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THE MAXIMAL REGULARITY OPERATOR ON TENT SPACES

Pascal Auscher

Univ. Paris-Sud, laboratoire de Mathématiques
UMR 8628, F-91405, Orsay; CNRS, F-91405, Orsay

Sylvie Monniaux

LATP-UMR 7353, FST Saint-Jérôme - Case Cour A
Univ. Aix-Marseille, F-13397 Marseille Cédex 20

Pierre Portal

Université Lille 1, Laboratoire Paul Painlevé
F-59655, Villeneuve d'Ascq
Current Address: Australian National University, Mathematical Sciences Institute
John Dedman Building, Acton ACT 0200, Australia

Abstract. Recently, Auscher and Axelsson gave a new approach to non-smooth boundary value problems with L^2 data, that relies on some appropriate weighted maximal regularity estimates. As part of the development of the corresponding L^p theory, we prove here the relevant weighted maximal estimates in tent spaces $T^{p,2}$ for p in a certain open range. We also study the case $p = \infty$.

1. **Introduction.** Let $-L$ be a densely defined closed linear operator acting on $L^2(\mathbb{R}^n)$ and generating a bounded analytic semigroup $(e^{-tL})_{t\geq 0}$. We consider the maximal regularity operator defined by

$$\mathcal{M}_L f(t,x) = \int_0^t Le^{-(t-s)L} f(s,.)(x)ds,$$

for functions $f \in C_c(\mathbb{R}_+ \times \mathbb{R}^n)$. The boundedness of this operator on $L^2(\mathbb{R}_+ \times \mathbb{R}^n)$ was established by de Simon in [16]. The $L^p(\mathbb{R}_+ \times \mathbb{R}^n)$ case, for $1 < p < \infty$, turned out, however, to be much more difficult. In [10], Kalton and Lancien proved that $\mathcal{M}_L$ could fail to be bounded on L^p as soon as $p \neq 2$. The necessary and sufficient assumption for L^p boundedness was then found by Weis [17] to be a vector-valued strengthening of analyticity, called R-analyticity. As many differential operators L turn out to generate R-analytic semigroups, the L^p boundedness of $\mathcal{M}_L$ has subsequently been successfully used in a variety of PDE situations (see [14] for a survey).

Recently, maximal regularity was used in a different manner as an important tool in [2], where a new approach to boundary value problems with L^2 data for divergence form elliptic systems on Lipschitz domains, is developed. More precisely, in [2], the authors establish and use the boundedness of $\mathcal{M}_L$ on weighted spaces $L^2(\mathbb{R}_+ \times \mathbb{R}^n; t^\beta dtdx)$, for certain values of $\beta \in \mathbb{R}$, under the additional assumption that L has bounded holomorphic functional calculus on $L^2(\mathbb{R}^n)$. This additional

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assumption was removed in [3, Theorem 1.3]. Here is the version when specializing the Hilbert space to be $L^2(\mathbb{R}^n)$.

Theorem 1.1. *With L as above, $\mathcal{M}_L$ extends to a bounded operator on $L^2(\mathbb{R}_+ \times \mathbb{R}^n; t^\beta dtdx)$ for all $\beta \in (-\infty, 1)$.*

The use of these weighted spaces is common in the study of boundary value problems, where they are seen as variants of the tent space $T^{2,2}$ which occurs for $\beta = -1$, introduced by Coifman, Meyer and Stein in [6]. For $p \neq 2$, the corresponding spaces are weighted versions of the tent spaces $T^{p,2}$, which are defined, for parameters $\beta \in \mathbb{R}$ and $m \in \mathbb{N}$, as the completion of $C_c(\mathbb{R}_+ \times \mathbb{R}^n)$ with respect to

$$\|g\|_{T^{p,2,m}(t^\beta dtdy)} = \left(\int_{\mathbb{R}^n} \Big(\int_0^\infty \int_{\mathbb{R}^n} \frac{1_{B(x,t^{\frac{1}{m}})}(y)}{t^{\frac{n}{m}}} \left|g(t,y)\right|^2 t^\beta dydt \Big)^{\frac{p}{2}} dx \right)^{\frac{1}{p}},$$

the classical case corresponding to $\beta = -1$, $m = 1$, and being denoted simply by $T^{p,2}$. The parameter m is used to allow various homogeneities, and thus to make these spaces relevant in the study of differential operators L of order m. To develop an analogue of [2] for L^p data, we need, among many other estimates yet to be proved, boundedness results for the maximal operator $\mathcal{M}_L$ on these tent spaces. This is the purpose of this note. Another motivation is well-posedness of non-autonomous Cauchy problems for operators with varying domains, which will be presented elsewhere. In the latter case, $\mathcal{M}_L$ can be seen as a model of the evolution operators involved. However, as $\mathcal{M}_L$ is an important operator on its own, we thought interesting to present this special case alone.

