

VARIOUS BOUNDARY CONDITIONS FOR NAVIER-STOKES EQUATIONS IN BOUNDED LIPSCHITZ DOMAINS

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ABSTRACT. We present here different boundary conditions for the Navier-Stokes equations in bounded Lipschitz domains in \mathbb{R}^3 , such as Dirichlet, Neumann or Hodge boundary conditions. We first study the linear Stokes operator associated to the boundary conditions. Then we show how the properties of the operator lead to local solutions or global solutions for small initial data.

1. **Introduction.** The aim of this paper is to describe how to find solutions of the Navier-Stokes equations

$$\begin{cases} \partial_t u - \Delta u + \nabla \pi + (u \cdot \nabla)u = 0 & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0 & \text{in } (0, T) \times \Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (1)$$

in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, and a time interval $(0, T)$ ($T \leq \infty$), for initial data u_0 in a critical space, with one of the following boundary conditions on $\partial\Omega$:

- Dirichlet boundary conditions:

$$u = 0, \quad (2)$$

- Neumann boundary conditions:

$$[\lambda(\nabla u) + (\nabla u)^\top]\nu - \pi\nu = 0, \quad \lambda \in (-1, 1], \quad (3)$$

- Hodge boundary conditions:

$$\nu \cdot u = 0, \quad \text{and } \nu \times \operatorname{curl} u = 0, \quad (4)$$

where $\nu(x)$ denotes the unit exterior normal vector on a point $x \in \partial\Omega$ (defined almost everywhere when $\partial\Omega$ is a Lipschitz boundary). The strategy is to find a functional setting in which the Fujita-Kato scheme applies, such as in their fundamental paper [4]. The paper is organized as follows. In Section 2, we define the Dirichlet-Stokes operator and then show the existence of a local solution of the system $\{(1), (2)\}$ for initial values in a critical space in the L^2 -Stokes scale. In Section 3, we adapt the previous proofs in the case of Neumann boundary conditions, i.e., for the system $\{(1), (3)\}$. In Section 4, we study (a slightly modified version of) the system $\{(1), (4)\}$ for initial conditions in the critical space

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$\{u \in L^3(\Omega; \mathbb{R}^3); \operatorname{div} u = 0 \text{ in } \Omega, \nu \cdot u = 0 \text{ on } \partial\Omega\}$. Finally, in Section 5, we discuss some open problems related to this subject.

2. Dirichlet boundary conditions. For a more complete exposition of the results in this section, as well as an extension to more general domains, the reader can refer to [16] and [11]. The case where Ω is smooth was solved by Fujita and Kato in [4]. In [2], the case of bounded Lipschitz domains Ω was studied for initial data not in a critical space.

2.1. The linear Dirichlet-Stokes operator. We start with a remark about L^2 vector fields on Ω .

Remark 1. For $\Omega \subset \mathbb{R}^3$ a bounded Lipschitz domain, let $u \in L^2(\Omega; \mathbb{R}^3)$ such that $\operatorname{div} u \in L^2(\Omega; \mathbb{R})$. Then we can define $\nu \cdot u$ on $\partial\Omega$ in the following weak sense in $H^{-\frac{1}{2}}(\partial\Omega; \mathbb{R})$: for $\phi \in H^1(\Omega; \mathbb{R})$,

$$\langle u, \nabla \phi \rangle_\Omega + \langle \operatorname{div} u, \phi \rangle_\Omega = \langle \nu \cdot u, \phi \rangle_{\partial\Omega} \tag{5}$$

where $\varphi = \operatorname{Tr}_{\partial\Omega} \phi$, the right hand-side of (5) depends only on φ on $\partial\Omega$ and not on the choice of ϕ , its extension to Ω . The notation $\langle \cdot, \cdot \rangle_E$ is for the L^2 -scalar product on E .

The space $L^2(\Omega; \mathbb{R}^3)$ is equal to the orthogonal direct sum $H_d \oplus^\perp G$ where

$$H_d = \{u \in L^2(\Omega; \mathbb{R}^3); \operatorname{div} u = 0 \text{ in } \Omega, \nu \cdot u = 0 \text{ on } \partial\Omega\} \tag{6}$$

and $G = \nabla H^1(\Omega; \mathbb{R})$. This follows from the following theorem.

Theorem 2.1 (de Rham). *Let T be a distribution in $\mathcal{C}_c^\infty(\Omega; \mathbb{R}^3)'$ such that $\langle T, \phi \rangle = 0$ for all $\phi \in \mathcal{C}_c^\infty(\Omega; \mathbb{R}^3)$ with $\operatorname{div} \phi = 0$ in Ω . Then there exists a distribution $S \in \mathcal{C}_c^\infty(\Omega; \mathbb{R})'$ such that $T = \nabla S$. Conversely, if $T = \nabla S$ with $S \in \mathcal{C}_c^\infty(\Omega; \mathbb{R})'$, then $\langle T, \phi \rangle = 0$ for all $\phi \in \mathcal{C}_c^\infty(\Omega; \mathbb{R}^3)$ with $\operatorname{div} \phi = 0$ in Ω .*

Remark 2. In the case of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, the space H_d coincides with the closure in $L^2(\Omega; \mathbb{R}^3)$ of the space of vector fields $u \in \mathcal{C}_c^\infty(\Omega; \mathbb{R}^3)$ with $\operatorname{div} u = 0$ in Ω .

We denote by $J : H_d \hookrightarrow L^2(\Omega; \mathbb{R}^3)$ the canonical embedding and $\mathbb{P} : L^2(\Omega; \mathbb{R}^3) \rightarrow H_d$ the orthogonal projection. It is clear that $\mathbb{P}J = \operatorname{Id}_{H_d}$. We define now the space $V_d = H_0^1(\Omega; \mathbb{R}^3) \cap H_d$. The embedding J restricted to V_d maps V_d to $H_0^1(\Omega; \mathbb{R}^3)$: we denote it by $J_0 : V_d \hookrightarrow H_0^1(\Omega; \mathbb{R}^3)$. Its adjoint $J_0' = \mathbb{P}_1 : H^{-1}(\Omega; \mathbb{R}^3) \rightarrow V_d'$ is then an extension of the orthogonal projection \mathbb{P} . We are now in the situation to define the Dirichlet-Stokes operator.

Definition 2.2. The Dirichlet-Stokes operator is defined as being the associated operator of the bilinear form

$$a : V_d \times V_d \rightarrow \mathbb{R}, \quad a(u, v) = \sum_{i=1}^3 \langle \partial_i J_0 u, \partial_i J_0 v \rangle.$$

Proposition 1. *The Dirichlet-Stokes operator A_d is the part in H_d of the bounded operator $A_{0,d} : V_d \rightarrow V_d'$ defined by $A_{0,d} u : V_d \rightarrow \mathbb{R}$, $(A_{0,d} u)(v) = a(u, v)$, and satisfies*

$$\begin{aligned} D(A_d) &= \{u \in V_d; \mathbb{P}_1(-\Delta_D^\Omega) J_0 u \in H_d\}, \\ A_d u &= \mathbb{P}_1(-\Delta_D^\Omega) J_0 u \quad u \in D(A_d), \end{aligned}$$

where Δ_D^Ω denotes the weak vector-valued Dirichlet-Laplacian in $L^2(\Omega; \mathbb{R}^3)$. The operator A_d is self-adjoint, invertible, $-A_d$ generates an analytic semigroup of contractions on H_d , $D(A_d^{\frac{1}{2}}) = V_d$ and for all $u \in D(A_d)$, there exists $\pi \in L^2(\Omega; \mathbb{R})$ such that

$$JA_d u = -\Delta J_0 u + \nabla \pi \tag{7}$$

and $D(A_d)$ admits the following description

$$D(A_d) = \{u \in V_d; \exists \pi \in L^2(\Omega; \mathbb{R}) : -\Delta J_0 u + \nabla \pi \in H_d\}.$$

