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The Incompressible Navier–Stokes System with Time-Dependent Robin-Type Boundary Conditions

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Abstract. We show that the incompressible 3D Navier–Stokes system in a $\mathscr{C}^{1,1}$ bounded domain or a bounded convex domain Ω with a non penetration condition $\nu \cdot u = 0$ at the boundary $\partial\Omega$ together with a time-dependent Robin boundary condition of the type $\nu \times \operatorname{curl} u = \beta(t)u$ on $\partial\Omega$ admits a solution with enough regularity provided the initial condition is small enough in an appropriate functional space.

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1. Introduction

We consider the following incompressible Navier–Stokes system in a (sufficiently smooth) bounded domain $\Omega \subset \mathbb{R}^3$ on a time interval $[0,\tau]$

$$\begin{cases} \partial_t u - \Delta u + \nabla p + (u \cdot \nabla)u = 0 & \text{in } [0,\tau] \times \Omega \\ \operatorname{div} u = 0 & \text{in } [0,\tau] \times \Omega \end{cases} \tag{NS}$$

where $\mathbb{S}(u,p) := \frac{1}{2}\big(\nabla u + (\nabla u)^\top\big) - p\,\mathrm{Id}$ is the Cauchy stress tensor applied to (u,p) supplemented with the conditions on the boundary $\partial\Omega$ (ν denotes the outer unit normal):

$$\begin{cases} \nu \cdot u = 0 & \text{on } [0,\tau] \times \partial\Omega \\ \big[\mathbb{S}(u,p)\nu\big]_{\tan} + Bu = 0 & \text{on } [0,\tau] \times \partial\Omega \end{cases} \tag{Nbc}$$

and the initial condition

$$u(0) = u_0 \quad \text{in } \Omega. \tag{IC}$$

As usual $[w]_{\tan}$ denotes the tangential part of w, that is $[w]_{\tan} = w - (\nu \cdot w)\nu$. The conditions (Nbc) are referred to in the literature as Navier's boundary conditions and were introduced by Navier in his lecture at the Académie royale des Sciences in 1822 [25]. They describe the fact that the fluid cannot escape from the domain Ω ($\nu \cdot u = 0$) and that the fluid slips with a friction described by a matrix B on $\partial\Omega$ ($\big[\mathbb{S}(u,p)\nu\big]_{\tan} + Bu = 0$). Such conditions have been recently derived from homogenization of rough boundaries; see, e.g., [4,6,11,13].

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First we transform the system (NS) with boundary conditions (Nbc) and initial condition (IC) into the following "Robin–Navier–Stokes" problem

$$\begin{cases} \partial_t u - \Delta u + \nabla\pi - u \times \operatorname{curl} u = 0 & \text{in } [0,\tau]\times\Omega \\ \operatorname{div} u = 0 & \text{in } [0,\tau]\times\Omega \\ \nu\cdot u \ = \ 0, \quad \nu\times\operatorname{curl} u = \beta u & \text{on } [0,\tau]\times\partial\Omega \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{RNS}$$

This is based on the identities

$$(u\cdot\nabla)u = -u\times\operatorname{curl} u + \frac{1}{2}\nabla|u|^2$$

and

$$\big[\mathbb{S}(u,p)\nu\big]_{\tan} = -\nu\times\operatorname{curl} u + 2\mathcal{W}u$$

on the boundary $\partial\Omega$ (see, e.g., [20, Section 2]), so that $\beta = 2\mathcal{W} + B$, and $\pi = p + \frac{1}{2}|u|^2$. Here $\mathcal{W}$ is the Weingarten map (for properties of $\mathcal{W}$, see, e.g., [20, Section 6]; in particular, $\mathcal{W}u = 0$ on flat parts of the boundary).

Our main objective in this paper is to prove existence and uniqueness of a solution of (RNS) in the case of time-dependent Robin boundary condition. That is, we allow β to depend on both time and space variables. We assume that $\beta : [0,\tau]\times\partial\Omega \to \mathscr{M}_3(\mathbb{R})$ is bounded measurable on $[0,\tau]\times\partial\Omega$ such that

$$0 \le \beta(t,x)\xi\cdot\xi \le M|\xi|^2 \text{ for almost all } (t,x)\in[0,\tau]\times\partial\Omega \quad \text{and all } \xi\in\mathbb{R}^3 \tag{1.1}$$

$$\beta(t,x) \text{ is symmetric for almost all } (t,x)\in[0,\tau]\times\partial\Omega, \tag{1.2}$$

$$\beta(t,x)\nu(x) = \lambda(t,x)\nu(x) \text{ for almost all } x\in\partial\Omega, t>0, \tag{1.3}$$

where $\lambda : [0,\tau]\times\partial\Omega \to \mathbb{R}$, so that a normal vector field transformed by $\beta = \beta^\top$ remains normal at the boundary.
Note that the condition $\beta \ge 0$ implies the geometric condition on the friction (symmetric) matrix B: $B \ge -2\mathcal{W}$. In particular, if Ω is convex, $\mathcal{W}\ge 0$, so that we can treat any nonnegative friction matrix B. Further, we assume that β is piecewise α-Hölder continuous for some $\alpha > 1/2$ with respect to the time variable. That is, there exist t_i, $0\le i\le n$, such that $[0,\tau] = \cup_{i=0}^n [t_i, t_{i+1}]$ and constants M_i such that on each interval (t_i, t_{i+1}), β is the restriction of some $\widetilde{\beta}$ such that for almost every $x\in\partial\Omega$

$$\|\widetilde{\beta}(t,x) - \widetilde{\beta}(s,x)\|_{\mathscr{M}_3} \le M_i|t-s|^\alpha \quad \text{for all } t,s\in[t_i,t_{i+1}]. \tag{1.4}$$

