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WELL-POSEDNESS RESULTS FOR THE NAVIER-STOKES EQUATIONS IN THE ROTATIONAL FRAMEWORK

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Dedicated to our good friend Jerry Goldstein on the occasion of his 70th birthday

Abstract. Consider the Navier-Stokes equations in the rotational framework either on $\mathbb{R}^3$ or on open sets $\Omega \subset \mathbb{R}^3$ subject to Dirichlet boundary conditions. This paper discusses recent well-posedness and ill-posedness results for both situations.

1. **Introduction.**

Well-posedness results for linear and non-linear differential equations are important aspects in many scientific articles by Jerry Goldstein. He contributed in many ways to this concept, either by investigating abstract Cauchy problems or by considering concrete partial differential equations in his many papers and in particular in his fundamental book on *Semigroups of Linear Operators and Applications*, see [8], which was published in 1985 by the Oxford University Press. Of fundamental importance in this approach are generators of semigroups and the variation of constant formula for inhomogeneous Cauchy problems in Banach spaces. Mapping properties of the semigroup combined with fixed point arguments are often the central tools for proving local or global existence results for non-linear evolution equations.

In this paper, we follow this approach and consider the Navier-Stokes equations with Coriolis force on $\Omega = \mathbb{R}^3$ or on domains $\Omega \subset \mathbb{R}^3$ subject to Dirichlet boundary conditions, i.e. we study the equation

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla\pi + \omega\, e \times u + (u\cdot\nabla)u &=& 0 & \text{in} \quad (0,\infty)\times\Omega, \\ \operatorname{div} u &=& 0 & \text{in} \quad (0,\infty)\times\Omega, \\ u &=& 0 & \text{on} \quad (0,\infty)\times\partial\Omega, \\ u(0) &=& u_0 & \text{in} \quad \Omega, \end{array}\right. \tag{1}$$

Here ω denotes the speed of rotation and e_3 is the unit vector in x_3-direction. If $\omega = 0$, the system (1) reduces to the classical Navier-Stokes system.

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This equation recently gained quite some attention due to its importance in applications to geophysical flows; in particular, large-scale atmospheric and oceanic flows are dominated by rotational effects, see e.g. [21] and [7].

Our main technique to study this equation follows the approach sketched above. In fact, considering first the linearization of equation (1), we show first that the so called Stokes-Coriolis operator generates a C_0-semigroup T on certain function spaces. Then, by the method which is nowadays called the Fujita-Kato method, well-posedness results for equation (1) in the L^2-seting will be obtained by considering the integral equation

$$u(t) = T(t)u_0 + \int_0^t T(t-s)\mathbb{P}\mathrm{div}\,(u \otimes u)(s)ds.$$

Here $\mathbb{P}$ denotes the Helmholtz projection onto the solinoidal vector fields of $L^2(\mathbb{R}^3)$. For the classical Navier-Stokes equations, this approach was generalized and pushed further to various scaling invariant function spaces. For details, we refer e.g. to the work of Kato [18], Koch and Tataru [19] and Cannone [5].

The first ill-posedness result for the classical Navier-Stokes equations is due to Bourgain and Pavlović [4] for initial data in the Besov space $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$. It means that the solution map sending an initial data to the solution given by the above formula, where T now denotes the classical Stokes semigroup, is now longer continuous with respect to this Besov-norm.

The above equation (1) was mainly studied so far in the case of $\Omega = \mathbb{R}^3$. It is a very remarkable fact that in this case the equation (1) allows a global, mild solution for arbitrary large data in the L^2-setting provided the speed Ω of rotation is fast enough, see [1], [2], [7] and [6]. More precisely, it was proved by Chemin, Desjardins, Gallagher and Grenier in [7] that for initial data $u_0 \in L^2(\mathbb{R}^2)^3 + H^{1/2}(\mathbb{R}^3)^3$ satisfying $\mathrm{div}\, u_0 = 0$, there exists a constant $\omega_0 > 0$ such that for every $\omega \geq \omega_0$ the equation (1) admits a unique, global mild solution. The case of periodic intial data was considered before by Babin, Mahalov and Nicolaenko in the papers [1] and [2].