Proof. By definition, for $u \in D(A_d)$, we have, for all $v \in V_d$,

$$\begin{aligned} \langle A_d u, v \rangle &= a(u, v) = \sum_{j=1}^n \langle \partial_j J_0 u, \partial_j J_0 v \rangle \\ &= - \sum_{j=1}^n {}_{H^{-1}} \langle \partial_j^2 J_0 u, J_0 v \rangle_{H_0^1} = {}_{H^{-1}} \langle (-\Delta) J_0 u, J_0 v \rangle_{H_0^1} \\ &= {}_{V_d'} \langle \mathbb{P}_1(-\Delta) J_0 u, v \rangle_{V_d}. \end{aligned}$$

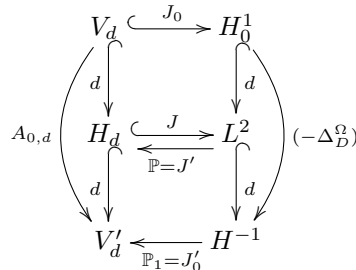
The third equality comes from the definition of weak derivatives in L^2 , the fourth equality comes from the fact that $\sum_{j=1}^n \partial_j^2 = \Delta$. The last equality is due to the fact that $J_0' = \mathbb{P}_1$. Therefore, $A_d u$ and $\mathbb{P}_1(-\Delta) J_0 u$ are two linear forms which coincide on V_d , they are then equal. So we proved here that $A_{0,d} = \mathbb{P}_1(-\Delta) J_0 : V_d \rightarrow V_d'$. Moreover, the fact that $u \in D(A_d)$ implies that $A_d u$ is a linear form on H_d , so that the linear form $\mathbb{P}_1(-\Delta) J_0 u$, originally defined on V_d , extends to a linear form on H_d (since V_d is dense in H_d by de Rham's theorem). The fact that A_d is self-adjoint and $-A_d$ generates an analytic semigroup of contractions comes from the properties of the form a : a is bilinear, symmetric, sectorial of angle 0, coercive on $V_d \times V_d$. The property that $D(A_d^{\frac{1}{2}}) = V_d$ is due to the fact that A_d is self-adjoint, applying a result by J.L. Lions [8, Théorème 5.3].

To prove the last assertions of this proposition, let $u \in D(A_d)$. Then $A_d u \in H_d$ and $\mathbb{P}_1 J(A_d u) = \mathbb{P} J(A_d u) = u$. Moreover, if $u \in D(A_d)$, u belongs, in particular, to V_d . Therefore, $J_0 u \in H_0^1(\Omega; \mathbb{R}^3)$ and $(-\Delta) J_0 u \in H^{-1}(\Omega; \mathbb{R}^3)$. We have then, the equalities taking place in V_d' ,

$$\mathbb{P}_1(J(A_d u) - (-\Delta) J_0 u) = \mathbb{P}_1 J(A_d u) - \mathbb{P}_1(-\Delta) J_0 u = A_d u - A_d u = 0.$$

By de Rham's theorem, this implies that there exists $p \in \mathcal{C}_c^\infty(\Omega; \mathbb{R})'$ such that $J(A_d u) - (-\Delta) J_0 u = \nabla p$: $\nabla p \in H^{-1}(\Omega; \mathbb{R}^3)$, which implies that $p \in L^2(\Omega; \mathbb{R})$. \square

The relations between the spaces and the operators are summarized in the following commutative diagram:



In the case of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, we also have the following property of $D(A_d^{\frac{3}{4}})$; see [11, Corollary 5.5].

Proposition 2. *The domain of $A_d^{\frac{3}{4}}$ is continuously embedded into $W_0^{1,3}(\Omega; \mathbb{R}^3)$.*

2.2. The nonlinear Dirichlet-Navier-Stokes equations. The system $\{(1), (2)\}$ is invariant under the scaling $u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$, $(\lambda^2 t, \lambda x) \in (0, T) \times \Omega$ ($\lambda > 0$): if u is a solution of $\{(1), (2)\}$ in $(0, T) \times \Omega$ for the initial value u_0 , then u_λ is a solution of $\{(1), (2)\}$ in $(0, \frac{T}{\lambda^2}) \times \frac{1}{\lambda} \Omega$ for the initial value $x \mapsto \lambda u_0(\lambda x)$.

We are interested in finding “mild” solutions of the system $\{(1), (2)\}$ for initial values u_0 in a critical space, in the same spirit as in [4].

Lemma 2.3. *The space $D(A_d^{\frac{1}{4}})$ is a critical space for the Navier-Stokes equations.*

Proof. We have to prove that $D(A_d^{\frac{1}{4}})$ is invariant under the scaling $u_\lambda(x) = \lambda u_0(\lambda x)$ for $x \in \frac{1}{\lambda} \Omega$, $\lambda > 0$. It suffices to check that $\|u_\lambda\|_2 = \lambda^{-\frac{1}{2}} \|u\|_2$ and $\|\nabla u_\lambda\|_2 = \lambda^{\frac{1}{2}} \|\nabla u\|_2$ and apply the fact that $D(A_d^{\frac{1}{4}})$ is the interpolation space (with coefficient $\frac{1}{2}$) between H_d and $V_d = D(A_d^{\frac{1}{2}})$. \square

For $T > 0$, define the space \mathcal{E}_T by

$$\mathcal{E}_T = \left\{ u \in \mathcal{C}_b([0, T]; D(A_d^{\frac{1}{4}})); u(t) \in D(A_d^{\frac{3}{4}}), u'(t) \in D(A_d^{\frac{1}{4}}) \text{ for all } t \in (0, T) \right. \\ \left. \text{and } \sup_{t \in (0, T)} \|t^{\frac{1}{2}} A_d^{\frac{3}{4}} u(t)\|_2 + \sup_{t \in (0, T)} \|t A_d^{\frac{1}{4}} u'(t)\|_2 < \infty \right\}$$

endowed with the norm

$$\|u\|_{\mathcal{E}_T} = \sup_{t \in (0, T)} \|A_d^{\frac{1}{4}} u(t)\|_2 + \sup_{t \in (0, T)} \|t^{\frac{1}{2}} A_d^{\frac{3}{4}} u(t)\|_2 + \sup_{t \in (0, T)} \|t A_d^{\frac{1}{4}} u'(t)\|_2.$$

The fact that \mathcal{E}_T is a Banach space is straightforward. Assume now that $u \in \mathcal{E}_T$, and that $(J_0 u, p)$ (with $p \in L^2(\Omega; \mathbb{R})$) satisfy $\{(1), (2)\}$ in $H^{-1}(\Omega; \mathbb{R}^3)$: indeed, every term ∇p , $\partial_t J_0 u$, $-\Delta J_0 u$ and $(J_0 u \cdot \nabla) J_0 u$ independently belong to $H^{-1}(\Omega; \mathbb{R}^3)$. We can then apply \mathbb{P}_1 to the equations and obtain

$$u'(t) + A_d u(t) = -\mathbb{P}_1((J_0 u \cdot \nabla) J_0 u)$$

since $\mathbb{P}_1 \nabla p = 0$ and $\mathbb{P}_1(-\Delta) J_0 u = A_{0,d} u$. We have then reduced the problem $\{(1), (2)\}$ into the abstract Cauchy problem

$$\begin{aligned} u'(t) + A_{0,d} u(t) &= -\mathbb{P}_1((J_0 u \cdot \nabla) J_0 u) \\ u(0) &= u_0, \quad u \in \mathcal{E}_T, \end{aligned} \tag{8}$$

for which a mild solution is given by the Duhamel formula: $u = \alpha + \phi(u, u)$, where $\alpha(t) = e^{-tA_d} u_0$ and

$$\phi(u, v)(t) = \int_0^t e^{-(t-s)A_d} \left(-\frac{1}{2} \mathbb{P}_1((J_0 u(s) \cdot \nabla) J_0 v(s) + (J_0 v(s) \cdot \nabla) J_0 u(s)) \right) ds.$$