Here $\|\cdot\|_{\mathscr{M}_3}$ denotes the operator norm in $\mathscr{M}_3$. Under these assumptions we prove the following regularity result.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^3$ be a bounded $\mathscr{C}^{1,1}$ or convex domain and let $\tau>0$. There exists $\epsilon>0$ such that for all initial condition $u_0\in L^2(\Omega,\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ in Ω, $\nu\cdot u_0 = 0$ on $\partial\Omega$ and $\operatorname{curl} u_0 \in L^2(\Omega,\mathbb{R}^3)$, $\|u_0\|_2 + \|\operatorname{curl} u_0\|_2 \le \epsilon$, there exists a unique (u,π) satisfying (RNS) for a.e. $(t,x)\in[0,\tau]\times\Omega$. In addition, $u\in H^1(0,\tau;L^2(\Omega,\mathbb{R}^3))$, $\Delta u\in L^2(0,\tau,L^2(\Omega,\mathbb{R}^3))$, $\pi\in L^2(0,\tau;H^1(\Omega))$ and there exists a constant C independent of u and π such that*

$$\|u\|_{H^1(0,\tau,L^2(\Omega,\mathbb{R}^3))} + \|-\Delta u\|_{L^2(0,\tau;L^2(\Omega,\mathbb{R}^3))} + \|\nabla\pi\|_{L^2(0,\tau;L^2(\Omega,\mathbb{R}^3))} \le C\epsilon.$$

In the case where $\beta(t,x) = 0$ for all $(t,x)\in[0,\tau]\times\partial\Omega$, the system (RNS) has been studied in [20], in the case of Lipschitz domains for initial conditions in L^3. See also [22] for related result on the Neumann type boundary conditions on smooth domains and [5] for the case β constant in $\mathscr{C}^{2,1}$ domains. For Dirichlet boundary conditions $u=0$ on $\partial\Omega$, which correspond to $\beta=\infty$, we refer to the classical results by Fujita and Kato [10] (see also [19,23] for the case of less regular domains).

The method to prove Theorem 1.1 relies on the study of operators defined by forms and recent results on maximal regularity for non-autonomous linear evolution equations. This latter property is the key ingredient to treat the non linearity by appealing to classical fixed point arguments. We also prove an existence result for small initial conditions u_0 less regular (i.e., in a critical space for the scaling properties of the Navier–Stokes equations) than those considered in Theorem 1.1; see Theorem 5.3 below.

The paper is organized as follows. Section 2 is devoted to analytical tools necessary for our approach of the problem. In Sect. 3, we define the (time dependent) Robin Stokes operator. We use recent results on maximal regularity in Sect. 4 in order to obtain regularity properties of the solution of the linearized (RNS) system. The proof of Theorem 1.1 is given in Sect. 5.

2. Background Material

Throughout this section, $\Omega \subset \mathbb{R}^3$ will be a bounded domain which is either convex or $\mathscr{C}^{1,1}$. We denote by $\partial\Omega$ its boundary. It is endowed with the surface measure $\mathrm{d}\sigma$. It is a classical fact (see, e.g., [16, Théorème 8.3] for smooth domains and [26, Ch. 2, Théorème 5.5] or [27, Ch. 2, Theorem 5.5] for Lipschitz domains) that

$$\mathrm{Tr}_{|\partial\Omega} : H^1(\Omega) \to H^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega, \mathrm{d}\sigma),$$

the latter embedding being compact.

(i) For $u \in L^2(\Omega, \mathbb{R}^3)$ such that $\operatorname{div} u \in L^2(\Omega)$, the normal component $\nu \cdot u$ of u on $\partial\Omega$ is defined in a weak sense in the negative Sobolev space $H^{-1/2}(\partial\Omega)$ by

$$_{H^{-1/2}(\partial\Omega)}\langle \nu \cdot u, \varphi \rangle_{H^{1/2}(\partial\Omega)} = \langle \operatorname{div} u, \phi \rangle_\Omega + \langle u, \nabla\phi \rangle_\Omega, \tag{2.1}$$

for all $\varphi \in H^{1/2}(\partial\Omega)$, where ϕ belongs to the Sobolev space $H^1(\Omega)$ with $\mathrm{Tr}_{|\partial\Omega}\phi = \varphi$. Here, $\langle \cdot, \cdot \rangle_\Omega$ denotes either the scalar or the vector-valued scalar product in L^2 defined over Ω. The notation $_{V'}\langle \cdot, \cdot \rangle_V$ means the duality between V' and V.

(ii) For $u \in L^2(\Omega, \mathbb{R}^3)$ such that $\operatorname{curl} u \in L^2(\Omega, \mathbb{R}^3)$, the tangential component $\nu \times u$ of u on $\partial\Omega$ is defined in a weak sense in $H^{-1/2}(\partial\Omega, \mathbb{R}^3)$ by

$$_{H^{-1/2}(\partial\Omega,\mathbb{R}^3)}\langle \nu \times u, \varphi \rangle_{H^{1/2}(\partial\Omega,\mathbb{R}^3)} = \langle \operatorname{curl} u, \phi \rangle_\Omega - \langle u, \operatorname{curl} \phi \rangle_\Omega, \tag{2.2}$$

for all $\varphi \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$ where $\phi \in H^1(\Omega, \mathbb{R}^3)$ with $\mathrm{Tr}_{|\partial\Omega}\phi = \varphi$. As before, $\langle \cdot, \cdot \rangle_\Omega$ denotes the vector-valued scalar product in L^2 defined over Ω.

The following result, valid for Lipschitz domains, can be found in [7] (see also [24]).