It is a natural question to ask whether, for given and fixed $\omega > 0$, there exists a unique, global mild solution to (1) provided the norm of the initial data is sufficiently small with respect to certain norms. In this context it is natural to extend the classical Fujita-Kato approach for the Navier-Stokes equations to the rotational setting. This was carried out first by Hieber and Shibata in [15] for the case of initial data belonging to $H^{1/2}(\mathbb{R}^3)$. Generalizations of this result to the case of Fourier-Besov spaces are due to Konieczny and Yoneda [20] and Iwabuchi and Takada [16]. These results will be discussed in some detail for the case $\Omega = \mathbb{R}^3$ in the following Section 2. We will further address the question of ill-posedness of (1). Starting point here is the pioniering paper by Bourgain and Pavlović, [4], who showed ill-posedness for the classical Navier-Stokes equation, i.e. for the case $\omega = 0$, in the Besov space $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$. It was recently shown by Iwabuchi and Takada [16] that equation (1) is also ill posed in certain Fourier-Besov spaces. All of these results rely on a good description of the Stokes-Coriolis semigroup.

In Section 4 we consider equation (1) on arbitrary domains $\Omega \subset \mathbb{R}^3$. It was shown in [13] that the Stokes-Coriolis operator, defined via form methods, generates a contraction semigroup on a certain subspace $\mathcal{H}$ of $L^2(\Omega)$. Note that $\mathcal{H}$ coincides with $L^2_\sigma(\Omega)$ in the case of domains with smooth boundaries.

The above equation was also studied in the setting of nondecaying initial data in a series of papers by Giga et al, see [10], [12]. More precisely, these authors prove local existence of mild solutions to the problem (1) for initial data u_0 belonging to $L^\infty_{\sigma,a}(\mathbb{R}^3)$, a suitable subspace of $L^\infty_\sigma(\mathbb{R}^3)$. Global existence results for initial data which may not decay at infinity were obtained by Giga, Inui, Mahalov and Saal in [11]. For more details, see Section 4. For extensions of this result we refer to [23].

2. **Linear theory for $\Omega = \mathbb{R}^3$.** We start this section by considering the linear equation on $\mathbb{R}^3$, i.e.

$$\begin{aligned} u_t + \Delta u + \omega e_3 \times u + \nabla p &= 0, \quad x \in \mathbb{R}^3, t > 0 \\ \operatorname{div} u &= 0, \quad x \in \mathbb{R}^3, t > 0, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}^3, \end{aligned} \tag{2}$$

and the corresponding resolvent equation in classical L^p spaces. To this end, let $\lambda \in \Sigma_\phi$ for some $\phi \in [0, \pi/2)$, where $\Sigma_\phi = \{z \in \mathbb{C}\backslash\{0\}, |\arg z| < \phi\}$ and let $f \in L^p_\sigma(\mathbb{R}^3)^3$, the space of all solenoidal vector fields belonging to $L^p(\mathbb{R}^3)$. Taking Fourier transforms with respect to x in the resolvent equation

$$\begin{aligned} \lambda u - \nu \Delta u + \omega e_3 \times u + \nabla p &= f, \quad x \in \mathbb{R}^3, \\ \operatorname{div} u &= 0, \quad x \in \mathbb{R}^3, \end{aligned} \tag{3}$$

yields

$$\begin{aligned} (\lambda + |\xi|^2)\widehat{u_1} - \omega \widehat{u_2} + i\xi_1 \widehat{p} &= \widehat{f}_1 \\ (\lambda + |\xi|^2)\widehat{u_2} + \omega \widehat{u_1} + i\xi_2 \widehat{p} &= \widehat{f}_2 \\ (\lambda + |\xi|^2)\widehat{u_3} + i\xi_3 \widehat{p} &= \widehat{f}_3. \end{aligned} \tag{4}$$

It follows that

$$\widehat{p}(\xi) = \frac{\omega}{|\xi|^2}[i\xi_2 \widehat{u_1}(\xi) - i\xi_1 \widehat{u_2}(\xi)]. \tag{5}$$