In Section 3 we state and prove the adequate boundedness results. The proof is based on recent results and methods developed in [9], building on ideas from [5] and [8]. In Section 2 we recall the relevant material from [9].

2. **Tools.**

When dealing with tent spaces, the key estimate needed is a change of aperture formula, i.e., a comparison between the $T^{p,2}$ norm and the norm

$$\|g\|_{T^{p,2}_\alpha} := \left(\int_{\mathbb{R}^n} \Big(\int_0^\infty \int_{\mathbb{R}^n} \frac{1_{B(x,\alpha t)}(y)}{t^n} \left|g(t,y)\right|^2 \frac{dydt}{t} \Big)^{\frac{p}{2}} dx \right)^{\frac{1}{p}},$$

for some parameter $\alpha > 0$. Such a result was first established in [6], building on similar estimates in [7], and analogues have since been developed in various contexts. Here we use the following version given in [9, Theorem 4.3].

Theorem 2.1. *Let $1 < p < \infty$ and $\alpha \geq 1$. There exists a constant $C > 0$ such that, for all $f \in T^{p,2}$,*

$$\|f\|_{T^{p,2}} \leq \|f\|_{T^{p,2}_\alpha} \leq C(1 + \log \alpha)\alpha^{n/\tau}\|f\|_{T^{p,2}},$$

where $\tau = \min(p, 2)$ and C depends only on n and p.

Theorem 2.1 is actually a special case of the Banach space valued result obtained in [9]. Note, however, that it improves the power of α appearing in the inequality from the n given in [6] to $\frac{n}{\tau}$. This is crucial in what follows, and has been shown to be optimal in [9].

Applying this to $(t,y) \mapsto t^{\frac{m(\beta+1)}{2}} f(t^m, y)$ instead of f, we also have the weighted result, where

$$\|g\|_{T^{p,2,m}_{\alpha}(t^\beta dtdy)} = \left(\int_{\mathbb{R}^n} \Big(\int_0^\infty \int_{\mathbb{R}^n} \frac{1_{B(x,\alpha t^{\frac{1}{m}})}(y)}{t^{\frac{n}{m}}} \left|g(t,y)\right|^2 t^\beta dydt \Big)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}.$$

Corollary 2.2. *Let $1 < p < \infty$, $m \in \mathbb{N}$, $\alpha \geq 1$, and $\beta \in \mathbb{R}$. There exists a constant $C > 0$ such that, for all $f \in T^{p,2,m}(t^\beta dtdy)$,*

$$\|f\|_{T^{p,2,m}(t^\beta dtdy)} \leq \|f\|_{T^{p,2,m}_{\alpha}(t^\beta dtdy)} \leq C(1+\log\alpha)\alpha^{n/\tau}\|f\|_{T^{p,2,m}(t^\beta dtdy)},$$

where $\tau = \min(p,2)$ and C depends only on n and p.

To take advantage of this result, one needs to deal with families of operators, that behave nicely with respect to tent norms. As pointed out in [9], this does not mean considering R-bounded families (which means R-analytic semigroups when one considers $(tLe^{-tL})_{t\geq 0}$ as in the $L^p(\mathbb{R}_+ \times \mathbb{R}^n)$ case, but tent bounded ones, i.e. families of operators with the following L^2 off-diagonal decay, also known as Gaffney-Davies estimates.

Definition 2.3. A family of bounded linear operators $(T_t)_{t\geq 0} \subset B(L^2(\mathbb{R}^n))$ is said to satisfy off-diagonal estimates of order M, with homogeneity m, if, for all Borel sets $E, F \subset \mathbb{R}^n$, all $t > 0$, and all $f \in L^2(\mathbb{R}^n)$:

$$\|1_E T_t 1_F f\|_2 \lesssim \Big(1 + \frac{dist(E,F)^m}{t}\Big)^{-M} \|1_F f\|_2.$$

In what follows $\|\cdot\|_2$ denotes the norm in $L^2(\mathbb{R}^n)$.