Proposition 2.1. *There exists a constant $C > 0$ such that for all $u \in L^2(\Omega, \mathbb{R}^3)$ satisfying $\operatorname{div} u \in L^2(\Omega, \mathbb{R})$, $\operatorname{curl} u \in L^2(\Omega, \mathbb{R}^3)$ and either $\nu \cdot u \in L^2(\partial\Omega)$ or $\nu \times u \in L^2(\partial\Omega, \mathbb{R}^3)$ we have $\mathrm{Tr}_{|\partial\Omega} u \in L^2(\partial\Omega, \mathbb{R}^3)$ with the estimate*

$$\begin{aligned}\|\mathrm{Tr}_{|\partial\Omega} u\|_{L^2(\partial\Omega,\mathbb{R}^3)} \leq & C \Big(\|u\|_2 + \|\operatorname{div} u\|_2 + \|\operatorname{curl} u\|_2 \\ & + \min\big\{ \|\nu \cdot u\|_{L^2(\partial\Omega)}, \|\nu \times u\|_{L^2(\partial\Omega,\mathbb{R}^3)} \big\} \Big).\end{aligned}$$

Moreover, $u \in H^{1/2}(\Omega, \mathbb{R}^3)$ and

$$\begin{aligned}\|u\|_{H^{1/2}(\Omega,\mathbb{R}^3)} \leq & C \Big(\|u\|_2 + \|\operatorname{div} u\|_2 + \|\operatorname{curl} u\|_2 \\ & + \min\big\{ \|\nu \cdot u\|_{L^2(\partial\Omega)}, \|\nu \times u\|_{L^2(\partial\Omega,\mathbb{R}^3)} \big\} \Big).\end{aligned} \tag{2.3}$$

Moving on, let W_T and W_N be the spaces defined by

$$\begin{aligned}W_T = & \big\{ u \in L^2(\Omega, \mathbb{R}^3); \operatorname{div} u \in L^2(\Omega), \operatorname{curl} u \in L^2(\Omega, \mathbb{R}^3) \\ & \text{and } \nu \cdot u = 0 \text{ on } \partial\Omega \big\}\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}W_N = & \big\{ u \in L^2(\Omega, \mathbb{R}^3); \operatorname{div} u \in L^2(\Omega), \operatorname{curl} u \in L^2(\Omega, \mathbb{R}^3) \\ & \text{and } \nu \times u = 0 \text{ on } \partial\Omega \big\}\end{aligned} \tag{2.5}$$

both endowed with the norm

$$\|u\|_W = \|u\|_{L^2(\Omega,\mathbb{R}^3)} + \|\operatorname{div} u\|_{L^2(\Omega)} + \|\operatorname{curl} u\|_{L^2(\Omega,\mathbb{R}^3)}, \quad u \in W_{T,N}. \tag{2.6}$$

It is easy to see that $W_{T,N}$ are Hilbert spaces. Note also that since Ω is either convex or $\mathscr{C}^{1,1}$, the spaces $W_{T,N}$ are contained in $H^1(\Omega,\mathbb{R}^3)$ (with continuous embedding). See [1, Theorem 2.9, Theorem 2.12 and Theorem 2.17]. Thus, there exists a constant $C>0$ such that for all $u \in W_{T,N}$

$$\|u\|_{H^1(\Omega)} \le C(\|u\|_{L^2(\Omega,\mathbb{R}^3)} + \|\operatorname{div} u\|_{L^2(\Omega)} + \|\operatorname{curl} u\|_{L^2(\Omega,\mathbb{R}^3)}). \tag{2.7}$$

In particular, the trace operator

$$\mathrm{Tr}_{|\partial\Omega} : W_{T,N} \to H^{1/2}(\partial\Omega,\mathbb{R}^3)$$

is continuous.

Next, we define the Hodge Laplacians with absolute and relative boundary conditions. Although these operators do not appear explicitly in our main results they will be useful for the proof of the description of the domain of Stokes operator with time dependent Robin boundary condition.

We define on $L^2(\Omega,\mathbb{R}^3)$ the two bilinear symmetric forms

$$\mathfrak{b}_0(u,v) = \langle \operatorname{div} u, \operatorname{div} v\rangle_\Omega + \langle \operatorname{curl} u, \operatorname{curl} v\rangle_\Omega, \quad u,v \in W_T \tag{2.8}$$

and

$$\mathfrak{b}_1(u,v) = \langle \operatorname{div} u, \operatorname{div} v\rangle_\Omega + \langle \operatorname{curl} u, \operatorname{curl} v\rangle_\Omega, \quad u,v \in W_N. \tag{2.9}$$

Both forms $\mathfrak{b}_0$ and $\mathfrak{b}_1$ are closed. Therefore, there exist two operators

$$B_{0,0} : W_T \to W_T', \quad B_{0,0}u = -\Delta u$$

associated with $\mathfrak{b}_0$ and

$$B_{1,0} : W_N \to W_N', \quad B_{1,0}u = -\Delta u$$

associated with $\mathfrak{b}_1$ in the sense that

$$\mathfrak{b}_0(u,v) = {}_{W_T'}\langle B_{0,0}u, v\rangle_{W_T}, \quad u,v \in W_T$$

and

$$\mathfrak{b}_1(u,v) = {}_{W_N'}\langle B_{1,0}u, v\rangle_{W_N}, \quad u,v \in W_N.$$

The part B_0 of $B_{0,0}$ on $L^2(\Omega,\mathbb{R}^3)$, i.e.,

$$\begin{aligned} D(B_0) &:= \big\{u \in W_T, \exists\, v \in L^2(\Omega,\mathbb{R}^3) : \mathfrak{b}_0(u,\phi) = \langle v,\phi\rangle_\Omega \ \forall\, \phi \in W_T\big\},\\ B_0 u &:= v, \end{aligned} \tag{2.10}$$

and the part B_1 of $B_{1,0}$ on $L^2(\Omega,\mathbb{R}^3)$, i.e.,

$$\begin{aligned} D(B_1) &:= \big\{u \in W_N, \exists\, v \in L^2(\Omega,\mathbb{R}^3) : \mathfrak{b}_1(u,\phi) = \langle v,\phi\rangle_\Omega \ \forall\, \phi \in W_N\big\},\\ B_1 u &:= v, \end{aligned} \tag{2.11}$$

are self-adjoint operators on $L^2(\Omega,\mathbb{R}^3)$.

