On the Navier-Stokes equations in unbounded domains

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Abstract

Existence of a global mild solution of the Navier-Stokes system in open sets of $\mathbb{R}^3$, no smoothness at the boundary required, for small initial data in a critical space, is proved.

1 Introduction

It has been claimed in a paper by the author [6] that for any open subset of $\mathbb{R}^3$, there exists a global mild solution of the Navier-Stokes system with Dirichlet boundary conditions for small initial data in a critical space and a local mild solution if no size condition is assumed on the initial data. In the case of unbounded domains, the proof of existence of global solutions proposed in [6] is not correct. We want to give here a correct proof and exhibit global mild solutions of the Navier-Stokes system with Dirichlet boundary conditions

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla\pi + (u\cdot\nabla)u &=& 0 & \text{in} \quad (0,\infty)\times\Omega, \\ \operatorname{div} u &=& 0 & \text{in} \quad (0,\infty)\times\Omega, \\ u &=& 0 & \text{on} \quad (0,\infty)\times\partial\Omega, \\ u(0) &=& u_0 & \text{in} \quad \Omega, \end{array}\right. \tag{1.1}$$

in an (unbounded) open set $\Omega\subset\mathbb{R}^3$, for initial data u_0 in a critical space.

The strategy in this note follows the lines of [6]: we describe a functional setting in which the (slightly modified) Fujita-Kato scheme applies, such as in their fundamental paper [1] where they treated the case of smooth bounded domains (global solutions); in the case of (smooth) unbounded domains, their method applies only to obtain local solutions (in a finite time interval). When $\Omega=\mathbb{R}^3$, classical Fourier analysis methods apply, and it can be proved that there exists a global mild solution of (1.1) if u_0 is small in $\dot H^{\frac12}(\mathbb{R}^3)$, the homogeneous Sobolev space (see e.g. [3, Theorem 15.2]). The case of smooth exterior domains in $\mathbb{R}^3$ has been treated by T. Miyakawa [5, Theorem 3.3]. When $\Omega\subset\mathbb{R}^3$ of class $\mathscr{C}^3$ is unbounded with $\partial\Omega$ bounded or unbounded, the existence of a global mild solution of (1.1) for small initial data u_0 in the domain of the fractional power $\frac14$ of the Stokes operator has been proved by H. Kozono and T. Ogawa [2, Theorem 1].

2 The linear Dirichlet-Stokes operator

Let Ω be an open set in $\mathbb{R}^3$ (bounded or unbounded) and define the vector-valued Hilbert space $H=L^2(\Omega;\mathbb{R}^3)$ by

$$H=\big\{u=(u_1,u_2,u_3); u_i\in L^2(\Omega;\mathbb{R}^3),\ \text{for all } i=1,2,3\big\}$$

endowed with the scalar product

$$\langle u,v\rangle=\int_\Omega u\cdot\overline{v}=\sum_{i=1}^3\int_\Omega u_i\,\overline{v_i}.$$

Define next

$$\mathcal{G}=\big\{\nabla p; p\in L^2_{\rm loc}(\Omega;\mathbb{R})\ \text{with}\ \nabla p\in L^2(\Omega;\mathbb{R}^3)\big\};$$

the set $\mathcal{G}$ is a closed subspace of H. Let now

$$\mathcal{H} = \mathcal{G}^{\perp} = \{u \in L^2(\Omega;\mathbb{R}^3); \langle u, g\rangle = 0 \text{ for all } g \in \mathcal{G}\}.$$

Let $J : \mathcal{H} \hookrightarrow H$ the canonical injection from $\mathcal{H}$ onto H and define a scalar product on $\mathcal{H}$ by

$$(u, v) \mapsto \langle Ju, Jv\rangle.$$

Endowed with this scalar product, $\mathcal{H}$ is a Hilbert space and the following Helmholtz decomposition holds:

$$H = \mathcal{H} \overset{\perp}{\oplus} \mathcal{G}.$$

We denote by $\mathbb{P}$ the orthogonal projection from H onto $\mathcal{H}$: $\mathbb{P}$ is equal to the adjoint J' of J and $\mathbb{P}J = \mathrm{Id}_{\mathcal{H}}$.