Inserting this expression for the pressure p in the above resolvent equation (4) yields that its solution is given by $\widehat{p}$ defined as in (5) and by $\widehat{u}$ defined by

$$\widehat{u} = \frac{\omega^2}{\det} I \widehat{f} + \frac{\omega}{\det} \frac{\xi_3}{|\xi|} R \widehat{f},$$

where I is the identity matrix and

$$R = R(\xi) = \begin{pmatrix} 0 & \frac{\xi_3}{|\xi|} & -\frac{\xi_2}{|\xi|} \\ -\frac{\xi_3}{|\xi|} & 0 & \frac{\xi_1}{|\xi|} \\ \frac{\xi_2}{|\xi|} & -\frac{\xi_1}{|\xi|} & 0 \end{pmatrix}$$

and

$$\det = \det\,(\xi) = \omega^4 + \omega^2 \frac{\xi_3^2}{|\xi|^2}.$$

It follows that the solution $\widehat{u}$ of the time dependent linear problem (2) in Fourier variables, i.e. of the problem

$$\begin{aligned} \widehat{u_t}(\xi) + \nu|\xi|^2 I \widehat{u}(\xi) + \omega \widehat{e_3 \times u}(\xi) + i\xi \widehat{p}(\xi) &= 0, \quad \xi \in \mathbb{R}^3, t > 0 \\ i\xi \cdot \widehat{u}(\xi) &= 0, \quad \xi \in \mathbb{R}^3, t > 0, \\ \widehat{u}(0,\xi) &= \widehat{u_0}(\xi), \quad \xi \in \mathbb{R}^3. \end{aligned} \tag{6}$$

is given by

$$\widehat{u}(t,\xi) = \cos(\omega\frac{\xi_3}{|\xi|}t)e^{-|\xi|^2 t}I\widehat{u_0}(\xi) + \sin(\omega\frac{\xi_3}{|\xi|}t)e^{-|\xi|^2 t}R(\xi)\widehat{u_0}(\xi), \quad t \geq 0, \xi \in \mathbb{R}^3.$$

This representation combined with Plancherel's theorem implies the following result.

Proposition 2.1 ([15], Prop. 2.1)**.** *The unique solution of equation (2) in $L^2_\sigma(\mathbb{R}^3)$ is given by the bounded C_0-semigroup T_2, which is explicitly given by*

$$T_2(t)f := \mathcal{F}^{-1}[\cos(\omega\frac{\xi_3}{|\xi|}t)e^{-|\xi|^2 t}I\widehat{f}(\xi) + \sin(\omega\frac{\xi_3}{|\xi|}t)e^{-|\xi|^2 t}R(\xi)\widehat{f}(\xi)],$$

for $t \geq 0$ and $f \in L^2_\sigma(\mathbb{R}^3)^3$.

The above semigroup is called the *Stokes-Coriolis semigroup.*

Writing

$$T_p(t)f := \mathcal{F}^{-1}[\cos(\omega\frac{\xi_3}{|\xi|}t)e^{-|\xi|^2 t}I\widehat{f}(\xi) + \sin(\omega\frac{\xi_3}{|\xi|}t)e^{-|\xi|^2 t}R(\xi)\widehat{f}(\xi)] \tag{7}$$

for $t \geq 0$, $f \in L^p_\sigma(\mathbb{R}^3)^3$ and $1 < p < \infty$, we may extend the semigroup T by Mikhlin's theorem to a C_0-semigroup on $L^p(\mathbb{R}^3)^3$.

Indeed, set $\widehat{R_3 f}(\xi) := \frac{\xi_3}{|\xi|}\widehat{f}\xi$ for $\xi \neq 0$. Then Mikhlin's theorem implies the following result.