The strategy to find $u \in \mathcal{E}_T$ satisfying $u = \alpha + \phi(u, u)$ is to apply a fixed point theorem. We have then to make sure that \mathcal{E}_T is a “good” space for the problem, i.e., $\alpha \in \mathcal{E}_T$ and $\phi(u, u) \in \mathcal{E}_T$. The fact that $\alpha \in \mathcal{E}_T$ follows directly from the properties of the Stokes operator A_d and the semigroup $(e^{-tA_d})_{t \geq 0}$.

Proposition 3. *The application $\phi : \mathcal{E}_T \times \mathcal{E}_T \rightarrow \mathcal{E}_T$ is bilinear, continuous and symmetric.*

Proof. The fact that ϕ is bilinear and symmetric is immediate, once we have proved that it is well-defined. For $u, v \in \mathcal{E}_T$, let

$$f(t) = -\frac{1}{2}\mathbb{P}_1((J_0u(t) \cdot \nabla)J_0v(t) + (J_0v(t) \cdot \nabla)J_0u(t)), \quad t \in (0, T). \tag{9}$$

By the definition of \mathcal{E}_T and Sobolev embeddings, it is easy to see that

$$(J_0u(t) \cdot \nabla)J_0v(t) + (J_0v(t) \cdot \nabla)J_0u(t) \in L^2(\Omega; \mathbb{R}^3)$$

and

$$\|(J_0u(t) \cdot \nabla)J_0v(t) + (J_0v(t) \cdot \nabla)J_0u(t)\|_2 \leq C t^{-\frac{3}{4}} \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}$$

where C is a constant independent from t . Indeed, by Proposition 2, if $u, v \in \mathcal{E}_T$, then $\nabla u, \nabla v \in L^3(\Omega, \mathbb{R}^{3 \times 3})$ with the estimates

$$\|\nabla u(t)\|_3 \leq t^{-\frac{1}{2}} \|u\|_{\mathcal{E}_T} \quad \text{and} \quad \|\nabla v(t)\|_3 \leq t^{-\frac{1}{2}} \|v\|_{\mathcal{E}_T} \quad \text{for all } t > 0.$$

Moreover, since $D(A^{\frac{1}{2}}) \hookrightarrow L^6(\Omega; \mathbb{R}^3)$, we also have

$$\|u(t)\|_6 \leq t^{-\frac{1}{4}} \|u\|_{\mathcal{E}_T} \quad \text{and} \quad \|v(t)\|_6 \leq t^{-\frac{1}{4}} \|v\|_{\mathcal{E}_T} \quad \text{for all } t > 0.$$

This, combined with the fact that $L^3 \cdot L^6 \hookrightarrow L^2$, gives the following estimate

$$\|f(t)\|_2 \leq C t^{-\frac{3}{4}} \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} \quad \text{for all } t > 0. \tag{10}$$

Therefore, we have

$$\begin{aligned} \|A_d^{\frac{1}{4}}\phi(u, v)(t)\|_2 &\leq \int_0^t \|A_d^{\frac{1}{4}}e^{-(t-s)A_d}\|_{\mathcal{L}(H_d)} C s^{-\frac{3}{4}} \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} ds \\ &\leq C \left(\int_0^t (t-s)^{-\frac{1}{4}} s^{-\frac{3}{4}} ds \right) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}, \end{aligned}$$

and since $\int_0^t (t-s)^{-\frac{1}{4}} s^{-\frac{3}{4}} ds = \int_0^1 (1-s)^{-\frac{1}{4}} s^{-\frac{3}{4}} ds$, we finally obtain the estimate

$$\|A_d^{\frac{1}{4}}\phi(u, v)(t)\|_2 \leq C \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}. \tag{11}$$

The proof of the continuity of $t \mapsto A_d^{\frac{1}{4}}\phi(u, v)(t)$ on H_d is straightforward once we have the estimate (11). The proof of the fact that

$$\|\sqrt{t}A_d^{\frac{3}{4}}\phi(u, v)(t)\|_2 \leq C \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} \tag{12}$$

is proved the same way, replacing $A_d^{\frac{1}{4}}$ by $A_d^{\frac{3}{4}}$ and using the fact that

$$\|A_d^{\frac{3}{4}}e^{-(t-s)A_d}\|_{\mathcal{L}(H_d)} \leq C (t-s)^{-\frac{3}{4}}$$

and

$$\int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{3}{4}} ds = t^{-\frac{1}{2}} \int_0^1 (1-s)^{-\frac{3}{4}} s^{-\frac{3}{4}} ds.$$

It remains to prove the estimate on the derivative with respect to t of $\phi(u, v)(t)$. Let us rewrite f as defined in (9) as follows:

$$f(s) = -\frac{1}{2}\mathbb{P}_1 \nabla \cdot (J_0u(s) \otimes J_0v(s) + J_0v(s) \otimes J_0u(s))$$

where $u \otimes v$ denotes the matrix $(u_i v_j)_{1 \leq i, j \leq 3}$ and $\nabla \cdot$ acts on matrices $M = (m_{i,j})_{1 \leq i, j \leq 3}$ the following way:

$$\nabla \cdot M = \left(\sum_{i=1}^3 \partial_i m_{i,j} \right)_{1 \leq j \leq 3}.$$

For $u, v \in \mathcal{E}_T$ and $s \in (0, T)$, we have

$$f'(s) = -\frac{1}{2}\mathbb{P}_1\nabla \cdot \left(Ju'(s) \otimes J_0v(s) + J_0u(s) \otimes Jv'(s) + Jv'(s) \otimes J_0u(s) + J_0v(s) \otimes Ju'(s) \right)$$

For all $s \in (0, T)$ we have

$$\begin{aligned} s^{\frac{5}{4}}\|Ju'(s) \otimes J_0v(s)\|_2 &\leq \|sJu'(s)\|_3 \|s^{\frac{1}{4}}J_0v(s)\|_6 \\ &\leq \|sA_d^{\frac{1}{4}}u'(s)\|_2 \|s^{\frac{1}{4}}A_d^{\frac{1}{2}}v(s)\|_2 \\ &\leq \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}, \end{aligned}$$

where the first inequality comes from the fact that $L^3 \cdot L^6 \hookrightarrow L^2$, the second comes from the Sobolev embeddings $D(A_d^{\frac{1}{4}}) \hookrightarrow L^3(\Omega; \mathbb{R}^3)$ and $D(A_d^{\frac{1}{2}}) \hookrightarrow L^6(\Omega; \mathbb{R}^3)$ and the third inequality follows directly from the definition of the space \mathcal{E}_T . Of course the same occurs for the other three terms $J_0u(s) \otimes Jv'(s)$, $Jv'(s) \otimes J_0u(s)$ and $J_0v(s) \otimes Ju'(s)$. Therefore, since $A_d^{-\frac{1}{2}}$ maps V'_d to H_d , we obtain

$$\sup_{0 < s < T} \|s^{\frac{5}{4}}A_d^{-\frac{1}{2}}f'(s)\|_2 \leq c\|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}. \tag{13}$$