As proven, for instance, in [4], many differential operators of order m, such as (for $m = 2$) divergence form elliptic operators with bounded measurable complex coefficients, are such that $(tLe^{-tL})_{t\geq 0}$ satisfies off-diagonal estimates of any order, with homogeneity m. This condition can, in fact, be seen as a replacement for the classical gaussian kernel estimates satisfied in the case of more regular coefficients.

3. **Results.**

Theorem 3.1. *Let $m \in \mathbb{N}$, $\beta \in (-\infty, 1)$, $p \in \big(\frac{2n}{n+m(1-\beta)}, \infty\big) \cap (1,\infty)$, and $\tau = \min(p,2)$. If $(tLe^{-tL})_{t\geq 0}$ satisfies off-diagonal estimates of order $M > \frac{n}{m\tau}$, with homogeneity m, then $\mathcal{M}_L$ extends to a bounded operator on $T^{p,2,m}(t^\beta dtdy)$.*

Proof. The proof is very much inspired by similar estimates in [5] and [9]. Let $f \in \mathscr{C}_c(\mathbb{R}_+ \times \mathbb{R}^n)$. Given $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, and $j \in \mathbb{Z}_+$, we consider

$$C_j(x,t) = \begin{cases} B(x,t) \text{ if } j = 0, \\ B(x, 2^j t) \backslash B(x, 2^{j-1}t) \text{ otherwise.} \end{cases}$$

We write $\|\mathcal{M}_L f\|_{T^{p,2}} \leq \sum\limits_{k=1}^{\infty}\sum\limits_{j=0}^{\infty} I_{k,j} + \sum\limits_{j=0}^{\infty} J_j$ where

$$I_{k,j} = \left(\int_{\mathbb{R}^n} \Big(\int_0^\infty \int_{\mathbb{R}^n} \frac{1_{B(x,t^{\frac{1}{m}})}(y)}{t^{\frac{n}{m}}} \Big| \int_{2^{-k-1}t}^{2^{-k}t} Le^{-(t-s)L}(1_{C_j(x,4t^{\frac{1}{m}})} f(s,.))(y) ds \Big|^2 t^\beta dydt \Big)^{\frac{p}{2}} dx \right)^{\frac{1}{p}},$$

$$J_j = \left(\int_{\mathbb{R}^n} \Big(\int_0^\infty \int_{\mathbb{R}^n} \frac{1_{B(x,t^{\frac{1}{m}})}(y)}{t^{\frac{n}{m}}} \Big| \int_{\frac{t}{2}}^{t} Le^{-(t-s)L}(1_{C_j(x,4s^{\frac{1}{m}})} f(s,.))(y) ds \Big|^2 t^\beta dydt \Big)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}.$$

Fixing $j \geq 0$, $k \geq 1$ we first estimate $I_{k,j}$ as follows. For fixed $x \in \mathbb{R}^n$,

$$\int_0^\infty \int_{B(x,t^{\frac{1}{m}})} \Big| \int_{2^{-k-1}t}^{2^{-k}t} Le^{-(t-s)L}(1_{C_j(x,4t^{\frac{1}{m}})} f(s,\cdot))(y)\, ds \Big|^2 t^{\beta-\frac{n}{m}} dy\, dt$$

$$\leq \int_0^\infty \int_{B(x,t^{\frac{1}{m}})} \Big(\int_{2^{-k-1}t}^{2^{-k}t} \Big| (t-s)Le^{-(t-s)L}(1_{C_j(x,4t^{\frac{1}{m}})} f(s,\cdot))(y) \Big| \frac{ds}{t-s} \Big)^2 t^{\beta-\frac{n}{m}} dy\, dt$$

$$\lesssim \int_0^\infty \int_{2^{-k-1}t}^{2^{-k}t} 2^{-k} t \Big(\int_{B(x,t^{\frac{1}{m}})} \Big| (t-s)Le^{-(t-s)L}(1_{C_j(x,4t^{\frac{1}{m}})} f(s,\cdot))(y) \Big|^2 dy \Big)\, t^{\beta-\frac{n}{m}-2} ds dt$$

$$\lesssim \int_0^\infty \int_{2^{-k-1}t}^{2^{-k}t} 2^{-k} \Big(1 + \frac{2^{jm}t}{t-s} \Big)^{-2M} \big\| 1_{B(x,2^{j+2}t^{\frac{1}{m}})} f(s,\cdot) \big\|_2^2\, t^{\beta-\frac{n}{m}-1} ds\, dt$$