Proposition 2.2. *The domains of B_0 and B_1 have the following description*

$$\begin{aligned} D(B_0) = \big\{&u \in L^2(\Omega,\mathbb{R}^3); \operatorname{div} u \in H^1(\Omega), \operatorname{curl} u \in L^2(\Omega,\mathbb{R}^3),\\ &\operatorname{curl}\operatorname{curl} u \in L^2(\Omega,\mathbb{R}^3) \text{ and } \nu\cdot u = 0, \nu\times\operatorname{curl} u = 0 \text{ on } \partial\Omega\big\} \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} D(B_1) = \big\{&u \in L^2(\Omega,\mathbb{R}^3); \operatorname{div} u \in H^1(\Omega), \operatorname{curl} u \in L^2(\Omega,\mathbb{R}^3),\\ &\operatorname{curl}\operatorname{curl} u \in L^2(\Omega,\mathbb{R}^3) \text{ and } \nu\times u = 0, \operatorname{div} u = 0 \text{ on } \partial\Omega\big\}. \end{aligned} \tag{2.13}$$

Moreover, for $u \in L^2(\Omega,\mathbb{R}^3)$ such that $\operatorname{curl} u \in L^2(\Omega,\mathbb{R}^3)$, the following commutator property occurs for all $\varepsilon > 0$

$$\operatorname{curl}(1+\varepsilon B_0)^{-1}u = (1+\varepsilon B_1)^{-1}\operatorname{curl} u. \tag{2.14}$$

Proof. The description of the domain of B_0 can be found in [21, (3.17) & (3.18)]. We can describe the domain of B_1 in the same way (see also [18, Theorem 7.1 & Theorem 7.3]). To prove (2.14), let $u \in L^2(\Omega,\mathbb{R}^3)$ such that $\operatorname{curl} u \in L^2(\Omega,\mathbb{R}^3)$. Let $u_\varepsilon = (1+\varepsilon B_0)^{-1}u$ and $w_\varepsilon = (1+\varepsilon B_1)^{-1}\operatorname{curl} u$.

`Step 1`: *We claim that* $\operatorname{curl} u_\varepsilon \in D(B_1)$.
By (2.12) we have

$$\begin{aligned}&\operatorname{curl} u_\varepsilon \in L^2(\Omega,\mathbb{R}^3), \operatorname{curl}\operatorname{curl} u_\varepsilon \in L^2(\Omega,\mathbb{R}^3), \operatorname{div}(\operatorname{curl} u_\varepsilon) = 0 \in H^1(\Omega),\\ &\nu \times \operatorname{curl} u_\varepsilon = 0 \text{ and } \operatorname{div}(\operatorname{curl} u_\varepsilon) = 0 \quad \text{on } \partial\Omega.\end{aligned}$$

To prove that $\operatorname{curl} u_\varepsilon \in D(B_1)$, it remains to show, thanks to (2.13), that $\operatorname{curl}\operatorname{curl}(\operatorname{curl} u_\varepsilon) \in L^2(\Omega,\mathbb{R}^3)$. This is due to the fact that

$$\operatorname{curl}\operatorname{curl}(\operatorname{curl} u_\varepsilon) = \operatorname{curl}(-\Delta u_\varepsilon) \quad \text{in } H^{-1}(\Omega,\mathbb{R}^3).$$

Since

$$-\Delta u_\varepsilon = B_0(1+\varepsilon B_0)^{-1}u = \frac{1}{\varepsilon}(u - u_\varepsilon)$$

and $\operatorname{curl} u_\varepsilon, \operatorname{curl} u \in L^2(\Omega,\mathbb{R}^3)$, the claim follows.

`Step 2`: *We claim now that* $\operatorname{curl} u_\varepsilon = w_\varepsilon$.
By Step 1, we know that $\operatorname{curl} u_\varepsilon \in D(B_1)$. Moreover, we have in the sense of distributions

$$(1+\varepsilon B_1)(\operatorname{curl} u_\varepsilon) = \operatorname{curl} u_\varepsilon - \varepsilon\Delta\operatorname{curl} u_\varepsilon = \operatorname{curl}\Big(u_\varepsilon - \varepsilon\Delta u_\varepsilon\Big) = \operatorname{curl} u$$

since $u_\varepsilon - \varepsilon\Delta u_\varepsilon = (1+\varepsilon B_0)(1+\varepsilon B_0)^{-1}u = u$. Therefore,

$$\operatorname{curl} u_\varepsilon = (1+\varepsilon B_1)^{-1}\operatorname{curl} u = w_\varepsilon$$

which proves the claim. □

The following lemma is inspired by [18, Proof of Proposition 2.4 (iii)].

Lemma 2.3. 1. *Let $g \in L^2(\partial\Omega,\mathbb{R}^3)$. Then there exists $w \in L^2(\Omega,\mathbb{R}^3)$ with $\operatorname{curl} w \in L^2(\Omega,\mathbb{R}^3)$ such that for all $\phi \in W_T$*

$$\langle g,\phi\rangle_{\partial\Omega} = \langle \operatorname{curl} w,\phi\rangle_\Omega - \langle w,\operatorname{curl}\phi\rangle_\Omega. \tag{2.15}$$

Moreover, there exists $C > 0$ such that

$$\|w\|_{L^2(\Omega,\mathbb{R}^3)} + \|\operatorname{curl} w\|_{L^2(\Omega,\mathbb{R}^3)} \le C\|g\|_{L^2(\partial\Omega,\mathbb{R}^3)}. \tag{2.16}$$

2. *If in addition $g \in L^2_{\tan}(\partial\Omega,\mathbb{R}^3)$ (which means that $g \in L^2(\partial\Omega,\mathbb{R}^3)$ and $\nu \cdot g = 0$ on $\partial\Omega$), then there exists $w \in L^2(\Omega,\mathbb{R}^3)$ such that $\operatorname{curl} w \in L^2(\Omega,\mathbb{R}^3)$ and (2.15) holds for all $\phi \in H^1(\Omega)$. And in that case $g = \nu \times w$ in $H^{-1/2}(\partial\Omega,\mathbb{R}^3)$.*