Let now $\mathscr{D} = \mathscr{C}_c^{\infty}(\Omega;\mathbb{R}^3)$. We next define

$$\mathcal{D} = \{u \in \mathscr{D}; \operatorname{div} u = 0 \text{ in } \Omega\},$$

closed subspace of $\mathscr{D}$. We denote by J_0 the canonical injection $J_0 : \mathcal{D} \hookrightarrow \mathscr{D}$: it is a restriction of the canonical injection J. Therefore, its adjoint $\mathbb{P}_1 = J_0' : \mathscr{D}' \to \mathcal{D}'$ is an extension of the Helmholtz projection $\mathbb{P}$. The following theorem characterizes the elements in $\ker \mathbb{P}_1$ (see e.g. [9, Proposition 1.1, p. 14]).

Theorem 2.1 (de Rham)**.** *Let T be a distribution in $\mathscr{D}'$ such that $\mathbb{P}_1 T = 0$ in $\mathcal{D}'$. Then there exists a distribution $S \in \mathscr{C}_c^{\infty}(\Omega;\mathbb{R})'$ such that $T = \nabla S$. Conversely, if $T = \nabla S$ with $S \in \mathscr{C}_c^{\infty}(\Omega;\mathbb{R})'$, then $\mathbb{P}_1 T = 0$ in $\mathcal{D}'$.*

To apply the framework of forms, we need a form-space $\mathcal{V}$: let us define $\mathcal{V}$ by $\mathcal{V} = \mathcal{H} \cap V$, where $V = H_0^1(\Omega;\mathbb{R}^3)$ is the closure of $\mathscr{D}$ with respect to the scalar product

$$(u, v) \mapsto \langle u, v\rangle_1 = \langle u, v\rangle + \sum_{i=1}^{3} \langle \partial_i u, \partial_i v\rangle.$$

The space $\mathcal{V}$ is a closed subspace of V; endowed with the scalar product $\langle\cdot,\cdot\rangle_1$, it is a Hilbert space. Moreover, $\mathcal{V}$ is dense in $\mathcal{H}$. Indeed, to prove that $\mathcal{V}$ is dense in $\mathcal{H}$, it suffices to prove that its orthogonal in $\mathcal{H}$ is equal to $\{0\}$. Let $u \in \mathcal{H}$, orthogonal to $\mathcal{V}$; i.e., $\langle u, v\rangle = 0$ for all $v \in \mathcal{V}$. Since $\mathcal{D} \subset \mathcal{V}$, this implies also that $\langle u, v\rangle = 0$ for all $v \in \mathcal{D}$ and then $Ju = T$, viewed as a distribution in $\mathscr{D}'$ satisfies

$$0 = {}_{\mathscr{D}'}\langle Ju, J_0 v\rangle_{\mathscr{D}} = {}_{\mathcal{D}'}\langle \mathbb{P}_1 T, v\rangle_{\mathcal{D}}$$

since $\mathbb{P}_1 = J_0'$. This means that $\mathbb{P}_1 T = 0$ on $\mathcal{D}$. By de Rham's theorem, this implies that there exists $S \in \mathscr{C}_c^{\infty}(\Omega)'$ such that $T = \nabla S$. Recall that $T = Ju \in H$, so that $\nabla S \in H$, and therefore, $T \in \mathcal{G}$. But $\mathcal{H} \cap \mathcal{G} = \{0\}$ (since they are orthogonal by definition), which implies then that $u = 0$ (since $u \in \mathcal{H}$ by assumption and we just proved that $Ju \in \mathcal{G}$).

Next, we denote by V' the dual space of V: $V' = H^{-1}(\Omega;\mathbb{R}^3)$ and by $\mathcal{V}'$ the dual space of $\mathcal{V}$. Let $\tilde{J}$ be the canonical injection $\mathcal{V} \hookrightarrow V$: it is a restriction of the canonical injection $J : \mathcal{H} \hookrightarrow H$, so that its adjoint $\tilde{\mathbb{P}} = \tilde{J}' : V' \to \mathcal{V}'$ is an extension of the Helmholtz projection $\mathbb{P} : H \to \mathcal{H}$.