$$\lesssim 2^{-k} 2^{-2jmM} \int_0^\infty \Big(\int_{2^k s}^{2^{k+1}s} t^{\beta-\frac{n}{m}-1} dt \Big) \big\| 1_{B(x,2^{j+\frac{k}{m}+3} s^{\frac{1}{m}})} f(s,\cdot) \big\|_2^2\, ds$$

$$\lesssim 2^{-k(\frac{n}{m}+1-\beta)} 2^{-2jmM} \int_0^\infty \big\| 1_{B(x,2^{j+\frac{k}{m}+3} s^{\frac{1}{m}})} f(s,\cdot) \big\|_2^2\, s^{\beta-\frac{n}{m}} ds.$$

In the second inequality, we use Cauchy-Schwarz inequality for the integral with respect to t, the fact that $t-s \sim t$ for $s \in \cup_{k \geq 1}[2^{-k-1}t, 2^{-k}t] \subset [0, \frac{t}{2}]$ and Fubini's theorem to exchange the integral in t and the integral in y. The next inequality follows from the off-diagonal estimate verified by $(t-s)Le^{-(t-s)L}$ and again the fact that $t-s \sim t$. By Corollary 2.2 this gives

$$I_{k,j} \lesssim (j+k) 2^{-k(\frac{1}{2}(\frac{n}{m}+1-\beta)-\frac{n}{m\tau})} 2^{-j(mM-\frac{n}{\tau})} \|f\|_{T^{p,2,m}(t^\beta dtdy)},$$

where $\tau = \min(p,2)$. It follows that $\sum\limits_{k=1}^{\infty} \sum\limits_{j=0}^{\infty} I_{k,j} \lesssim \|f\|_{T^{p,2,m}(t^\beta dtdy)}$ since $M > \frac{n}{m\tau}$ and $\frac{n}{m} + 1 - \beta > \frac{2n}{m\tau}$ (Note that for $p \geq 2$, this requires $\beta < 1$).

We now turn to J_0 and remark that $J_0 \leq \big(\int_{\mathbb{R}^n} J_0(x)^{\frac{p}{2}} dx \big)^{\frac{1}{p}}$, where

$$J_0(x) = \int_0^\infty \int_{\mathbb{R}^n} \Big| \int_{\frac{t}{2}}^t Le^{-(t-s)L}(g(s,\cdot)(y) ds \Big|^2 t^{\beta-\frac{n}{m}} dy\, dt$$

with $g(s,y) = 1_{B(x,4s^{\frac{1}{m}})}(y) f(s,y)$. The inside integral can be rewritten as

$$\mathcal{M}_L g(t,\cdot) - e^{-\frac{t}{2}L} \mathcal{M}_L g(\frac{t}{2},\cdot).$$

As $\mathcal{M}_L$ is bounded on $L^2(\mathbb{R}_+ \times \mathbb{R}^n; t^{\beta-\frac{n}{m}} dydt)$ by Theorem 1.1 and $(e^{-tL})_{t \geq 0}$ is uniformly bounded on $L^2(\mathbb{R}^n)$, we get

$$J_0(x) \lesssim \int_0^\infty \big\| 1_{B(x,4s^{\frac{1}{m}})} f(s,\cdot) \big\|_2^2\, s^{\beta-\frac{n}{m}} ds.$$

We finally turn to J_j, for $j \geq 1$. For fixed $x \in \mathbb{R}^n$,

$$\int_0^\infty \int_{\mathbb{R}^n} 1_{B(x,t^{\frac{1}{m}})}(y) \Big| \int_{\frac{t}{2}}^t Le^{-(t-s)L}(1_{C_j(x,4s^{\frac{1}{m}})} f(s,.))(y) ds \Big|^2 t^{\beta - \frac{n}{m}} dy\, dt$$

$$\leq \int_0^\infty \int_{\mathbb{R}^n} 1_{B(x,t^{\frac{1}{m}})}(y) \Big(\int_{\frac{t}{2}}^t \big|(t-s)Le^{-(t-s)L}(1_{C_j(x,4s^{\frac{1}{m}})} f(s,.))(y)\big| \frac{ds}{t-s} \Big)^2 t^{\beta-\frac{n}{m}} dydt$$