We have

$$\phi(u, v)(t) = \int_0^{\frac{t}{2}} e^{-sA_d} f(t-s) ds + \int_0^{\frac{t}{2}} e^{-(t-s)A_d} f(s) ds \quad t \in (0, T),$$

and therefore

$$\begin{aligned} \phi(u, v)'(t) &= e^{-\frac{t}{2}A_d} f\left(\frac{t}{2}\right) + \int_0^{\frac{t}{2}} A_d^{\frac{1}{2}} e^{-sA_d} A_{0,d}^{-\frac{1}{2}} f'(t-s) ds \\ &\quad + \int_0^{\frac{t}{2}} -A_d e^{-(t-s)A_d} f(s) ds, \end{aligned}$$

which yields

$$\begin{aligned} \|A_d^{\frac{1}{4}}\phi(u, v)'(t)\|_2 &\leq \frac{c}{t^{\frac{1}{4}}} \|f\left(\frac{t}{2}\right)\|_2 + c \left(\int_0^{\frac{t}{2}} \frac{1}{s^{\frac{3}{4}}} \frac{1}{(t-s)^{\frac{5}{4}}} ds \right) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} \\ &\quad + c \left(\int_0^{\frac{t}{2}} \frac{1}{(t-s)^{\frac{5}{4}}} \frac{1}{s^{\frac{3}{4}}} ds \right) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} \\ &\leq \frac{c}{t} \left(1 + \int_0^{\frac{1}{2}} \frac{d\sigma}{(1-\sigma)^{\frac{5}{4}} \sigma^{\frac{3}{4}}} \right) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}, \end{aligned}$$

where we used the estimates (10), (13), and the fact that $-A_d$ generates a bounded analytic semigroup, so that $\|A_d^\alpha e^{-tA_d}\|_{\mathcal{L}(H_d)} \leq C t^{-\alpha}$. This last inequality together with (11) and (12) ensure that $\phi(u, v) \in \mathcal{E}_T$ whenever $u, v \in \mathcal{E}_T$. \square

We conclude this section by applying Picard’s fixed point theorem (see e.g. [17, Theorem A.1]) to obtain the following existence result for the system $\{(1), (2)\}$.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $u_0 \in D(A_d^{\frac{1}{4}})$. Let α and ϕ be defined as above.*

- (i) *If $\|A_d^{\frac{1}{4}}u_0\|_2$ is small enough, then there exists a unique $u \in \mathcal{E}_\infty$ solution of $u = \alpha + \phi(u, u)$.*

(ii) For all $u_0 \in D(A_d^{\frac{1}{4}})$, there exists $T > 0$ and a unique $u \in \mathcal{E}_T$ solution of $u = \alpha + \phi(u, u)$.

Uniqueness in the larger space $\mathcal{C}_b([0, T]; D(A_d^{\frac{1}{4}}))$ can be obtained, applying [15, Theorem 1.1].

3. Neumann boundary conditions. In this section, we study the system $\{(1), (3)\}$. We will only survey the results proved in [14], the method to prove existence of solutions being similar to what we done in Section 2.

3.1. The linear Neumann-Stokes operator. The boundary conditions (3) are indexed by $\lambda \in (-1, 1]$: if $\lambda = 1$, (3) becomes

$$T(u, \pi)\nu = 0 \text{ on } (0, T) \times \partial\Omega, \tag{14}$$

where $T(u, \pi) = \nabla u + (\nabla u)^\top - \pi \text{Id}$ denotes the stress tensor; if $\lambda = 0$, (3) becomes

$$\partial_\nu u - \pi\nu = 0 \text{ on } (0, T) \times \partial\Omega. \tag{15}$$

Before defining the Neumann-Stokes operator, we need the following integration by parts formula.

Lemma 3.1. *Let $\lambda \in \mathbb{R}$, $u, w : \Omega \rightarrow \mathbb{R}^3$, $\pi, \rho : \Omega \rightarrow \mathbb{R}$ sufficiently nice functions defined on the Lipschitz domain $\Omega \subset \mathbb{R}^3$. Let $L_\lambda u = \Delta u + \lambda \nabla(\text{div } u)$ and define the conormal derivative*

$$\partial_\nu^\lambda(u, \pi) = (\nabla u + (\nabla u)^\top)\nu - \pi\nu \text{ on } \partial\Omega. \tag{16}$$

Then the following integration by parts formula hold

$$\begin{aligned} \int_\Omega (L_\lambda u - \nabla\pi) \cdot w \, dx &= - \int_\Omega [I_\lambda(\nabla u, \nabla w) - \pi \text{div } w] \, dx \\ &\quad + \int_{\partial\Omega} \partial_\nu^\lambda(u, \pi) \cdot w \, d\sigma \end{aligned} \tag{17}$$

$$\begin{aligned} &= \int_\Omega (L_\lambda w - \nabla\rho) \cdot u \, dx + \int_\Omega [\pi \text{div } w - \rho \text{div } u] \, dx \\ &\quad + \int_{\partial\Omega} [\partial_\nu^\lambda(u, \pi) \cdot w - \partial_\nu^\lambda(w, \rho) \cdot u] \, d\sigma, \end{aligned} \tag{18}$$

where

$$I_\lambda(\xi, \zeta) = \sum_{i,j=1}^3 (\xi_{i,j}\zeta_{i,j} + \lambda\xi_{i,j}\zeta_{j,i}), \text{ for } \xi = (\xi_{i,j})_{1 \leq i,j \leq 3} \text{ and } \zeta = (\zeta_{i,j})_{1 \leq i,j \leq 3}.$$

Recall that $\nabla u = (\partial_i u_j)_{1 \leq i,j \leq 3}$.

The space $L^2(\Omega; \mathbb{R}^3)$ admits the following orthogonal decomposition: $H_n \overset{\perp}{\oplus} G_0$, where $G_0 = \{\nabla\pi; \pi \in H_0^1(\Omega; \mathbb{R})\}$ and

$$H_n = \{u \in L^2(\Omega; \mathbb{R}^3); \text{div } u = 0\}. \tag{19}$$

Following the steps of the previous section, we define $V_n = H^1(\Omega; \mathbb{R}^3) \cap H_n$ and $J_n : H_n \hookrightarrow L^2(\Omega; \mathbb{R}^3)$ the canonical embedding, $\mathbb{P}_n = J_n' : L^2(\Omega; \mathbb{R}^3) \rightarrow H_n$ the orthogonal projection, $\tilde{J}_n : V_n \hookrightarrow H^1(\Omega; \mathbb{R}^3)$ the restriction of J_n on V_n and $\tilde{J}_n' = \tilde{\mathbb{P}}_n : (H^1(\Omega; \mathbb{R}^3))' \rightarrow V_n'$, extension of \mathbb{P}_n to $(H^1(\Omega; \mathbb{R}^3))'$. We can now define the Neumann-Stokes operator.

Definition 3.2. Let $\lambda \in \mathbb{R}$. The Neumann-Stokes operator is defined as being the associated operator of the bilinear form

$$a_\lambda : V_n \times V_n \rightarrow \mathbb{R}, \quad a_\lambda(u, v) = \int_\Omega I_\lambda(\nabla \tilde{J}_n u, \nabla \tilde{J}_n v) \, dx$$

In the case where $\lambda \in (-1, 1]$, the bilinear form a_λ is continuous, symmetric, coercive and sectorial. So its associated operator is self-adjoint, invertible and the negative generator of an analytic semigroup of contractions on H_n .