On $\mathcal{V} \times \mathcal{V}$, we define the sesquilinear form

$$a(u, v) = \sum_{j=1}^{n} \langle \partial_j \tilde{J}u, \partial_j \tilde{J}v\rangle \quad u, v \in \mathcal{V}.$$

The Dirichlet-Stokes operator A in $\mathcal{H}$ is the associated operator of the form a. It is defined by

$$\begin{aligned} D(A) &= \{u \in \mathcal{V}; \tilde{\mathbb{P}}(-\Delta_D^{\Omega})\tilde{J}u \in \mathcal{H}\} \\ Au &= \tilde{\mathbb{P}}(-\Delta_D^{\Omega})\tilde{J}u \end{aligned}$$

where Δ_D^Ω denotes the Dirichlet-Laplacian on V. From the theory of operators associated to forms on Hilbert spaces (see e.g., [8]), it is immediate that $-A$ generates an analytic semigroup of contractions of angle $\frac{\pi}{2}$, $(e^{-tA})_{t\geq 0}$. Since a is symmetric, the operator A is self-adjoint, $D(A^{\frac{1}{2}}) = V$ (by [4, Corollaire 5.2]). Note also that the operator $\delta\,\mathrm{Id} + A$ is invertible for all $\delta > 0$, and the following estimate holds

$$\|A(\delta\,\mathrm{Id} + A)^{-1}\|_{\mathscr{L}(\mathcal{H})} \leq 2, \quad \text{for all } \delta > 0. \tag{2.1}$$

Moreover by de Rham's theorem, for $u \in D(A)$, there exists $p \in L^2_{\mathrm{loc}}(\Omega;\mathbb{C})$ such that

$$J(Au) = -\Delta u + \nabla p,$$

so that we can equivalently define $D(A)$ by

$$D(A) = \big\{u \in V; \exists\, p \in L^2_{\mathrm{loc}}(\Omega;\mathbb{R}) : -\Delta \tilde{J}u + \nabla p \in \mathcal{H}\big\}.$$

The relations between the spaces and the operators are summarized in the following diagram:

$$\begin{tikzcd}
\mathcal{D} \arrow[r, hook, "J_0"] & \mathscr{D} \\
\mathcal{V} \arrow[r, hook, "\tilde{J}"] \arrow[dd, bend right, "A"'] & V \arrow[dd, bend left, "(-\Delta_D^\Omega)"] \\
\mathcal{H} \arrow[r, hook, shift left, "J"] & H \arrow[l, shift left, "\mathbb{P}=J'"] \\
\mathcal{V}' & V' \arrow[l, "\tilde{\mathbb{P}}=\tilde{J}'"] \\
\mathcal{D}' & \mathscr{D}' \arrow[l, "\mathbb{P}_1=J_0'"]
\end{tikzcd}$$

The following estimates will be used to treat the nonlinear term and the initial condition in (1.1).

Proposition 2.2. *For all $\alpha \geq 0$ and all $f \in D(A^\alpha)$,*

$$\big\|t \mapsto A^\alpha T(t) f\big\|_{L^\infty(0,\infty;\mathcal{H})} \leq \|A^\alpha f\|_2 \quad \textit{and} \quad \Big(\int_0^\infty \big\|A^{\frac{1+2\alpha}{2}} T(\tfrac{t}{2}) f\big\|_2^2\, dt\Big)^{\frac{1}{2}} \leq \|A^\alpha f\|_2. \tag{2.2}$$

Corollary 2.3. *The semigroup $(T(t))_{t\geq 0}$ satisfies*

$$\big\|t \mapsto A^{\frac{1}{2}} T(t)\big\|_{L^2(0,\infty;\mathscr{L}(\mathcal{H}))} \leq \frac{1}{\sqrt{2}} \quad \textit{and} \quad \big\|t \mapsto A^{\frac{3}{4}} T(t)\big\|_{L^{\frac{4}{3}}(0,\infty;\mathscr{L}(\mathcal{H}))} \leq 1 \tag{2.3}$$

