1}. \tag{3.45}$$

Consider the harmonic function $\nabla_x \nabla_y G(\cdot, y)$ in $B(x, \frac{1}{2}|x-y|)$ which, by Theorem 2.5, is bounded by $C|x-y|^{-n}$ in this ball. This and interior estimates further imply that, under the current assumptions on x and y,

$$|\nabla_x^2 \nabla_y G(x,y)| \leqslant C|x-y|^{-1}|x-y|^{-n} = C|x-y|^{-n-1}. \tag{3.46}$$

This completes the proof of (3.45) and, consequently, the proof of Step I.

Step II. Proof of the fact that if (3.40) holds for some $k \in \mathbb{N}_0$ when $x, y \in \Omega$, then (3.40) also holds for $k+1$ when $x, y \in \Omega$.

Case (a). $x, y \in \Omega$ with $\delta(x) \leqslant |x-y|$. Under the current assumptions on x and y, it follows that for every $z \in B(x, \frac{1}{2}\delta(x))$ we have $|z-y| \geqslant \frac{1}{2}\delta(x)$ and $\delta(z) \geqslant \frac{1}{2}\delta(x)$, so that $\min\{\delta(z), |z-y|\} \geqslant \frac{1}{2}\delta(x)$. Also, $|z-y| \geqslant |x-y| - |z-x| \geqslant |x-y| - \frac{1}{2}\delta(x) \geqslant \frac{1}{2}|x-y|$. Hence, by invoking the induction hypothesis, we obtain

$$\begin{aligned} |\nabla_x^k \nabla_y G(z,y)| &\leqslant C \frac{|z-y|^{-n}}{\min\{\delta(z), |z-y|\}^k} \\ &\leqslant C\delta(x)^{-k}|x-y|^{-n} \quad \forall\, z \in B\big(x, \frac{1}{2}\delta(x)\big). \end{aligned} \tag{3.47}$$

Thus, if we now use interior estimates for the harmonic function $\nabla_x^k \nabla_y G(\cdot, y)$ in $B(x, \frac{1}{2}\delta(x))$ combined with (3.47), we arrive at

$$\begin{aligned} |\nabla_x^{k+1} \nabla_y G(x,y)| &\leqslant C\delta(x)^{-1}\delta(x)^{-k}|x-y|^{-n} \\ &= \frac{C|x-y|^{-n}}{\min\{\delta(x), |x-y|\}^{k+1}}. \end{aligned} \tag{3.48}$$

This concludes the proof of case (a) in step II.

Case (b). $x, y \in \Omega$ with $\delta(x) > |x-y|$. Under the current assumptions on x and y, it follows that for every $z \in B(x, \frac{1}{2}|x-y|)$ we have $\delta(z) \geqslant \frac{1}{2}|x-y|$ and $|z-y| \geqslant |x-y| - |z-x| \geqslant |x-y| - \frac{1}{2}|x-y| \geqslant \frac{1}{2}|x-y|$, so that $\min\{\delta(z), |z-y|\} \geqslant \frac{1}{2}|x-y|$. The latter, together with the induction hypothesis, implies that

$$\begin{aligned}|\nabla_x^k \nabla_y G(z,y)| &\leqslant C \frac{|z-y|^{-n}}{\min\{\delta(z), |z-y|\}^k} \\ &\leqslant C|x-y|^{-n-k} \quad \forall\, z \in B(x, \frac{1}{2}|x-y|).\end{aligned} \tag{3.49}$$

Employing interior estimates for the harmonic function $\nabla_x^k \nabla_y G(\cdot, y)$ this time in $B(x, \frac{1}{2}|x-y|)$, and then recalling (3.49), we write

$$|\nabla_x^{k+1} \nabla_y G(x,y)| \leqslant C|x-y|^{-n-k-1} = C \frac{|x-y|^{-n}}{\min\{\delta(x), |z-y|\}^{k+1}}. \tag{3.50}$$

This completes the proof of case (b) in step II.