Proof. 1. We define the space $X := \{(\phi,\operatorname{curl}\phi); \phi \in W_T\}$. It is a closed subspace of $L^2(\Omega,\mathbb{R}^3)\times L^2(\Omega,\mathbb{R}^3)$. By classical trace theorems (see, e.g., [16, Théorème 8.3], [26, Ch. 2, Théorème 5.5] or [27, Ch. 2, Theorem 5.5 with $k = 1$ and $p = 2$]), we have that $\nu \times \phi \in L^2(\partial\Omega,\mathbb{R}^3)$ for all $\phi \in W_T \subset H^1(\Omega,\mathbb{R}^3)$. Since $g \in L^2(\partial\Omega,\mathbb{R}^3)$, it is immediate that $\nu \times g \in L^2(\partial\Omega,\mathbb{R}^3) = \big(L^2(\partial\Omega,\mathbb{R}^3)\big)'$. Thus, $\nu \times g$ acts as a linear functional on X as follows:

$$(\nu \times g)(\phi,\operatorname{curl}\phi) := \langle \nu\times g,\nu\times\phi\rangle_{\partial\Omega} \quad \text{for all } \phi \in W_T.$$

By the Hahn–Banach theorem, there exist $(v_1,v_2) \in L^2(\Omega,\mathbb{R}^3)\times L^2(\Omega,\mathbb{R}^3)$ such that

$$(\nu\times g)(\phi,\operatorname{curl}\phi) = \langle v_1,\operatorname{curl}\phi\rangle_\Omega + \langle v_2,\phi\rangle_\Omega \quad \text{for all } \phi \in W_T,$$

where we have identified $\big(L^2(\Omega,\mathbb{R}^3)\times L^2(\Omega,\mathbb{R}^3)\big)'$ with $L^2(\Omega,\mathbb{R}^3)\times L^2(\Omega,\mathbb{R}^3)$. We can choose $\phi \in H^1_0(\Omega,\mathbb{R}^3) \subset W_T$ and obtain that

$$0 = {}_{H^{-1}}\langle \operatorname{curl} v_1 + v_2,\phi\rangle_{H^1_0}.$$

This gives that $\operatorname{curl} v_1 + v_2 = 0$ in $H^{-1}(\Omega, \mathbb{R}^3)$. We set $w := -v_1 \in L^2(\Omega, \mathbb{R}^3)$, we have $\operatorname{curl} w = v_2 \in L^2(\Omega, \mathbb{R}^3)$ and

$$\langle \nu \times g, \nu \times \phi \rangle_{\partial\Omega} = -\langle w, \operatorname{curl} \phi \rangle_\Omega + \langle \operatorname{curl} w, \phi \rangle_\Omega \quad \text{for all } \phi \in W_T. \tag{2.17}$$

Since $\phi \in W_T$, $\mathrm{Tr}_{|\partial\Omega} \phi \in L^2_{\tan}(\partial\Omega, \mathbb{R}^3)$ it is clear[1]that $\phi = (\nu \times \phi) \times \nu$, so that the left-hand side of (2.17) coincides with

$$\langle g, \phi \rangle_{\partial\Omega} \qquad \text{for all } \phi \in W_T, \tag{2.18}$$

which proves (2.15).

The existence of $C > 0$ such that (2.16) holds follows from the Closed Graph Theorem since $\{u \in L^2(\Omega, \mathbb{R}^3); \operatorname{curl} u \in L^2(\Omega, \mathbb{R}^3)\}$ is complete for the norm $\|u\|_2 + \|\operatorname{curl} u\|_2$.

2. Assume now that $g \in L^2_{\tan}(\partial\Omega, \mathbb{R}^3)$. Let $w \in L^2(\Omega, \mathbb{R}^3)$ such that $\operatorname{curl} w \in L^2(\Omega, \mathbb{R}^3)$ and (2.15) holds. Since $\nu \times g \in L^2(\partial\Omega, \mathbb{R}^3)$, we can approach it in $L^2(\partial\Omega, \mathbb{R}^3)$ by a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of vector fields $\varphi_n \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$. In particular,

$$\varphi_n \times \nu \longrightarrow (\nu \times g) \times \nu = g \quad \text{in } L^2(\partial\Omega, \mathbb{R}^3) \text{ as } n \to \infty.$$

By assertion 2.3, for each $n \in \mathbb{N}$ there exists $w_n \in L^2(\Omega, \mathbb{R}^3)$ such that $\operatorname{curl} w_n \in L^2(\Omega, \mathbb{R}^3)$ satisfying

$$\langle \varphi_n \times \nu, \phi \rangle_{\partial\Omega} = \langle \operatorname{curl} w_n, \phi \rangle_\Omega - \langle w_n, \operatorname{curl} \phi \rangle_\Omega \quad \text{for all } \phi \in W_T.$$

Thanks to the estimate (2.16), it is immediate that

$$w_n \longrightarrow w \ \text{ and } \ \operatorname{curl} w_n \longrightarrow \operatorname{curl} w \quad \text{in } L^2(\Omega, \mathbb{R}^3) \text{ as } n \to \infty.$$

Let now $\phi \in H^1(\Omega, \mathbb{R}^3)$. For $\varepsilon > 0$, let $\phi_\varepsilon = (1 + \varepsilon B_0)^{-1}\phi$ with B_0 as in Proposition 2.2. Then $\phi_\varepsilon \in W_T$ and thanks to (2.14)

$$\phi_\varepsilon \longrightarrow \phi \ \text{ and } \ \operatorname{curl} \phi_\varepsilon = (1 + \varepsilon B_1)^{-1} \operatorname{curl} \phi \longrightarrow \operatorname{curl} \phi \quad \text{in } L^2(\Omega, \mathbb{R}^3) \text{ as } \varepsilon \to 0.$$

This implies also that

$$\nu \times \phi_\varepsilon \longrightarrow \nu \times \phi \quad \text{in } H^{-1/2}(\partial\Omega, \mathbb{R}^3) \text{ as } \varepsilon \to 0.$$

Therefore, we have for all $\varepsilon > 0$ and $n \in \mathbb{N}$

$$\langle \nu \times \phi_\varepsilon, \varphi_n \rangle_{\partial\Omega} = \langle \varphi_n \times \nu, \phi_\varepsilon \rangle_{\partial\Omega} = \langle \operatorname{curl} w_n, \phi_\varepsilon \rangle_\Omega - \langle w_n, \operatorname{curl} \phi_\varepsilon \rangle_\Omega.$$