Proof of Proposition 2.2. The property (2.2) comes from the energy equality. Assume first that $f \in D(A^{2\alpha}) \cap D(A)$ and define $u(t) = T(t)f$. Then $u(0) = f$ and u is solution of $u'(s) + Au(s) = 0$: taking the scalar product in $\mathcal{H}$ of this equation with $A^{2\alpha}u(s)$, we have

$$\langle u'(s), A^{2\alpha}u(s)\rangle + \langle Au(s), A^{2\alpha}u(s)\rangle = 0, \quad s \geq 0. \tag{2.4}$$

Since the operator A is self-adjoint, integrating (2.4) between 0 and t, we obtain

$$\|A^\alpha u(t)\|_2^2 + 2\int_0^t \|A^{\frac{1+2\alpha}{2}} u(s)\|_2^2\, ds = \|A^\alpha f\|_2^2, \quad \text{for all } t > 0. \tag{2.5}$$

Since $D(A^{2\alpha}) \cap D(A)$ is dense in $D(A^\alpha)$, (2.5) holds for all $f \in D(A^\alpha)$. Therefore, we have

$$\sup_{t\geq 0} \|A^\alpha T(t) f\|_2 \leq \|A^\alpha f\|_2 \quad \text{and} \quad \Big(\int_0^\infty \|A^{\frac{1+2\alpha}{2}} T\big(\tfrac{s}{2}\big) f\|_2^2\, ds\Big)^{\frac{1}{2}} \leq \|A^\alpha f\|_2, \quad f \in \mathcal{H}, \tag{2.6}$$

which gives (2.2). $\square$

Proof of Corollary 2.3. The first part of (2.3) is contained in (2.2) for $\alpha = 0$. Moreover, by (2.2) for $\alpha = 0$, we have

$$\big\| t \mapsto T(t) \big\|_{L^\infty(0,\infty;\mathscr{L}(\mathcal{H}))} \leq 1$$

and

$$\big\| t \mapsto A^{\frac{1}{2}} T\big(\tfrac{t}{2}\big) \big\|_{L^2(0,\infty;\mathscr{L}(\mathcal{H}))} \leq 1. \tag{2.7}$$

Interpolating between these two estimates, we obtain that

$$\big\| t \mapsto A^{\frac{1}{4}} T\big(\tfrac{t}{2}\big) \big\|_{L^4(0,\infty;\mathscr{L}(\mathcal{H}))} \leq 1. \tag{2.8}$$

Now, with the two estimates (2.7) and (2.8), the proof of the second part of (2.3) is immediate. Indeed, the equality $A^{\frac{3}{4}} T(t) = A^{\frac{1}{2}} T\big(\frac{t}{2}\big) A^{\frac{1}{4}} T\big(\frac{t}{2}\big)$ holds for all $t > 0$ by the semigroup property, and for all $f \in L^2$ and $g \in L^4$, the product fg belongs to $L^{\frac{4}{3}}$ and $\|fg\|_{\frac{4}{3}} \leq \|f\|_2 \|g\|_4$ (since $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$). $\square$

3 The Dirichlet-Navier-Stokes system

We define the space $\mathscr{E}$ by

$$\mathscr{E} = \mathscr{C}_b\big([0,\infty); \dot{D}(A^{\frac{1}{4}})\big) \cap L^4\big(0,\infty; \dot{H}^1_0(\Omega;\mathbb{R}^3)\big), \tag{3.1}$$

where $\dot{D}(A^{\frac{1}{4}})$ denotes the homogeneous $D(A^{\frac{1}{4}})$-space, i.e., the completion of $D(A^{\frac{1}{4}})$ with respect to the (semi-)norm $f \mapsto \|A^{\frac{1}{4}} f\|_2$, and $\mathscr{C}_b$ denotes the space of bounded continuous functions. We define the following norm on $\mathscr{E}$

$$\|u\|_{\mathscr{E}} = \|A^{\frac{1}{4}} u\|_{L^\infty(0,\infty;\mathcal{H})} + \|\nabla u\|_{L^4(0,\infty;L^2(\Omega;\mathbb{R}^3))}.$$