Proposition 2.2. *Let $1 < p < \infty$ and let T_p be defined as in (7). Then T_p is a C_0-semigroup on $L^p_\sigma(\mathbb{R}^3)^3$ satisfying*

$$\|T_p(t)f\|_p \leq M_p\omega^2 t^2\|f\|_p, \quad t \geq 1, f \in L^p_\sigma(\mathbb{R}^3)^3$$

for some constant M_p. Moreover, T_p may be represented as

$$T_p(t) = [\cos(\omega R_3 t)I + \sin(\omega R_3 t)R]e^{t\Delta}f, \quad t \geq 0, f \in L^p_\sigma(\mathbb{R}^3)^3].$$

Aiming for global existence results for equation (1), it is interesting to look for function spaces on which the Stokes-Coriolis semigroup defines a *bounded* semigroup. To this end, let $\Phi \in \mathcal{S}(\mathbb{R}^3)$ such that $0 \leq \widehat{\Phi}(\xi) \leq 1$, supp $\widehat{\Phi} \subset \{\xi \in \mathbb{R}^3 : 2^{-1} \leq |\xi| \leq 2\}$ and

$$\sum_{j\in\mathbb{Z}} \widehat{\Phi_j}(\xi) = 1, \quad \xi \in \mathbb{R}^3\backslash\{0\},$$

where $\Phi_j(x) := 2^{jn}\Phi(2^j x)$. Then the Fourier-Besov space $F\dot{B}^s_{p,q}(\mathbb{R}^3)$ is defined as follows: let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. Then the space $F\dot{B}^s_{p,q}(\mathbb{R}^3)$ is defined by

$$F\dot{B}^s_{p,q}(\mathbb{R}^3) := \{f \in \mathcal{S}(\mathbb{R}^3) : \widehat{f} \in L^1_{loc}(\mathbb{R}^3) \text{ and } \|f\|_{F\dot{B}^s_{p,q}(\mathbb{R}^3)} < \infty\},$$

where

$$\|f\|_{F\dot{B}^s_{p,q}(\mathbb{R}^3)} := \|\{2^{sj}\|\widehat{\Phi_j}\widehat{f}\|_p\}_{j\in\mathbb{Z}}\|_{l^q}.$$

Given the representation of Proposition 2.1 it in now not difficult to verify the following assertion.

Proposition 2.3 ([16],Lemma 2.1)**.** *There exists a constant $C > 0$ such that*

$$\|T(t)f\|_{F\dot{B}^{-1}_{1,2}(\mathbb{R}^3)} \leq C\|f\|_{F\dot{B}^{-1}_{1,2}(\mathbb{R}^3)}, \quad \omega \geq 0.$$

The following L^p–L^q as well as $H^{\frac{1}{2}}$–L^q smoothing properties for the semigroup T are established by [15] and are essential in their approach for the nonlinear problem (1).

Proposition 2.4. *Let $1 \leq p \leq 2 \leq q \leq \infty$. Then for $m \in \mathbb{N}_0$ there exists a constant $C > 0$ such that*

$$\|\nabla^m T(t)f\|_q \leq Ct^{-\frac{m}{2}-\frac{3}{2}(1/p-1/q)}\|f\|_p, \quad t > 0, f \in L^p(\mathbb{R}^3), \tag{8}$$

$$\|\Delta^{\frac{1}{4}} T(t)f\|_2 \leq Ct^{-\frac{1}{4}-\frac{3}{2}(1/p-1/2)}\|f\|_p, \quad t > 0, f \in L^p(\mathbb{R}^3). \tag{9}$$

Moreover, there exists a constant $C > 0$ such that

$$\|T(t)f\|_{\frac{1}{2}} \leq C\|f\|_{\frac{1}{2}}, \quad t > 0, f \in H^{\frac{1}{2}}(\mathbb{R}^3), \tag{10}$$

$$\|\nabla T(t)f\|_2 \leq Ct^{-\frac{1}{4}}\|f\|_{\frac{1}{2}}, \quad t > 0, f \in H^{\frac{1}{2}}(\mathbb{R}^3), \tag{11}$$

and for $q > 3$ there exists $C > 0$ such that

$$\|T(t)f\|_q \leq Ct^{-\frac{1}{2}+\frac{3}{2q}}\|f\|_{\frac{1}{2}}, \quad t > 0, f \in H^{\frac{1}{2}}(\mathbb{R}^3). \tag{12}$$