The following proposition is a consequence of the integration by parts formula (17), [14, Theorem 6.8] and [8, Théorème 5.3].

Proposition 4. Let $\lambda \in (-1, 1]$. The Neumann-Stokes operator A_λ is the part in H_n of the bounded operator $A_{0,\lambda} : V_n \rightarrow V'_n$ defined by $(A_{0,\lambda}u)(v) = a_\lambda(u, v)$. The operator A_λ is self-adjoint, invertible, $-A_\lambda$ generates an analytic semigroup of contractions on H_n , $D(A_\lambda^{\frac{1}{2}}) = V_n$ and for all $u \in D(A_\lambda)$, there exists $\pi \in L^2(\Omega; \mathbb{R})$ such that

$$J_n A_\lambda u = -\Delta \tilde{J}_n u + \nabla \pi \tag{20}$$

and $D(A_\lambda)$ admits the following description

$$D(A_\lambda) = \{u \in V_n; \exists \pi \in L^2(\Omega; \mathbb{R}) : f = -\Delta \tilde{J}_n u + \nabla \pi \in H_n \text{ and } \partial_\nu^\lambda(u, \pi)_f = 0\},$$

where $\partial_\nu^\lambda(u, \pi)_f$ is defined in a weak sense for all $f \in (H^1(\Omega; \mathbb{R}^3))'$ by

$$\langle \partial_\nu^\lambda(u, \pi)_f, \psi \rangle_{\partial\Omega} = {}_{(H^1)'} \langle f, \Psi \rangle_{H^1} + \int_\Omega I_\lambda(\nabla \tilde{J}_n u, \nabla \Psi) \, dx - {}_{L^2} \langle \pi, \operatorname{div} \Psi \rangle_{L^2}$$

for $\Psi \in H^1(\Omega)$ and $\psi = \operatorname{Tr}_{\partial\Omega} \Psi$.

Remark 3. If $f \in (H^1(\Omega; \mathbb{R}^3))'$, the quantity $\partial_\nu^\lambda(u, \pi)_f$ exists in the Besov space $B_{-\frac{1}{2}}^{2,2}(\partial\Omega; \mathbb{R}^3) = H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^3)$ according to [14, Proposition 3.6].

Thanks to [14, Sections 9 & 10], we have a good description of the domain of fractional powers of the Neumann-Stokes operator A_λ . In particular, in [14, Corollary 10.6] it was established that

$$D(A_\lambda^{\frac{3}{4}}) \text{ is continuously embedded into } W^{1,3}(\Omega; \mathbb{R}^3). \tag{21}$$

3.2. The nonlinear Neumann-Navier-Stokes equations. The results in 3.1 allow us to prove a result similar to Theorem 2.4 for the system $\{(1), (3)\}$. As in the previous section, it is not difficult to see that $D(A_\lambda^{\frac{1}{4}}) \hookrightarrow L^3(\Omega; \mathbb{R}^3)$ is a critical space for the system. For $T \in (0, \infty]$, following the definition of \mathcal{E}_T in Section 2, we define

$$\begin{aligned} \mathcal{F}_T = & \left\{ u \in \mathcal{C}_b([0, T]; D(A_\lambda^{\frac{1}{4}})); u(t) \in D(A_\lambda^{\frac{3}{4}}), u'(t) \in D(A_\lambda^{\frac{1}{4}}) \text{ for all } t \in (0, T] \right. \\ & \left. \text{and } \sup_{t \in (0, T)} \|t^{\frac{1}{2}} A_\lambda^{\frac{3}{4}} u(t)\|_2 + \sup_{t \in (0, T)} \|t A_\lambda^{\frac{1}{4}} u'(t)\|_2 < \infty \right\} \end{aligned}$$

endowed with the norm

$$\|u\|_{\mathcal{F}_T} = \sup_{t \in (0, T)} \|A_\lambda^{\frac{1}{4}} u(t)\|_2 + \sup_{t \in (0, T)} \|t^{\frac{1}{2}} A_\lambda^{\frac{3}{4}} u(t)\|_2 + \sup_{t \in (0, T)} \|t A_\lambda^{\frac{1}{4}} u'(t)\|_2.$$

The same tools as in 2.2 apply, so we can prove the following result (see [14, Theorem 11.3]).

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $u_0 \in D(A_\lambda^{\frac{1}{4}})$. Let β and ψ be defined by*

$$\beta(t) = e^{-tA_\lambda} u_0, \quad t \geq 0,$$

and for $u, v \in \mathcal{F}_T$ and $t \in (0, T)$,

$$\psi(u, v)(t) = \int_0^t e^{-(t-s)A_\lambda} \left(-\frac{1}{2}\mathbb{P}_n\right) \left((J_n u(s) \cdot \nabla) \tilde{J}_n v(s) + J_n v(s) \cdot \nabla \tilde{J}_n u(s) \right) ds.$$

- (i) *If $\|A_\lambda^{\frac{1}{4}} u_0\|_2$ is small enough, then there exists a unique $u \in \mathcal{F}_\infty$ solution of $u = \beta + \psi(u, u)$.*
- (ii) *For all $u_0 \in D(A_\lambda^{\frac{1}{4}})$, there exists $T > 0$ and a unique $u \in \mathcal{F}_T$ solution of $u = \beta + \psi(u, u)$.*

A comment here may be necessary to link the solution u obtained in Theorem 3.3 and a solution of the system $\{(1), (3)\}$. If $u \in \mathcal{F}_T$, then $u' \in H_n$ and $(J_n u \cdot \nabla) \tilde{J}_n u \in L^2(\Omega; \mathbb{R}^n)$. Moreover, if u satisfies the equation $u = \beta + \psi(u, u)$, then u is a mild solution of

$$A_\lambda u = -u' - \mathbb{P}_n((J_n u \cdot \nabla) \tilde{J}_n u) \in H_n.$$

Going further, we may write

$$J_n \mathbb{P}_n((J_n u \cdot \nabla) \tilde{J}_n u) = (J_n u \cdot \nabla) \tilde{J}_n u - \nabla q$$

where $q \in H_0^1(\Omega; \mathbb{R})$ satisfies

$$\Delta q = \operatorname{div} (J_n u \cdot \nabla) \tilde{J}_n u \in H^{-1}(\Omega; \mathbb{R}^n).$$

Therefore, we have by definition of A_λ , there exists $\pi \in L^2(\Omega, \mathbb{R})$ such that

$$-\Delta \tilde{J}_n u + \nabla \pi = J_n(A_\lambda u) = -J_n u' - (J_n u \cdot \nabla) \tilde{J}_n u + \nabla q$$

and at the boundary, (u, π) satisfies (3) in the weak sense as in Proposition 4. Since $q \in H_0^1(\Omega; \mathbb{R})$, $(u, \pi - q)$ satisfies also (3). This proves that $(u, \pi - q)$ is a solution of the system $\{(1), (3)\}$.

The uniqueness is true in a larger space than \mathcal{F}_T : for each $u_0 \in D(A_\lambda^{\frac{1}{4}})$, there is at most one $u \in \mathcal{C}_b([0, T]; D(A_\lambda^{\frac{1}{4}}))$, mild solution of the system $\{(1), (3)\}$. For a more precise statement, see [14, Theorem 11.8].