We reduce the problem of finding mild solutions of (1.1) by solving

$$\begin{aligned} u'(t) + Au(t) &= -\mathbb{P}_1\big((J_0 u \cdot \nabla) J_0 u\big) \\ u(0) &= u_0, \quad u \in \mathscr{E}, \end{aligned} \tag{3.2}$$

for which a mild solution is given by the Duhamel formula: $u = \alpha + \phi(u,u)$, where, for $t > 0$,

$$\begin{aligned} &\alpha(t) = T(t) u_0 \quad \text{and} \\ &\phi(u,v)(t) = \int_0^t T(t-s)\Big(-\tfrac{1}{2}\mathbb{P}_1\big((J_0 u(s) \cdot \nabla) J_0 v(s) + (J_0 v(s) \cdot \nabla) J_0 u(s)\big)\Big)\, ds. \end{aligned} \tag{3.3}$$

The strategy to find $u \in \mathscr{E}$ satisfying $u = \alpha + \phi(u,u)$ is to apply a fixed point theorem. We have then to make sure that $\mathscr{E}$ is a "good" space for the problem, i.e., $\alpha \in \mathscr{E}$ and $\phi(u,u) \in \mathscr{E}$. The fact that α is continuous in time comes from the strong continuity of the Stokes semigroup and the fact that $A^{\frac{1}{4}}$ commutes with the Stokes semigroup on $\dot{D}(A^{\frac{1}{4}})$. Moreover, $\alpha \in \mathscr{E}$ by (2.2) for $\alpha = \frac{1}{4}$ and interpolation, and the following estimate holds

$$\|\alpha\|_{\mathscr{E}} \leq (1 + 2^{-\frac{1}{4}})\, \|A^{\frac{1}{4}} f\|_2 \tag{3.4}$$

Proposition 3.1. *The application $\phi : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ is bilinear, continuous and symmetric. We denote by M its norm:*

$$M = \sup\big\{ \|\phi(u,v)\|_{\mathscr{E}}; u, v \in \mathscr{E}, \|u\|_{\mathscr{E}}, \|v\|_{\mathscr{E}} \leq 1 \big\}.$$

Proof. The fact that ϕ is bilinear and symmetric is immediate, once we have proved that it is well-defined. For $u, v \in \mathscr{E}$, let

$$f(t) = -\tfrac{1}{2}\mathbb{P}_1\big((J_0 u(t)\cdot\nabla)J_0 v(t) + (J_0 v(t)\cdot\nabla)J_0 u(t)\big), \quad t \in (0,\infty). \tag{3.5}$$

By the definition of $\mathscr{E}$ and Sobolev embeddings, it is easy to see that

$$(J_0 u(t)\cdot\nabla)J_0 v(t) + (J_0 v(t)\cdot\nabla)J_0 u(t) \in L^{\frac{3}{2}}(\Omega;\mathbb{R}^3).$$

By (2.1), for all $\lambda_t > 0$,

$$\big\|(\lambda_t \operatorname{Id} + A)^{-\frac{1}{4}} f(t)\big\|_2 \le 2\,C\,\|\nabla u(t)\|_2 \|\nabla v(t)\|_2$$

where C is the norm of the continuous embedding $\dot{H}^1_0(\Omega) \hookrightarrow L^6(\Omega)$, which gives that $t \mapsto (\lambda_t \operatorname{Id} + A)^{-\frac{1}{4}} f(t)$ belongs to $L^2(0,\infty;\mathcal{H})$ with the following estimate

$$\Big(\int_0^\infty \big\|(\lambda_t \operatorname{Id} + A)^{-\frac{1}{4}} f(t)\big\|_2^2\,dt\Big)^{\frac{1}{2}} \le 2\,C\,\|u\|_{\mathscr{E}}\|v\|_{\mathscr{E}} \tag{3.6}$$