3. **Linear theory for domains.** In this section we define the Stokes and the Stokes-Coriolis operator by the theory of forms. To this end, let $\Omega \subset \mathbb{R}^3$ be an open set and let $H = L^2(\Omega;\mathbb{R}^3)$ by defined by

$$H = \big\{u = (u_1, u_2, u_3); u_i \in L^2(\Omega;\mathbb{R}^3), \text{ for all } i = 1, 2, 3\big\},$$

endowed with the usual scalar product $\langle\cdot,\cdot\rangle$. Observe that the set $\mathcal{G}$ defined by

$$\mathcal{G} := \big\{\nabla p; p \in L^2_{\text{loc}}(\Omega;\mathbb{R}) \text{ with } \nabla p \in L^2(\Omega;\mathbb{R}^3)\big\};$$

is a closed subspace of H. Set

$$\mathcal{H} := \mathcal{G}^\perp = \big\{u \in L^2(\Omega;\mathbb{R}^3); \langle u, g\rangle = 0 \text{ for all } g \in \mathcal{G}\big\}.$$

Denote by $J : \mathcal{H} \hookrightarrow H$ the canonical injection from $\mathcal{H}$ onto H and define a scalar product on $\mathcal{H}$ by

$$(u, v) \mapsto \langle Ju, Jv\rangle.$$

Endowed with this scalar product, $\mathcal{H}$ is a Hilbert space and we have

$$H = \mathcal{H} \overset{\perp}{\oplus} \mathcal{G}.$$

Finally, denote by $\mathbb{P}$ the orthogonal projection from H onto $\mathcal{H}$. Then $\mathbb{P}$ is equal to the adjoint J' of J and $\mathbb{P}J = \mathrm{Id}_{\mathcal{H}}$.

We are now in the position to apply the theory of forms as follows. Define $\mathcal{V}$ by $\mathcal{V} := \mathcal{H} \cap V$, where $V = H^1_0(\Omega;\mathbb{R}^3)$. Then $\mathcal{V}$ is a closed subspace of V a hence a Hilbert space. It follows from De Rham's theorem that $\mathcal{V}$ is dense in $\mathcal{H}$.

Next, denote by V' the dual space of V, i.e. $V' = H^{-1}(\Omega;\mathbb{R}^3)$ and let $\mathcal{V}'$ be the dual space of $\mathcal{V}$. Let $\tilde{J}$ be the canonical injection $\mathcal{V} \hookrightarrow V$. It is a restriction of the canonical injection $J : \mathcal{H} \hookrightarrow H$, so that its adjoint $\tilde{\mathbb{P}} = \tilde{J}' : V' \to \mathcal{V}'$ is an extension of the Helmholtz projection $\mathbb{P} : H \to \mathcal{H}$.

Given $\omega \geq 0$, we define sesquilinear forms a and b on $\mathcal{V} \times \mathcal{V}$ by

$$\begin{aligned} a(u, v) &:= \sum_{j=1}^n \langle\partial_j \tilde{J}u, \partial_j \tilde{J}v\rangle \quad u, v \in \mathcal{V}, \textit{and} \\ b(u, v) &:= \omega\langle e \times Ju, Jv\rangle. \end{aligned}$$

The Stokes operator A in $\mathcal{H}$ is then defined to be the operator associated with the form a, i.e. A is given by

$$\begin{aligned} D(A) &:= \left\{u \in \mathcal{V}; \tilde{\mathbb{P}}(-\Delta_D^\Omega)\tilde{J}u \in \mathcal{H}\right\} \\ Au &:= \tilde{\mathbb{P}}(-\Delta_D^\Omega)\tilde{J}u. \end{aligned}$$

Here, Δ_D^Ω denotes the Dirichlet-Laplacian on V.

Remark 3.1. The Stokes operator A has the following properties:

a) A is a self-adjoint operator in $\mathcal{H}$.
b) $D(A^{\frac{1}{2}}) = V$.
c) $-A$ generates an analytic semigroup of contractions on $\mathcal{H}$ of angle $\frac{\pi}{2}$.
d) $D(A) = \left\{u \in V; \exists p \in L^2_{\mathrm{loc}}(\Omega;\mathbb{R}) : -\Delta\tilde{J}u + \nabla p \in \mathcal{H}\right\}$.