Combining steps I and II, we obtian the desired result. □

It should be remarked that, with a little more effort, the conclusion in the above theorem can be seen to hold in any nontangentially accessible domain $\Omega \subseteq \mathbb{R}^n$ (in the sense of D. Jerison and C. Kenig) with an Ahlfors regular boundary, and which satisfies a UEBC.

We conclude this section with an application to the mapping properties of the Poisson integral operator on Besov–Triebel–Lizorkin spaces, which has obvious implications for the solvability of the Dirichlet problem for the Laplacian in this context (the interested reader is referred to [5, 6, 9, 12, 15, 16, 17, 18, 13, 14] and the references therein).

Theorem 3.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with outward unit normal ν and surface measure σ on $\partial\Omega$. Let $G(\cdot,\cdot)$ denote the Green function for the Laplacian in Ω. Define*

$$(\mathrm{PI}f)(y) := -\int_{\partial\Omega} \partial_{\nu(x)} G(x,y) f(x)\, d\sigma(x), \quad y \in \Omega. \tag{3.51}$$

Then, if Ω satisfies a UEBC, it follows that the operators

(i) $\mathrm{PI} : B_s^{p,q}(\partial\Omega) \to B_{s+\frac{1}{p}}^{p,q}(\Omega)$

(ii) $\mathrm{PI} : B_s^{p,p}(\partial\Omega) \to F_{s+\frac{1}{p}}^{p,q}(\Omega)$

are bounded whenever $0 < p, q \leqslant \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$, with the additional condition that $p, q \neq \infty$ in the case of Triebel–Lizorkin spaces.

Proof. By Theorem 3.3 (with the roles of x and y reversed), we know that for every $k \in \mathbb{N}_0$ there exists $C > 0$ and $E \subseteq \partial\Omega$ with $\sigma(E) = 0$ such that

$$\left|\nabla_y^{k+1}\left[\partial_{\nu(x)}G(x,y)\right]\right| \leqslant \frac{C}{\delta(y)^k|x-y|^n} \quad \forall\, y \in \Omega,\ \forall\, x \in \partial\Omega \setminus E. \tag{3.52}$$

Furthermore, it is not difficult to check that

$$\mathrm{PI}(1) = 1 \quad \text{in} \quad \Omega \quad \text{and} \quad \Delta \circ \mathrm{PI} = 0 \quad \text{in} \quad \Omega. \tag{3.53}$$

As a consequence, we can apply Theorem 1.1 in order to conclude that if $\frac{n-1}{n} < p \leqslant \infty$ and $(n-1)(\frac{1}{p}-1)_+ < s < 1$, then for every $k \in \mathbb{N}$ there exists $C > 0$ such that

$$\|\delta^{k+1-\frac{1}{p}-s}|\nabla^{k+1}\mathrm{PI}f\|_{L^p(\Omega)} + \sum_{j=0}^{k}\|\nabla^j \mathrm{PI}f\|_{L^p(\Omega)} \leqslant C\|f\|_{B_s^{p,p}(\partial\Omega)} \tag{3.54}$$

for all $f \in B_s^{p,p}(\partial\Omega)$. Next, we recall from Theorem 1.2 that if $0 < p, q \leqslant \infty$ and $\alpha \in \mathbb{R}$, then (recall (1.15))

$$\Delta u = 0 \text{ in } \Omega, \quad \delta^{\langle\alpha\rangle-\alpha}|\nabla^{\langle\alpha\rangle}u| \in L^p(\Omega) \implies u \in F_\alpha^{p,q}(\Omega) \cap B_\alpha^{p,p}(\Omega) \tag{3.55}$$

with the extra assumption that $p, q < \infty$ in the case of the Triebel–Lizorkin spaces. Now, if we chose $\alpha := s + \frac{1}{p}$ and $k := \langle\alpha\rangle - 1$, then (3.53), (3.54), and (3.55) imply that the operator in (ii) is bounded. The boundedness of the operator in (i) now follows from this and real interpolation (cf. Propositions 2.1 and 2.3). □

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