Therefore, we have, choosing $\lambda_s = \frac{1}{1+(t-s)^2}$,

$$\|A^{\frac{1}{4}}\phi(u,v)(t)\|_2 \le 2\,C\int_0^t \|A^{\frac{1}{4}}T(t-s)\big(\tfrac{1}{1+(t-s)^2}\operatorname{Id} + A\big)^{\frac{1}{4}}\|_{\mathscr{L}(H)}\|\nabla u(s)\|_2\|\nabla v(s)\|_2\,ds. \tag{3.7}$$

and

$$\|\nabla\phi(u,v)(t)\|_2 \le 2\,C\int_0^t \|A^{\frac{1}{2}}T(t-s)\big(\tfrac{1}{1+(t-s)^2}\operatorname{Id} + A\big)^{\frac{1}{4}}\|_{\mathscr{L}(H)}\|\nabla u(s)\|_2\|\nabla v(s)\|_2\,ds. \tag{3.8}$$

Let $k_1(t) = \|A^{\frac{1}{4}}(\frac{1}{1+t^2}\operatorname{Id} + A)^{\frac{1}{4}}T(t)\|_{\mathscr{L}(H)}$ and $k_2(t) = \|A^{\frac{1}{2}}(\frac{1}{1+t^2}\operatorname{Id} + A)^{\frac{1}{4}}T(t)\|_{\mathscr{L}(H)}$: by Corollary 2.3 and the fact that $t \mapsto \big(\frac{1}{1+t^2}\big)^{\frac{1}{4}}$ belongs to $L^4(0,\infty)$, the function k_1 belongs to $L^2(0,\infty)$ and the function k_2 belongs to $L^{\frac{4}{3}}(0,\infty)$. Moreover, for $u, v \in \mathscr{E}$, the function $\varphi : t \mapsto \|\nabla u(t)\|_2\|\nabla v(t)\|_2$ belongs to $L^2(0,\infty)$. Therefore, by Young's inequality (since $\frac{1}{2}+\frac{1}{2}-1 = \frac{1}{\infty}$ and $\frac{3}{4}+\frac{1}{2}-1=\frac{1}{4}$),

$$t \mapsto k_1 \star \varphi(t) = \int_0^t k_1(t-s)\varphi(s)\,ds \in L^\infty(0,\infty)$$

and

$$t \mapsto k_2 \star \varphi(t) = \int_0^t k_2(t-s)\varphi(s)\,ds \in L^4(0,\infty),$$

which proves, together with (3.7) and (3.8), that $\phi(u,v) \in \mathscr{E}$ and

$$\|\phi(u,v)\|_{\mathscr{E}} \le M\,\|u\|_{\mathscr{E}}\|v\|_{\mathscr{E}}.$$

The bilinear symmetric form ϕ is then continuous from $\mathscr{E}\times\mathscr{E}$ to $\mathscr{E}$: the fact that $t \mapsto A^{\frac{1}{4}}\phi(u,v)(t)$ is continuous from $[0,\infty)$ to $\mathcal{H}$ follows directly from the strong continuity of the Stokes semigroup and the representation of $A^{\frac{1}{4}}\phi(u,v)$ as a convolution. Note that the constant M does not depend on the size nor on the regularity of Ω. □

We conclude by applying Picard's fixed point theorem (see e.g. [3, Theorem 13.2] or [7, Theorem A.1]) to obtain the following existence result for the system (1.1).

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^3$ be an open set. Then for all $u_0 \in D(\dot{A}^{\frac{1}{4}})$ with $\|A^{\frac{1}{4}}u_0\|_2 < \frac{1}{4M}$, there exists a unique $u \in \mathscr{E}$ with $\|u\|_{\mathscr{E}} < \frac{1}{2M}$ solution of $u = \alpha + \phi(u,u)$, where α and ϕ were defined in (3.3).*

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