4. Hodge boundary conditions. Most of the results presented here are proved thoroughly in [12] for the linear theory and [13] for the nonlinear system. We start with the study of the linear Hodge-Laplacian on L^p -spaces and then move to the Hodge-Stokes operator before applying the properties of this operator to prove the existence of mild solutions of the Hodge-Navier-Stokes system in L^3 .

4.1. The Hodge-Laplacian. Let $H = L^2(\Omega; \mathbb{R}^3)$ and

$$V = \{u \in H; \operatorname{curl} u \in H, \operatorname{div} u \in L^2(\Omega; \mathbb{R}) \text{ and } \nu \cdot u = 0 \text{ on } \partial\Omega\}.$$

We start by defining on $V \times V$ the following form

$$b : V \times V \rightarrow \mathbb{R}, \quad b(u, v) = \langle \operatorname{curl} u, \operatorname{curl} v \rangle + \langle \operatorname{div} u, \operatorname{div} v \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes either the scalar or the vector-valued L^2 -pairing.

Remark 4. Contrary to the case of smooth bounded domains (with a $\mathcal{C}^{1,1}$ boundary), the space V is not contained in $H^1(\Omega; \mathbb{R}^3)$. The Sobolev embedding associated to the space V is as follows: $V \hookrightarrow H^{\frac{1}{2}}(\Omega; \mathbb{R}^3)$ with the estimate

$$\|u\|_{H^{\frac{1}{2}}} \leq C [\|u\|_2 + \|\operatorname{curl} u\|_2 + \|\operatorname{div} u\|_2], \quad u \in V; \tag{22}$$

see for instance [10].

Proposition 5. *The Hodge-Laplacian operator B , defined as the associated operator in H of the form b , satisfies*

$$\begin{aligned} D(B) &= \{u \in V; \nabla \operatorname{div} u \in H, \operatorname{curl} \operatorname{curl} u \in H \text{ and } \nu \times \operatorname{curl} u = 0 \text{ on } \partial\Omega\} \\ Bu &= -\Delta u, \quad u \in D(B). \end{aligned}$$

Since the form b is continuous, bilinear, symmetric, coercive and sectorial, the operator $-B$ generates an analytic semigroup of contractions on H , B is self-adjoint and $D(B^{\frac{1}{2}}) = V$.

Remark 5. As in Remark 1 for a bounded Lipschitz domain Ω and a vector field $w \in H$ satisfying $\operatorname{curl} w \in H$, we can define $\nu \times w$ on $\partial\Omega$ in the following weak sense in $H^{-\frac{1}{2}}(\partial\Omega; \mathbb{R}^3)$: for $\phi \in H^1(\Omega; \mathbb{R}^3)$,

$$\langle \operatorname{curl} w, \phi \rangle_\Omega - \langle w, \operatorname{curl} \phi \rangle_\Omega = \langle \nu \times w, \phi \rangle_{\partial\Omega} \tag{23}$$

where $\varphi = \operatorname{Tr}_{\partial\Omega} \phi$, the right hand-side of (23) depends only on φ on $\partial\Omega$ and not on the choice of ϕ , its extension to Ω .

To prove that B extends to L^p -spaces, we prove that its resolvent admits $L^2 - L^2$ off-diagonal estimates. This was proved in [12, Section 6]

Proposition 6. *There exist two constants $C, c > 0$ such that for any open sets $E, F \subset \Omega$ such that $\operatorname{dist}(E, F) > 0$ and for all $t > 0$, $f \in H$ and*

$$u = (\operatorname{Id} + t^2 B)^{-1}(\chi_F f),$$

we have

$$\|\chi_E u\|_2 + t\|\chi_E \operatorname{div} u\|_2 + t\|\chi_E \operatorname{curl} u\|_2 \leq C e^{-c \frac{\operatorname{dist}(E, F)}{t}} \|\chi_F f\|_2. \tag{24}$$

Proof. We start by choosing a smooth cut-off function $\xi : \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfying $\xi = 1$ on E , $\xi = 0$ on F and $\|\nabla \xi\|_\infty \leq \frac{k}{\operatorname{dist}(E, F)}$. We then define $\eta = e^{\alpha \xi}$ where $\alpha > 0$ is to be chosen later. Next, we take the scalar product of the equation

$$u - t^2 \Delta u = \chi_F f, \quad u \in D(B)$$

with the function $v = \eta^2 u$. Since $\eta = 1$ on F and $\|u\|_2 \leq \|\chi_F f\|_2$, it is easy to check then that

$$\begin{aligned} &\|\eta u\|_2^2 + t^2 \|\eta \operatorname{div} u\|_2^2 + t^2 \|\eta \operatorname{curl} u\|_2^2 \\ &\leq \|\chi_F f\|_2^2 + 2\alpha \|\nabla \xi\|_\infty t^2 \|\eta u\|_2 (\|\eta \operatorname{div} u\|_2 + \|\eta \operatorname{curl} u\|_2) \end{aligned}$$

and therefore, using the estimate on $\|\nabla \xi\|_\infty$ and choosing $\alpha = \frac{\operatorname{dist}(E, F)}{4kt}$, we obtain

$$\|\eta u\|_2^2 + t^2 \|\eta \operatorname{div} u\|_2^2 + t^2 \|\eta \operatorname{curl} u\|_2^2 \leq 2\|\chi_F f\|_2^2.$$

Using now the fact that $\eta = e^\alpha$ on E , we finally get

$$\|\chi_E u\|_2 + t\|\chi_E \operatorname{div} u\|_2 + t\|\chi_E \operatorname{curl} u\|_2 \leq \sqrt{2} e^{-\frac{\operatorname{dist}(E, F)}{4kt}} \|\chi_F f\|_2,$$

which gives (24) with $C = \sqrt{2}$ and $c = \frac{1}{4k}$. □

With a slight modification of the proof, we can show that for all $\theta \in (0, \pi)$ there exist two constants $C, c > 0$ such that for any open sets $E, F \subset \Omega$ such that $\text{dist}(E, F) > 0$ and for all $z \in \Sigma_{\pi-\theta} = \{\omega \in \mathbb{C} \setminus \{0\}; |\arg z| < \pi - \theta\}$, $f \in H$ and

$$u = (z\text{Id} + B)^{-1}(\chi_F f),$$

we have

$$|z| \|\chi_E u\|_2 + |z|^{\frac{1}{2}} \|\chi_E \text{div } u\|_2 + |z|^{\frac{1}{2}} \|\chi_E \text{curl } u\|_2 \leq C e^{-c \text{dist}(E, F) |z|^{\frac{1}{2}}} \|\chi_F f\|_2. \tag{25}$$

With that in hand and the Sobolev embedding (22), together with the rescaled Sobolev inequality

$$R^{\frac{1}{2}} \|\chi_E u\|_3 \leq C (\|\chi_E u\|_2 + R \|\chi_E \text{div } u\|_2 + R \|\chi_E \text{curl } u\|_2) \tag{26}$$

where $R = \text{diam } E$, we can prove that, choosing $E = \Omega \cap B(x, |z|^{-\frac{1}{2}})$ and $F_j = \Omega \cap (B(x, 2^{j+1}|z|^{-\frac{1}{2}}) \setminus B(x, 2^j|z|^{-\frac{1}{2}}))$ for $x \in \Omega$ and $j \in \mathbb{N}$:

$$|z| \|\chi_E u\|_3 \leq C |z|^{-\frac{1}{4}} e^{-c 2^j} \|f_j\|_2 \tag{27}$$

where $f_j = \chi_{F_j} f$.