Define the Stokes-Coriolis operator A_C in $\mathcal{H}$ to be the operator associated to the form $a+b$. We then have the following result.

Proposition 3.2 ([13])**.** *The operator $-A_C$ generates a semigroup of contractions $(T_C(t))_{t\geq 0}$ on $\mathcal{H}$ satisfying the properties*

a) $\big(t \mapsto T_C(t)\big) \in L^\infty(0,\infty;\mathscr{L}(\mathcal{H}))$ *and* $\big(t \mapsto A^{\frac{1}{2}}T_C(t)\big) \in L^2(0,\infty;\mathscr{L}(\mathcal{H}))$ *with norms less than or equal to* 1*;*
b) $\big(t \mapsto A^{\frac{1}{4}}T_C(t)A^{\frac{1}{4}}\big) \in L^2(0,\infty;\mathscr{L}(\mathcal{H}))$ *and* $\big(t \mapsto A^{\frac{1}{2}}T_C(t)A^{\frac{1}{4}}\big) \in L^{\frac{4}{3}}(0,\infty;\mathscr{L}(\mathcal{H}))$ *with norms less than or equal to 1.*

4. **Global existence results for the non-linear equation in $\Omega = \mathbb{R}^3$.** We start this section with a global existence result for data being small in the $H^{1/2}$-norm. More precisely, we have the following result.

Theorem 4.1 ([15], Theorem 3.1)**.** *There exists $\varepsilon > 0$, independent of ω, such that for any $u_0 \in H_\sigma^{\frac{1}{2}}(\mathbb{R}^3)$ with $\|u_0\|_{\frac{1}{2}} \leq \varepsilon$, the equation (1) admits a unique, mild solution $u \in C^0([0,\infty), H_\sigma^{\frac{1}{2}}(\mathbb{R}^3))^3$ satisfying $\lim_{t\to 0+}\|u(\cdot,t) - u_0\|_{1/2} = 0$.*

The proof of the above theorem is based on the Fujita-Kato argument applied to the integral equation

$$u(t) = T(t)u_0 - \int_0^t T(t-s)P[(u(s)\cdot\nabla)u(s)]\,ds, \tag{13}$$

where P denotes the Helmholtz projection. In order to do so, one has to estimate the bilinear form associated with the second term on the right hand side above in certain function spaces. For details, see [15].

The above result was recently generalized to the setting to Fourier-Besov spaces by Konieczny and Yoneda [20] and Iwabuchi and Takada [16]. In fact, their results read as follows.

Theorem 4.2 ([20], Theorem 2.2)**.** *Let $3 < p \leq \infty$. Then there exists a constant $\delta > 0$, independent of ω, such that for all $u_0 \in F\dot{B}^{2-3/p}_{p,\infty}(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ and $\|u_0\|_{F\dot{B}^{2-3/p}_{p,\infty}(\mathbb{R}^3)} \leq \delta$, the equation (1) admits a unique, global mild solution*

$$u \in C^w([0,\infty); F\dot{B}^{2-3/p}_{p,\infty}) \cap L^\infty(0,\infty; F\dot{B}^{2-3/p}_{p,\infty}).$$

Moreover, if u_0 *is small in* $X_0 = F\dot{B}B^{-1}_{1,1}(\mathbb{R}^3) \cap F\dot{B}^0_{1,1}(\mathbb{R}^3)$*, then there exists a unique global solution* u *to* (1) *such that*

$$u \in C([0,\infty); F\dot{B}^{-1}_{1,1}) \cap L^2(0,\infty; X_0).$$

Theorem 4.3 ([16], Theorem 1.2)**.** *For all* $\alpha \in (0,1)$*, there exist constants* $C, \delta > 0$*, independent of* ω*, such that for all* $u_0 \in F\dot{B}^{-1}_{1,2}(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$ *and* $\|u_0\|_{F\dot{B}^{-1}_{1,2}(\mathbb{R}^3)} \leq \delta$*, the equation* (1) *admits a unique, global mild solution* $u \in X^\alpha$*, where*