We first take the limit as ε goes to 0 and obtain (recall that φ_n belongs to $H^{1/2}(\partial\Omega, \mathbb{R}^3)$)

$$_{H^{-1/2}}\langle \nu \times \phi, \varphi_n \rangle_{H^{1/2}} = \langle \operatorname{curl} w_n, \phi \rangle_\Omega - \langle w_n, \operatorname{curl} \phi \rangle_\Omega.$$

Since $\phi \in H^1(\Omega, \mathbb{R}^3)$, the first term of the latter equation is also equal to $\langle \varphi_n \times \nu, \phi \rangle_{\partial\Omega}$. Taking the limit as n goes to ∞ yields

$$\langle g, \phi \rangle_{\partial\Omega} = \langle \operatorname{curl} w, \phi \rangle_\Omega - \langle w, \operatorname{curl} \phi \rangle_\Omega$$

which proves the claim made in 2.3. □

Lemma 2.4. *Let $\varphi \in H^{1/2}(\partial\Omega, \mathbb{R}^3) \cap L^2_{\tan}(\partial\Omega, \mathbb{R}^3)$. Then there exists $v \in H^1(\Omega, \mathbb{R}^3)$ such that* $\operatorname{div} v = 0$ *on Ω and $v_{|\partial\Omega} = \varphi$.*

Proof. Let $\varphi \in H^{1/2}(\partial\Omega, \mathbb{R}^3) \cap L^2_{\tan}(\partial\Omega, \mathbb{R}^3)$. Since the trace operator $\mathrm{Tr}_{|\partial\Omega} : H^1(\Omega, \mathbb{R}^3) \to H^{1/2}(\partial\Omega, \mathbb{R}^3)$ is onto there exists $w \in H^1(\Omega, \mathbb{R}^3)$ such that $w_{|\partial\Omega} = \varphi$. By [8, Theorem 4.6], there exist three operators $R : L^2(\Omega, \mathbb{R}^3) \to H^1_0(\Omega, \mathbb{R}^3)$, $S : L^2(\Omega) \to H^1_0(\Omega, \mathbb{R}^3)$ and $T : L^2(\Omega, \mathbb{R}^3) \to H^1_0(\Omega, \mathbb{R}^3)$ such that

$$\operatorname{curl} Tu + S \operatorname{div} u = u - Ru \quad \text{for all } u \in H^1(\Omega, \mathbb{R}^3) \text{ with } \nu \cdot u = 0 \text{ on } \partial\Omega$$

---

[1] Recall that for $a, b, c \in \mathbb{R}^3$, the following identities hold:

$$(a \times b) \cdot c = (b \times c) \cdot a, \quad a \times b = -b \times a, \quad |a|^2 b = (a \times b) \times a + (a \cdot b)a.$$

(choose $n = 3$, $T = T_2$, $S = T_3$ and $R = L_2$ in [8, Theorem 4.6]). We apply this result to $u = w$ and we define

$$v := \operatorname{curl} Tw = w - S\operatorname{div} w - Rw;$$

v satisfies $\operatorname{div} v = 0$, $v \in H^1(\Omega, \mathbb{R}^3)$ and $v_{|\partial\Omega} = w_{|\partial\Omega} = \varphi$. $\square$

The classical Hodge-Helmholtz decomposition asserts that the space $L^2(\Omega, \mathbb{R}^3)$ is the orthogonal direct sum $H \overset{\perp}{\oplus} G$ where

$$H := \left\{u \in L^2(\Omega, \mathbb{R}^3); \operatorname{div} u = 0 \text{ in } \Omega, \nu \cdot u = 0 \text{ on } \partial\Omega\right\} \tag{2.19}$$

and $G := \nabla H^1(\Omega, \mathbb{R})$.

Remark 2.5. The space H coincides with the closure in $L^2(\Omega, \mathbb{R}^3)$ of the space of vector fields $u \in \mathscr{C}_c^\infty(\Omega, \mathbb{R}^3)$ with $\operatorname{div} u = 0$ in Ω which we denote by $\mathscr{D}(\Omega)$. See, e.g., [30, Theorem 1.4].

We denote by $J : H \hookrightarrow L^2(\Omega; \mathbb{R}^3)$ the canonical embedding and $\mathbb{P} : L^2(\Omega; \mathbb{R}^3) \to H$ the orthogonal projection. Recall that for $u \in L^2(\Omega, \mathbb{R}^3)$, there exists $p \in H^1(\Omega)$ so that $\mathbb{P}u = u - \nabla p$. It is clear that $\mathbb{P}J = \mathrm{Id}_H$ and that

$$\langle u, \mathbb{P}v\rangle_\Omega = \langle \mathbb{P}u, v\rangle_\Omega \quad \text{for all } u, v \in L^2(\Omega; \mathbb{R}^3). \tag{2.20}$$

Define now the space $V := W_T \cap H$. Thus, for every $v \in W_T$, $\mathbb{P}v \in V$. The space V will be used to define the Stokes operator with Robin boundary conditions in the next section.

3. The Robin–Stokes Operator

In this section we define the Stokes operator with Robin boundary conditions on $\partial\Omega$. In order to do this we use the method of sesquilinear forms; see e.g., [9, Example 3, p. 449]. We start by defining the Hodge-Laplacian with Robin boundary conditions. As in the previous section, Ω is a bounded domain of $\mathbb{R}^3$ and we suppose that it is either convex or has a $\mathscr{C}^{1,1}$-boundary.

Fix $\tau \in (0, \infty)$ and let $\beta : [0, \tau] \times \partial\Omega \to \mathscr{M}_3(\mathbb{R})$ be bounded measurable on $[0, \tau] \times \partial\Omega$. We assume that (1.1)–(1.3) are satisfied.