Proposition 7. *There exists a constant $C > 0$ such that for all $f \in L^2(\Omega; \mathbb{R}^3) \cap L^3(\Omega; \mathbb{R}^3)$, $z \in \Sigma_{\pi-\theta}$, the following estimate holds:*

$$|z| \|(z\text{Id} + B)^{-1} f\|_3 \leq C \|f\|_3. \tag{28}$$

Proof. For $x \in \Omega$ and $r > 0$, denote by $B_\Omega(x, r)$ the ball centered in x with radius r intersected with Ω . Let $u = (z\text{Id} + B)^{-1} f$. For $x \in \Omega$, let $f_j = \chi_{F_j} f$ for $F_j = B_\Omega(x, 2^{j+1}|z|^{-\frac{1}{2}}) \setminus B_\Omega(x, 2^j|z|^{-\frac{1}{2}})$ and $u_j = (z\text{Id} + B)^{-1} f_j$. From (27) and Fubini's theorem, keeping in mind that a Lipschitz domain in \mathbb{R}^n is a n -set in the terminology of [7] (which means that balls centered in Ω with radius r intersected with Ω have a volume equivalent to r^n), we have

$$\begin{aligned} |z| \|u\|_3 &\leq C |z| \left[\int_\Omega \left(|B_\Omega(x, t)|^{-1} \int_{B_\Omega(x, t)} |u(y)|^3 dy \right) dx \right]^{\frac{1}{3}} \\ &\leq C |z| \left[\int_\Omega \left[\left(|B_\Omega(x, t)|^{-1} \int_{B_\Omega(x, t)} |u(y)|^3 dy \right)^{\frac{1}{3}} \right]^3 dx \right]^{\frac{1}{3}} \\ &\leq C |z| \left[\int_\Omega \left[\sum_{j=0}^\infty \left(|B_\Omega(x, t)|^{-1} \int_{B_\Omega(x, t)} |u_j(y)|^3 dy \right)^{\frac{1}{3}} \right]^3 dx \right]^{\frac{1}{3}} \\ &\leq C \left[\int_\Omega \left(\sum_{j=0}^\infty C e^{-c 2^j} 2^{\frac{3j}{2}} \left(|B_\Omega(x, 2^j t)|^{-1} \int_{B_\Omega(x, 2^j t)} |f(y)|^2 dy \right)^{\frac{1}{2}} \right)^3 dx \right]^{\frac{1}{3}} \\ &\leq C \left(\sum_{j=0}^\infty e^{-c 2^j} 2^{\frac{3j}{2}} \right) \|M(|f|^2)\|_{L^{\frac{3}{2}}(\Omega; \mathbb{R})}^{\frac{1}{2}} \\ &\leq C \|f\|_3 \end{aligned}$$

where we used the notation $t = |z|^{-\frac{1}{2}}$ and M denotes the Hardy-Littlewood maximal operator (which is bounded on L^p for all $p \in (1, \infty)$). □

Corollary 1. *The semigroup $(e^{-tB})_{t \geq 0}$ extends to a bounded analytic semigroup on $L^p(\Omega; \mathbb{R}^3)$ for $p \in [\frac{3}{2}, 3]$.*

Proof. For $p = 3$, this comes directly from Proposition 7. We obtain the result for all $p \in [2, 3]$ by interpolation and for all $p \in [\frac{3}{2}, 2]$ by duality (since the operator B is self-adjoint). □

We can actually prove that the semigroup $(e^{-tB})_{t \geq 0}$ extends to a bounded analytic semigroup on $L^p(\Omega; \mathbb{R}^3)$ for p in an interval containing $[\frac{6}{5}, 6]$. In an open interval (p_Ω, q_Ω) containing $[\frac{3}{2}, 3]$, the negative generator B_p of this semigroup satisfies

$$\begin{aligned} D(B_p) &= \{u \in L^p(\Omega; \mathbb{R}^3); \operatorname{div} u \in W^{1,p}(\Omega; \mathbb{R}^3), \operatorname{curl} u \in L^p(\Omega; \mathbb{R}^3), \\ &\quad \operatorname{curl} \operatorname{curl} u \in L^p(\Omega; \mathbb{R}^3), \nu \cdot u = 0 \text{ and } \nu \times \operatorname{curl} u = 0 \text{ on } \partial\Omega\} \\ B_p u &= -\Delta u, \quad u \in D(B_p). \end{aligned}$$

To obtain estimates in L^p for $p > 3$, the method is in the same spirit as what we have just done, combined with a bootstrap argument and regularity results for B . For a complete proof, the reader may refer to [12, Section 5].

4.2. The nonlinear Hodge-Navier-Stokes equations. Granted that u is a sufficiently smooth vector field, we have the following identification

$$(u \cdot \nabla)u = \frac{1}{2} \nabla |u|^2 + u \times \operatorname{curl} u.$$

That is, replacing π in (1) by $\pi + \frac{1}{2}|u|^2$, the system $\{(1), (4)\}$ reads

$$\left\{ \begin{array}{ll} \partial_t u - \Delta u + \nabla \pi + u \times \operatorname{curl} u = 0 & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0 & \text{in } (0, T) \times \Omega, \\ u(0) = u_0 & \text{in } \Omega, \\ \nu \cdot u = 0 & \text{on } (0, T) \times \partial\Omega, \\ \nu \times \operatorname{curl} u = 0 & \text{on } (0, T) \times \partial\Omega. \end{array} \right. \tag{29}$$

Before trying to solve this system, we need some facts about the Hodge-Stokes operator. In [3], it was proved that the orthogonal projection \mathbb{P} defined in Section 2 on $L^2(\Omega; \mathbb{R}^3)$ extends to a bounded projection on $L^p(\Omega; \mathbb{R}^3)$ for p in an open interval (p_Ω, q_Ω) containing $[\frac{3}{2}, 3]$; denote it by \mathbb{P}_p . In [12, Lemma 3.7], it was proved that \mathbb{P}_p and B_p , the Hodge-Laplacian in $L^p(\Omega; \mathbb{R}^3)$ commute on $D(B_p)$. This allows us to define the Hodge-Stokes operator A_p on

$$H_p = \{u \in L^p(\Omega; \mathbb{R}^3); \operatorname{div} u = 0 \text{ in } \Omega, \text{ and } \nu \cdot u = 0 \text{ on } \partial\Omega\}.$$

The results we proved for the Hodge-Laplacian naturally extend to the Hodge-Stokes operator as stated in the following theorem.