$$X^\alpha = \{u \in C([0,\infty); F\dot{B}^{-1}_{1,2}(\mathbb{R}^3)) : \|u\|_{X^\alpha} \leq 2C\delta, \operatorname{div} u = 0\}$$

and $\|u\|_{X^\alpha}$ *is defined as*

$$\|u\|_{X^\alpha} := \sup_{t>0} \|u(t)\|_{F\dot{B}^{-1}_{1,2}} + \|u\|_{Z^\alpha} + \|u\|_{Z^{-\alpha}},$$

where $\|u\|_{Z^{\pm\alpha}} = \{\sum_{j\in\mathbb{Z}}(2^{\alpha j}\|\widehat{\Phi_j}\widehat{u}\|_{L^{\frac{2}{1+\alpha}}(0,\infty;L^1(\mathbb{R}^3))})^2\}^{1/2}$.

Remark 4.4. Note that due to the embedding

$$\dot{H}^{1/2}(\mathbb{R}^3) \hookrightarrow F\dot{B}^{-1}_{1,2}(\mathbb{R}^3)$$

the above Theorem 4.3 generalizes in particular the corresponding Theorem 4.1.

Given Theorem 4.3, it is natural to ask whether equation (1) is well posed in larger function spaces as $F\dot{B}^{-1}_{1,2}(\mathbb{R}^3)$. The following result is hence very interesting.

Theorem 4.5 ([16], Theorem 1.5)**.** *For* $q \in (2,\infty]$*, the equation* (1) *is ill posed in* $F\dot{B}^{-1}_{1,q}(\mathbb{R}^3)$ *in the sense that the solution map from the initial data to the solution is not continuous.*

The proof of Theorem 4.5 is based on an ill-posedness approach for nonlinear Schrödinger equations, due to Bejenaru and Tao; see [3].

Observe that the above result is true even also in the case $\omega = 0$. It is interesting to compare Theorem 4.5 with the ill posedness result for the Navier-Stokes equations in the Besov space $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$, due to Bourgain and Pavlović [4]. In fact, since $F\dot{B}^{-1}_{1,q}(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1}_{\infty,q}(\mathbb{R}^3)$ for all $q \in [1,\infty]$, the above Theorem 4.5 generalizes the Bourgain-Pavlović result to the case $\dot{B}^{-1}_{\infty,q}(\mathbb{R}^3)$ for $q \in (2,\infty]$.

In order to describe the situation for nondecaying initial data u_0 denote first by M the space of finite $\mathbb{C}^3$-valued Radon measures on $\mathbb{R}^3$ and by FM its Fourier transform. Equipped with the norm $\|f\|_{FM} := \|\mathcal{F}^{-1}f\|_M$, the space FM becomes a Banach space, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Furthermore, denote by $\mathfrak{F}^3$ the class of all sum-closed frequency sets in $\mathbb{R}^3$; for details see [11]. Moreover, sum-closed frequency set with distance $\delta > 0$ from zero are denoted by F_δ. For $F_\delta \in \mathfrak{F}^3$ define the space

$$FM_{\sigma,\delta} := \{f \in FM : \operatorname{div} f = 0, \operatorname{supp} \widehat{f} \subset F_\delta\}.$$

The following result is due to Giga, Inui, Maholov and Saal.

Theorem 4.6 ([11],Thm.1.2)**.** *Let* $\delta > 0$, $\omega \in \mathbb{R}$, $F_\delta \in \mathfrak{F}^3$ *and* $u_0 \in FM_{\sigma,\delta}$*. If*

$$\|u_0\|_{FM} < \frac{\delta}{K}$$

for a certain $K > 0$*, then there exists a global, mild solution* $u \in BC([0,\infty), FM_{\sigma,\delta})$ *to equation* (1) *satisfying* $\|u(t) - u_0\|_{FM} \to 0$ *for* $t \to 0$ *and* $\|u(t)\|_{FM} \leq 2e^{-\delta^2 t}\|u_0\|_{FM}$ *for* $t \geq 0$.