$$\lesssim \int_0^\infty \int_{\mathbb{R}^n} 1_{B(x,t^{\frac{1}{m}})}(y) \int_{\frac{t}{2}}^t \big|(t-s)Le^{-(t-s)L}(1_{C_j(x,4s^{\frac{1}{m}})} f(s,.))(y)\big|^2 \frac{ds}{(t-s)^2} t^{\beta-\frac{n}{m}+1} dydt$$

$$\lesssim \int_0^\infty \int_{\frac{t}{2}}^t (t-s)^{-2} \Big(1 + \frac{2^{jm}t}{t-s}\Big)^{-2M} \big\|1_{B(x,2^{j+2}s^{\frac{1}{m}})} f(s,.)\big\|_2^2 s^{\beta - \frac{n}{m}+1} dsdt$$

$$\lesssim 2^{-jm(2M-2)} \int_0^\infty \left(\int_s^{2s} s(t-s)^{-2}\Big(1+\frac{2^{jm}t}{t-s}\Big)^{-2} dt \right) \big\|1_{B(x,2^{j+2}s^{\frac{1}{m}})} f(s,.)\big\|_2^2 s^{\beta-\frac{n}{m}} ds$$

$$\lesssim 2^{-2jmM} \int_0^\infty \big\|1_{B(x,2^{j+2}s^{\frac{1}{m}})} f(s,.)\big\|_2^2 s^{\beta - \frac{n}{m}} ds,$$

where we have used Cauchy-Schwarz inequality in the second inequality, the off-diagonal estimates and the fact that $s \leq t$ in the third, Fubini's theorem and the fact that $s \geq \frac{t}{2}$ in the fourth, and the change of variable $\sigma = \frac{t}{t-s}$ in the last. An application of Corollary 2.2, then gives

$$J_j \lesssim 2^{-jmM} j 2^{j\frac{n}{\tau}} \|f\|_{T^{p,2,m}(t^\beta dtdy)} = j2^{-j(mM-\frac{n}{\tau})} \|f\|_{T^{p,2,m}(t^\beta dtdy)},$$

and the proof is concluded by summing the estimates. □

An end-point result holds for $p = \infty$. In this context the appropriate tent space consists of functions such that $|g(t,y)|^2 \frac{dydt}{t}$ is a Carleson measure, and is defined as the completion of the space $\mathscr{C}_c(\mathbb{R}_+ \times \mathbb{R}^n)$ with respect to

$$\|g\|^2_{T^{\infty,2}} = \sup_{(x,r) \in \mathbb{R}^n \times \mathbb{R}_+} r^{-n} \int_{B(x,r)} \int_0^r |g(t,y)|^2 \frac{dydt}{t}.$$

We also consider the weighted version defined by

$$\|g\|^2_{T^{\infty,2,m}(t^\beta dtdy)} := \sup_{(x,r)\in\mathbb{R}^n\times\mathbb{R}_+} r^{-\frac{n}{m}} \int_{B(x,r^{\frac{1}{m}})} \int_0^r |g(t,y)|^2 t^\beta dydt.$$

Theorem 3.2. *Let $m \in \mathbb{N}$, and $\beta \in (-\infty, 1)$. If $(tLe^{-tL})_{t\geq 0}$ satisfies off-diagonal estimates of order $M > \frac{n}{2m}$, with homogeneity m, then $\mathcal{M}_L$ extends to a bounded operator on $T^{\infty,2,m}(t^\beta dtdy)$.*

Proof. Pick a ball $B(z, r^{\frac{1}{m}})$. Let

$$I^2 = \int_{B(z,r^{\frac{1}{m}})} \int_0^r |(\mathcal{M}_L f)(t,x)|^2 t^\beta dxdt.$$

We want to show that $I^2 \lesssim r^{\frac{n}{m}} \|f\|^2_{T^{\infty,2}(t^\beta dtdy)}$. We set

$$I_j^2 = \int_{B(z,r^{\frac{1}{m}})} \int_0^r |(\mathcal{M}_L f_j)(t,x)|^2 t^\beta dxdt$$

where $f_j(s,x) = f(s,x)1_{C_j(z,4r^{\frac{1}{m}})}(x)1_{(0,r)}(s)$ for $j \geq 0$. Thus by Minkowsky inequality, $I \leq \sum I_j$. For I_0 we use again Theorem 1.1 which implies that $\mathcal{M}_L$ is bounded on $L^2(\mathbb{R}_+ \times \mathbb{R}^n, t^\beta dxdt)$. Thus

$$I_0^2 \lesssim \int_{B(z,4r^{\frac{1}{m}})} \int_0^r |f(t,x)|^2 t^\beta dxdt \lesssim r^{\frac{n}{m}} \|f\|^2_{T^{\infty,2,m}(t^\beta dtdy)}.$$