Theorem 4.1. *Let $p \in (p_\Omega, q_\Omega)$. The Hodge-Stokes operator A_p defined on H_p by*

$$\begin{aligned} D(A_p) &= \{u \in H_p; \operatorname{curl} u \in L^p(\Omega; \mathbb{R}^3), \operatorname{curl} \operatorname{curl} u \in L^p(\Omega; \mathbb{R}^3) \\ &\quad \text{and } \nu \times \operatorname{curl} u = 0 \text{ on } \partial\Omega\} \\ A_p u &= -\mathbb{P}_p \Delta u = -\Delta u, \quad u \in D(A_p) \end{aligned}$$

is the negative generator of a bounded analytic semigroup on H_p defined by

$$e^{-tA_p}u = \mathbb{P}_p e^{-tB_p}u = e^{-tB_p}\mathbb{P}_p u = e^{-tB_p}u, \quad u \in H_p.$$

Moreover, this semigroup satisfies the uniform estimate

$$\begin{aligned} \sup_{t>0} \left(\|e^{-tA_p}\|_{\mathcal{L}(H_p)} + \|\sqrt{t} \operatorname{curl} e^{-tA_p}\|_{\mathcal{L}(H_p, L^p)} \right. \\ \left. + \|t \operatorname{curl} \operatorname{curl} e^{-tA_p}\|_{\mathcal{L}(H_p, L^p)} \right) < \infty. \end{aligned} \tag{30}$$

We now rewrite the nonlinear Hodge-Navier-Stokes system for initial data in the critical space H_3 in the abstract form

$$u'(t) + A_p u(t) + \mathbb{P}_p(u(t) \times \operatorname{curl} u(t)) = 0, \quad u_0 \in H_3, \tag{31}$$

for p to be determined. The idea to solve (31) is to apply the same method as in Sections 2 & 3. To do so, we need a regularizing property of the Hodge-Stokes semigroup, which was proved in [13, Theorem 3.1 and Theorem 4.1]: the Hodge-Stokes semigroup satisfies the estimate

$$\sup_{t>0} \left(\|t^{\frac{\alpha}{2}} e^{-tA_p}\|_{\mathcal{L}(H_p, L^q)} + \|t^{\frac{1+\alpha}{2}} \operatorname{curl} e^{-tA_p}\|_{\mathcal{L}(H_p, L^q)} \right) < \infty \tag{32}$$

whenever $p \in (p_\Omega, q_\Omega)$, $q \in (p, q_\Omega)$ with $\frac{1}{p} - \frac{\alpha}{3} = \frac{1}{q}$ for some $\alpha \in (0, 1)$. The proof of this results relies on the possibility to find an “inverse of the curl” modulo gradient vectors and uses results proved in [9].

With these properties of the Hodge-Stokes semigroup in hand, the following existence result for (31) is almost immediate. For $T \in (0, \infty]$, we define the space \mathcal{G}_T by

$$\begin{aligned} \mathcal{G}_T = \left\{ u \in \mathcal{C}_b([0, T], H_3) \cap \mathcal{C}((0, T), H_{3(1+\varepsilon)}); \operatorname{curl} u \in \mathcal{C}((0, T), L^3(\Omega, \mathbb{R}^3)) \right. \\ \left. \text{with } \sup_{0 < t < T} (\|s^{\frac{\varepsilon}{2(1+\varepsilon)}} u(s)\|_{3(1+\varepsilon)} + \|\sqrt{s} \operatorname{curl} u(s)\|_3) < \infty \right\} \end{aligned}$$

endowed with the norm

$$\|u\|_{\mathcal{G}_T} = \sup_{0 < t < T} (\|u(s)\|_3 \|s^{\frac{\varepsilon}{2(1+\varepsilon)}} u(s)\|_{3(1+\varepsilon)} + \|\sqrt{s} \operatorname{curl} u(s)\|_3),$$

where $\varepsilon > 0$ is such that $3(1 + \varepsilon) < q_\Omega$.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $u_0 \in H_3$. Let γ and Φ be defined by*

$$\gamma(t) = e^{-tA_p}u_0, \quad t \geq 0,$$

and for $u, v \in \mathcal{G}_T$, and $t \in (0, T)$,

$$\Phi(u, v)(t) = \int_0^t e^{-(t-s)A_{\frac{3}{2}}} \left(-\frac{1}{2}\mathbb{P}_{\frac{3}{2}}\right) ((u(s) \times \operatorname{curl} v(s) + v(s) \times \operatorname{curl} u(s)) ds.$$

- (i) *If $\|u_0\|_3$ is small enough, then there exists a unique $u \in \mathcal{G}_\infty$ solution of $u = \gamma + \Phi(u, u)$.*
- (ii) *For all $u_0 \in H_3$, there exists $T > 0$ and a unique $u \in \mathcal{G}_T$ solution of $u = \gamma + \Phi(u, u)$.*

For a complete proof of this theorem, we refer to [13, Section 5].

5. Remarks and open problems.

5.1. Comparison between the boundary conditions. The boundary conditions (2), (3) and (4) can be decomposed, for sufficiently regular vector fields u , into their normal part and their tangential part as follows

(i) (2) becomes

$$\nu \cdot u = 0 \quad \text{and} \quad \nu \times u = 0 \quad \text{on } \partial\Omega, \quad (33)$$

(ii) (3) becomes

$$\nu \cdot [(\nabla u + \nabla u^\top)\nu] = \pi \quad \text{and} \quad [(\nabla u + \nabla u^\top)\nu]_{tan} = 0 \quad \text{on } \partial\Omega \text{ if } \lambda = 1, \quad (34)$$

(iii) the Navier's slip boundary conditions read

$$\nu \cdot u = 0 \quad \text{and} \quad [(\nabla u + \nabla u^\top)\nu]_{tan} = 0 \quad \text{on } \partial\Omega, \quad (35)$$

(iv) (4) is already decomposed into its normal part $\nu \cdot u = 0$ and its tangential part $\nu \cdot \operatorname{curl} u = 0$ on $\partial\Omega$.

It is common to identify the Navier's slip boundary conditions (35) with the Hodge boundary conditions (4). This is true only on flat parts of the boundary. In the case of a \mathcal{C}^2 domain Ω , it can be proved that (35) and (4) differ only by a zero-order term. For more informations on this subject, the interested reader could refer to [13, Section 2].

5.2. Open problems. In the case of a smooth bounded domain in \mathbb{R}^n , it was proved by Y. Giga and T. Miyakawa in [6] that the Dirichlet-Navier-Stokes system admits a local mild solution for initial values in L^n (critical space for the system in dimension n). Their method relies on the fact that the Dirichlet-Stokes operator, as defined in Section 2, extends to all L^p spaces and is the negative generator of an analytic semigroup there, which was proved in [5]. The situation in Lipschitz domains is different. For instance, P. Deuring provided in [1] an example of a domain with one conical singularity such that the Dirichlet-Stokes semigroup does not extend to an analytic semigroup in L^p for p large (or p small), away from 2.

As already mentioned, E. Fabes, O. Mendez and M. Mitrea proved in [3] that the orthogonal projection \mathbb{P} defined in Section 2 on $L^2(\Omega; \mathbb{R}^3)$ extends to a bounded projection on $L^p(\Omega; \mathbb{R}^3)$ for p in an open interval containing $[\frac{3}{2}, 3]$ (if Ω is \mathcal{C}^1 , then this interval is $(1, \infty)$). This led M. Taylor in [18] to formulate the conjecture that the Dirichlet-Stokes semigroup defined originally on H_d extends to an analytic semigroup on L^p for p in the same interval as in [3].

Remark 6. This conjecture is actually true when, instead of considering Dirichlet boundary conditions, we consider Hodge boundary conditions, as proved in Section 4.

In the same paper [3], the authors proved that the orthogonal projection \mathbb{P}_n defined in Section 3 on $L^2(\Omega; \mathbb{R}^3)$ also extends to a bounded projection on $L^p(\Omega; \mathbb{R}^3)$ for p in the same open interval containing $[\frac{3}{2}, 3]$. This leads to the conjecture similar to Taylor's that the Neumann-Stokes semigroup defined originally on H_n extends to an analytic semigroup on L^p for p in the same interval.

As for now, no positive result is known in L^p for $p \neq 2$ for these two conjectures. To apply the Fujita-Kato scheme as in Sections 2 & 3, proving that the Stokes semigroup extends to an analytic semigroup in L^3 seems to be the first step to obtain mild solutions of the Navier-Stokes system with either Dirichlet or Neumann boundary conditions.

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