5. **Global existence results for the non-linear equation in domains.** We define the space $\mathscr{E}$ by

$$\mathscr{E} = \Big\{u \in L^4(0,\infty; \dot{H}^1_0(\Omega;\mathbb{R}^3)); \operatorname{div} u = 0 \text{ in } (0,\infty)\times\Omega\Big\}, \tag{14}$$

endowed with the norm

$$\|u\|_{\mathscr{E}} = \|\nabla u\|_{L^4(0,\infty;L^2(\Omega;\mathbb{R}^3))}.$$

We reduce the problem of finding mild solutions for (1) by solving the equation

$$\begin{aligned} u'(t) + Au(t) + Bu(t) &= -\tilde{\mathbb{P}}\big((\tilde{J}u\cdot\nabla)\tilde{J}u\big) \\ u(0) &= u_0, \quad u \in \mathscr{E}, \end{aligned} \tag{15}$$

for which a mild solution u is given by $u = \alpha + \phi(u,u)$, where, for $t > 0$, $\alpha(t) = T_C(t)u_0$, and

$$\phi(u,v)(t) = \int_0^t T_C(t-s)\Big(-\tfrac{1}{2}\tilde{\mathbb{P}}\big((\tilde{J}u(s)\cdot\nabla)\tilde{J}v(s) + (\tilde{J}v(s)\cdot\nabla)\tilde{J}u(s)\big)\Big)\,ds. \tag{16}$$

The strategy for finding $u \in \mathscr{E}$ satisfying $u = \alpha + \phi(u,u)$ is to apply the contration principle in a suitable "good" space $\mathscr{E}$, for which $\alpha \in \mathscr{E}$ and $\phi(u,u) \in \mathscr{E}$.

Proposition 5.1. *The application $\phi : \mathscr{E}\times\mathscr{E} \to \mathscr{E}$ is bilinear, continuous and symmetric.*

Denote by M the norm of the above mapping ϕ, i.e.

$$M = \sup\big\{\|\phi(u,v)\|_{\mathscr{E}}; u,v \in \mathscr{E}, \|u\|_{\mathscr{E}}, \|v\|_{\mathscr{E}} \le 1\big\}.$$

Observe that M is independent on ω.

We define now the space X where we will consider the initial values u_0 for (1) by

$$X := \big\{f \in \mathcal{H}; t \mapsto \nabla T_C(t)f \in L^4(0,\infty;L^2(\Omega;\mathbb{R}^3)\big\}, \tag{17}$$

endowed with its norm

$$\|f\|_X = \Big(\int_0^\infty \|\nabla T_C(t)f\|_2^4\,dt\Big)^{\frac{1}{4}}.$$

Then the following result holds.

Theorem 5.2 ([13])**.** *Let $\Omega \subset \mathbb{R}^3$ be an open set. Then, for all $u_0 \in X$ with $\|u_0\|_X < \frac{1}{4M}$, there exists a unique $u \in \mathscr{E}$ satisfying $\|u\|_{\mathscr{E}} < \frac{1}{2M}$ and such that u is the solution of $u = \alpha + \phi(u,u)$, where α and ϕ are defined as in (16). Moreover, in this case, $t \mapsto A^{\frac{1}{4}}(u(t)-\alpha(t))$ belongs to $L^\infty(0,\infty;\mathcal{H})$.*

Remark 5.3. We remark that Theorem 5.2 generalizes in particular Theorem 4.1, since in the case $\Omega = \mathbb{R}^3$, it is not difficult to verify that

$$\|A^{\frac{1}{4}}\alpha\|_{\mathscr{C}([0,\infty);\mathcal{H})} \le \|A^{\frac{1}{4}}u_0\|_{\mathcal{H}} = \|u_0\|_{\frac{1}{2}} \text{ and,}$$

$$X = D(\dot{A}^{\frac{1}{4}}) = \dot{H}^{\frac{1}{2}}_\sigma \text{ and } \quad \|f\|_X = \|A^{\frac{1}{4}}f\|_{\mathcal{H}} = \|f\|_{\frac{1}{2}}, \quad f \in X.$$

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