Recall that $V = W_T \cap H$ and that the embedding J restricted to V maps V to W_T. We denote this restriction by $J_0 : V \hookrightarrow W_T$. Its adjoint $J_0' =: \mathbb{P}_1 : W_T' \to V'$ is then an extension of the orthogonal projection $\mathbb{P}$.

Lemma 3.1. *The projection $\mathbb{P}$ restricted to W_T takes its values in V, so that $\mathbb{P}J_0 = \mathrm{Id}_V$ holds.*

Proof. Let $w \in W_T$. Since $W_T \subset L^2(\Omega, \mathbb{R}^3)$, there exists $\pi \in H^1(\Omega)$ such that $w = J\mathbb{P}w + \nabla\pi$ and π satisfies $\Delta\pi = \operatorname{div} w \in L^2(\Omega)$ and $\partial_\nu\pi = \nu \cdot w = 0$ on $\partial\Omega$. Moreover, $\operatorname{curl} \nabla\pi = 0$ in Ω, so that $\nabla\pi \in W_T$. Therefore, $\operatorname{div} J\mathbb{P}w = 0$ in Ω, $\operatorname{curl} J\mathbb{P}w = \operatorname{curl} w \in L^2(\Omega, \mathbb{R}^3)$ and $\nu \cdot J\mathbb{P}w = 0$ on $\partial\Omega$, which proves that $\mathbb{P}w \in V$. $\square$

We are now in the situation to define the Stokes operator with Robin boundary conditions. We consider on the Hilbert space H the bilinear symmetric form

$$\begin{aligned} \mathfrak{a}_\beta : V \times V &\longrightarrow \mathbb{R} \\ \mathfrak{a}_\beta(u, v) &:= \langle \operatorname{curl} J_0u, \operatorname{curl} J_0v\rangle_\Omega + \langle \beta \operatorname{Tr}_{|\partial\Omega} J_0u, \operatorname{Tr}_{|\partial\Omega} J_0v\rangle_{\partial\Omega}. \end{aligned} \tag{3.1}$$

Using the fact that $\mathbb{P}J_0 = \mathrm{Id}_V$ we see that the form $\mathfrak{a}_\beta$ is closed. Therefore, there exists an operator $A_{\beta,0} : V \to V'$ associated with $\mathfrak{a}_\beta$ in the sense that

$$\mathfrak{a}_\beta(u, v) = {}_{V'}\langle A_{\beta,0}u, v\rangle_V, \quad u, v \in V.$$

The part A_β of $A_{\beta,0}$ on H, i.e.,

$$D(A_\beta) := \{u \in V, \exists\, v \in H : \mathfrak{a}_\beta(u, \phi) = \langle v, \phi\rangle_\Omega \ \forall\, \phi \in V\}, \quad A_\beta u := v$$

is a self-adjoint operator on H. We call A_β the Robin–Stokes operator.

From now on, since J and J_0 are embedding operators, we will omit to write them to avoid too pedantic an exposition.

Theorem 3.2. *The operator A_β is given by*

$$\begin{aligned} D(A_\beta) &= \big\{u \in V; \operatorname{curl}\operatorname{curl} u \in L^2(\Omega, \mathbb{R}^3), \nu \times \operatorname{curl} u = \beta u \text{ on } \partial\Omega\big\}, \\ A_\beta u = \mathbb{P}(\operatorname{curl}\operatorname{curl} u) &= -\Delta u + \nabla p, \qquad u \in D(A_\beta), \end{aligned} \tag{3.2}$$

for some $p \in H^1(\Omega)$.

In addition, $-A_\beta$ generates an analytic semigroup of contractions on H and $D(A_\beta^{\frac{1}{2}}) = V$.

Proof. Let D_β be the space on the right-hand side of (3.2). First note that, thanks to the condition (1.3) on β, $\beta \mathrm{Tr}_{|\partial\Omega} u \in L^2_{\tan}(\partial\Omega, \mathbb{R}^3)$ whenever $u \in W_T$. Next, remark that for $u \in D_\beta$, since

$$\operatorname{curl} u \in L^2(\Omega, \mathbb{R}^3) \quad \text{and} \quad \operatorname{curl}\operatorname{curl} u \in L^2(\Omega, \mathbb{R}^3),$$

the integration by parts (2.2) allows to define

$$\nu \times \operatorname{curl} u \in H^{-1/2}(\partial\Omega, \mathbb{R}^3).$$

Moreover, the condition $\nu \times \operatorname{curl} u = \beta u$ on $\partial\Omega$ implies that $\nu \times \operatorname{curl} u \in L^2(\partial\Omega, \mathbb{R}^3)$ and by the obvious fact that $\operatorname{div}\operatorname{curl} u = 0 \in L^2(\Omega)$, Proposition 2.1 yields $\mathrm{Tr}_{|\partial\Omega}(\operatorname{curl} u) \in L^2(\partial\Omega, \mathbb{R}^3)$.

If $u \in D_\beta$, then $-\Delta u = \operatorname{curl}\operatorname{curl} u \in L^2(\Omega, \mathbb{R}^3)$ and for all $v \in V$, we have by (2.2)

$$\begin{aligned} \mathfrak{a}_\beta(u, v) &= \langle \operatorname{curl} u, \operatorname{curl} v\rangle_\Omega + \langle \beta u, v\rangle_{\partial\Omega} &(3.3)\\ &= \langle \operatorname{curl}\operatorname{curl} u, v\rangle_\Omega - \langle \nu \times \operatorname{curl} u, v\rangle_{\partial\Omega} + \langle \beta u, v\rangle_{\partial\Omega} &(3.4)\\ &= \langle \mathbb{P}(\operatorname{curl}\operatorname{curl} u), v\rangle_\Omega. &(3.5)\end{aligned}$$

Since $\mathbb{P}(\operatorname{curl}\operatorname{curl} u) \in H$, we have then proved that for all $u \in D_\beta$, $u \in D(A_\beta)$ and $A_\beta u = \mathbb{P}(\operatorname{curl}\operatorname{curl} u)$.

