A three lines proof for traces of H^1 functions on special Lipschitz domains

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1 Introduction

It is well known (see [2, Theorem 1.2]) that for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, the trace operator $\mathrm{Tr}_{|\partial\Omega} : \mathscr{C}(\overline{\Omega}) \to \mathscr{C}(\partial\Omega)$ restricted to $\mathscr{C}(\overline{\Omega}) \cap H^1(\Omega)$ extends to a bounded operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$ and the following estimate holds:

$$\|\mathrm{Tr}_{|\partial\Omega}\, u\|_{L^2(\partial\Omega)} \leq C \left(\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega,\mathbb{R}^n)}\right) \qquad \text{for all } u \in H^1(\Omega), \tag{1.1}$$

where $C = C(\Omega) > 0$ is a constant depending on the domain Ω. This result can be proved via a simple integration by parts and Cauchy-Schwarz inequality if the domain is the upper graph of a Lipschitz function, i.e.,

$$\Omega = \left\{x = (x_h, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; x_n > \omega(x_h)\right\} \tag{1.2}$$

where $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$ is a globally Lipschitz function.

2 The result

Let $\Omega \subset \mathbb{R}^n$ be a domain of the form (1.2). The exterior unit normal ν of Ω at a point $x = (x_h, \omega(x_h))$ on the boundary Γ of Ω:

$$\Gamma := \left\{x = (x_h, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; x_n = \omega(x_h)\right\}$$

is given by

$$\nu(x_h, \omega(x_h)) = \frac{1}{\sqrt{1 + |\nabla_h \omega(x_h)|^2}} \left(\nabla_h \omega(x_h), -1\right)$$

(∇_h denotes the "horizontal gradient" on $\mathbb{R}^{n-1}$ acting on the "horizontal variable" x_h). We denote by $\theta \in [0, \frac{\pi}{2})$ the angle

$$\theta = \arccos\Big(\inf_{x_h \in \mathbb{R}^{n-1}} \frac{1}{\sqrt{1 + |\nabla_h \omega(x_h)|^2}}\Big), \tag{2.1}$$

so that in particular for $e = (0_{\mathbb{R}^{n-1}}, 1)$ the "vertical" direction, we have

$$-e \cdot \nu(x_h, \omega(x_h)) = \frac{1}{\sqrt{1 + |\nabla_h \omega(x_h)|^2}} \geq \cos\theta > 0, \quad \text{for all } x_h \in \mathbb{R}^{n-1}. \tag{2.2}$$

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be as above. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support. Then*

$$\int_\Gamma |\varphi|^2 \,\mathrm{d}\sigma \leq \frac{2}{\cos\theta} \|\varphi\|_{L^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega,\mathbb{R}^n)}, \tag{2.3}$$

where θ has been defined in (2.1).

Proof. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support, and apply the divergence theorem in Ω with $u = \varphi^2\, e$ where $e = (0_{\mathbb{R}^{n-1}}, 1)$. Since $\operatorname{div}(\varphi^2\, e) = 2\,\varphi\,(e \cdot \nabla\varphi)$, we obtain

$$\int_\Omega 2\,\varphi\,(e \cdot \nabla\varphi)\,\mathrm{d}x = \int_\Omega \operatorname{div}(\varphi^2\, e)\,\mathrm{d}x = \int_\Gamma \nu \cdot (\varphi^2\, e)\,\mathrm{d}\sigma.$$

Therefore using (2.2) and Cauchy-Schwarz inequality, we get

$$\cos\theta \int_\Gamma \varphi^2\,\mathrm{d}\sigma \le -2 \int_\Omega \varphi\,(e \cdot \nabla\varphi)\,\mathrm{d}x \le 2\,\|\varphi\|_{L^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega,\mathbb{R}^n)},$$

which gives the estimate (2.3). □

Corollary 2.2. *There exists a unique operator $T \in \mathscr{L}(H^1(\Omega), L^2(\Gamma))$ satisfying*

$$T\varphi = \mathrm{Tr}_{|\Gamma}\varphi, \quad \textit{for all } \varphi \in H^1(\Omega) \cap \mathscr{C}(\overline{\Omega})$$

and

$$\|T\|_{\mathscr{L}(H^1(\Omega),L^2(\Gamma))} \le \frac{1}{\sqrt{\cos\theta}}. \tag{2.4}$$

Proof. The existence and uniqueness of the operator T follow from Theorem 2.1 the density of $\mathscr{C}_c^\infty(\overline{\Omega})$ in $H^1(\Omega)$ (see, e.g., [1, Theorem 4.7, p. 248]). Moreover, (2.3) implies

$$\|\varphi\|^2_{L^2(\Gamma,\mathrm{d}\sigma)} \le \frac{1}{\cos\theta}\left(\|\varphi\|^2_{L^2(\Omega)} + \|\nabla\varphi\|^2_{L^2(\Omega,\mathbb{R}^n)}\right), \quad \text{for all } \varphi \in \mathscr{C}_c^\infty(\overline{\Omega}),$$

which proves (2.4). □

References

[1] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1987, Oxford Science Publications.

[2] Jindřich Nečas, *Direct methods in the theory of elliptic equations*, Springer Monographs in Mathematics, Springer, Heidelberg, 2012, Translated from the 1967 French original by Gerard Tronel and Alois Kufner, Editorial coordination and preface by Šárka Nečasová and a contribution by Christian G. Simader.