Next, for $j \neq 0$, we proceed as in the proof of Theorem 3.1 to obtain

$$\begin{aligned} I_j^2 \quad &\lesssim \quad \sum_{k=1}^{\infty} \int_0^r \int_{2^{-k-1}t}^{2^{-k}t} 2^{-k} t \Big(1 + \frac{2^{jm} r}{t-s}\Big)^{-2M} \|f_j(s,.)\|^2_{L^2} t^{\beta-2} ds\, dt \\ &\quad + \int_0^r \int_{\frac{t}{2}}^t t(t-s)^{-2} \Big(1 + \frac{2^{jm} r}{t-s}\Big)^{-2M} \|f_j(s,.)\|^2_{L^2} t^\beta ds\, dt. \end{aligned}$$

Exchanging the order of integration, and using the fact that $t \sim t - s$ in the first part and that $t \sim s$ in the second, we have the following.

$$\begin{aligned} I_j^2 \quad &\lesssim \quad \sum_{k=1}^{\infty} 2^{-k} 2^{-2jmM} r^{-2M} \int_0^{2^{-k}r} \int_{2^k s}^{2^{k+1}s} t^{\beta+2M-1} \|f_j(s,.)\|^2_{L^2} dtds \\ &\quad + \int_0^r \int_s^{2s} r(t-s)^{-2} \Big(1 + \frac{2^{jm} r}{t-s}\Big)^{-2M} \|f_j(s,.)\|^2_{L^2} s^\beta dtds \\ &\lesssim \quad \sum_{k=1}^{\infty} 2^{-k} 2^{-2jmM} \int_0^{2^{-k}r} (2^k s)^\beta \|f_j(s,.)\|^2_{L^2} ds \\ &\quad + \int_0^r \int_1^\infty \big(1 + 2^{jm}\sigma\big)^{-2M} \|f_j(s,.)\|^2_{L^2} s^\beta d\sigma ds \\ &\lesssim \quad 2^{-2jmM} \int_0^r \|f_j(s,.)\|^2_{L^2} s^\beta ds, \end{aligned}$$

where we used $\beta < 1$. We thus have

$$I_j^2 \lesssim 2^{-2jmM} (2^j r^{\frac{1}{m}})^n \|f\|^2_{T^{\infty,2,m}(t^\beta dtdy)},$$

and the condition $M > \frac{n}{2m}$ allows us to sum these estimates. □

Remark 3.3. Assuming off-diagonal estimates, instead of kernel estimates, allows to deal with differential operators L with rough coefficients. The harmonic analytic objects associated with L then fall outside the Calderón-Zygmund class, and it is common (see for instance [1]) for their boundedness range to be a proper subset of $(1, \infty)$. Here, our range $(\frac{2n}{n+m(1-\beta)}, \infty]$ includes $[2, \infty]$ as $\beta < 1$, which is consistent with [2]. In the case of classical tent spaces, i.e., $m = 1$ and $\beta = -1$, it is the range $(2_*, \infty]$, where 2_* denotes the Sobolev exponent $\frac{2n}{n+2}$. We do not know, however, if this range is optimal.

Remark 3.4. Theorem 3.2 is a maximal regularity result for parabolic Carleson measure norms. This is quite natural from the point of view of non-linear parabolic PDE (where maximal regularity is often used), and such norm have, actually, already been used in the context of Navier-Stokes equations in [11], and, subsequently, for some geometric non-linear PDE in [12]. Theorem 3.1 is also reminiscent of Krylov's Littlewood-Paley estimates [13], and of their recent far-reaching generalization in [15]. In fact, the methods and results from [9], on which this paper relies, use the same circle of ideas (R-boundedness, Kalton-Weis γ multiplier theorem...) as [15]. The combination of these ideas into a "conical square function" approach to stochastic maximal regularity will be the subject of a forthcoming paper.

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E-mail address: pascal.auscher@math.u-psud.fr
E-mail address: sylvie.monniaux@univ-amu.fr
E-mail address: pierre.portal@math.univ-lille1.fr