Conversely, let $u \in V \subset W_T$ and set $g := \beta \mathrm{Tr}_{|\partial\Omega} u$. As already mentioned, $g \in L^2_{\tan}(\partial\Omega, \mathbb{R}^3)$ thanks to (1.3). We can then apply Lemma 2.3 to obtain $w \in L^2(\Omega, \mathbb{R}^3)$ with $\operatorname{curl} w \in L^2(\Omega, \mathbb{R}^3)$ satisfying

$$\langle g, v\rangle_{\partial\Omega} = \langle \operatorname{curl} w, v\rangle_\Omega - \langle w, \operatorname{curl} v\rangle_\Omega \quad \text{for all } v \in W_T. \tag{3.6}$$

Therefore, for a fixed $u \in V$, we can rewrite $\mathfrak{a}_\beta(u, \cdot)$ as follows:

$$\mathfrak{a}_\beta(u, v) = \langle \operatorname{curl} u, \operatorname{curl} v\rangle_\Omega + \langle \operatorname{curl} w, v\rangle_\Omega - \langle w, \operatorname{curl} v\rangle_\Omega \quad \text{for all } v \in V. \tag{3.7}$$

We assume now that $u \in D(A_\beta)$. Since $A_\beta u \in H \subset L^2(\Omega, \mathbb{R}^3)$ and $\mathbb{P}v \in V$ for $v \in W_T$, we can write

$$\begin{aligned} \langle A_\beta u, v\rangle_\Omega &= \langle A_\beta u, \mathbb{P}v\rangle_\Omega = \mathfrak{a}_\beta(u, \mathbb{P}v) &(3.8)\\ &= \langle \operatorname{curl} u, \operatorname{curl} \mathbb{P}v\rangle_\Omega + \langle \operatorname{curl} w, \mathbb{P}v\rangle_\Omega - \langle w, \operatorname{curl} \mathbb{P}v\rangle_\Omega &(3.9)\\ &= \langle \operatorname{curl} u - w, \operatorname{curl} v\rangle_\Omega + \langle \mathbb{P}\operatorname{curl} w, v\rangle_\Omega. &(3.10)\end{aligned}$$

The last equality (3.10) comes from (2.20) and the fact that $\operatorname{curl}\mathbb{P}v = \operatorname{curl} v$. Therefore we obtain

$$\langle A_\beta u - \mathbb{P}\operatorname{curl} w, v\rangle_\Omega = \langle \operatorname{curl} u - w, \operatorname{curl} v\rangle_\Omega \quad \text{for all } v \in W_T. \tag{3.11}$$

For all $v \in H^1_0(\Omega, \mathbb{R}^3) \subset W_T$, (3.11) becomes

$$\langle A_\beta u - \mathbb{P}\operatorname{curl} w, v\rangle_\Omega = {}_{H^{-1}}\langle \operatorname{curl}(\operatorname{curl} u - w), v\rangle_{H^1_0},$$

which implies that $\operatorname{curl}(\operatorname{curl} u - w) \in L^2(\Omega, \mathbb{R}^3)$ and ultimately, since $\operatorname{curl} w \in L^2(\Omega, \mathbb{R}^3)$, $\operatorname{curl}\operatorname{curl} u \in L^2(\Omega, \mathbb{R}^3)$.

We have proved that for $u \in D(A_\beta)$, $\operatorname{curl}\operatorname{curl} u \in L^2(\Omega, \mathbb{R}^3)$. It remains to identify $A_\beta u$ and the boundary condition $\nu \times \operatorname{curl} u = \beta u$ on $\partial\Omega$ for $u \in D(A_\beta)$. Note that this condition is well defined thanks to (2.2) since $\operatorname{curl} u \in L^2(\Omega, \mathbb{R}^3)$ ($u \in D(A_\beta) \subset V \subset W_T$) and $\operatorname{curl}\operatorname{curl} u \in L^2(\Omega, \mathbb{R}^3)$. By definition (3.1) of $\mathfrak{a}_\beta$ and thanks to (2.20), we have for all $v \in \mathscr{D}(\Omega)$ (recall that $\mathscr{D}(\Omega) = \big\{w \in \mathscr{C}^\infty_c(\Omega, \mathbb{R}^3), \operatorname{div} w = 0 \text{ in } \Omega\big\}$ has been defined in Remark 2.5)

$$\langle A_\beta u, v\rangle_\Omega = \mathfrak{a}_\beta(u, v) = \langle \operatorname{curl} u, \operatorname{curl} v\rangle_\Omega$$

$$= \langle \operatorname{curl}\operatorname{curl} u, v\rangle_\Omega = \langle \operatorname{curl}\operatorname{curl} u, \mathbb{P}v\rangle_\Omega$$
$$= \langle \mathbb{P}(\operatorname{curl}\operatorname{curl} u), v\rangle_\Omega, \tag{3.12}$$

since $\mathbb{P}v = v$. This proves that $A_\beta u = \mathbb{P}(\operatorname{curl}\operatorname{curl} u)$ since $\mathscr{D}(\Omega)$ is dense in H (see Remark 2.5).

Now, let $v \in V$ and recall that $\mathrm{Tr}_{|\partial\Omega} v \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$. We have then by (2.2)

$$\begin{aligned}\langle \mathbb{P}(\operatorname{curl}\operatorname{curl} u), v\rangle_\Omega &= \langle A_\beta u, v\rangle_\Omega = \mathfrak{a}_\beta(u, v)\\ &= \langle \operatorname{curl} u, \operatorname{curl} v\rangle_\Omega + \langle \beta u, v\rangle_{\partial\Omega}\\ &= \langle \operatorname{curl}\operatorname{curl} u, v\rangle_\Omega - {}_{H^{-1/2}}\langle \nu\times\operatorname{curl} u, v\rangle_{H^{1/2}} + \langle \beta u, v\rangle_{\partial\Omega}\\ &= \langle \mathbb{P}(\operatorname{curl}\operatorname{curl} u), v\rangle_\Omega - {}_{H^{-1/2}}\langle \nu\times\operatorname{curl} u, v\rangle_{H^{1/2}} + \langle \beta u, v\rangle_{\partial\Omega},\end{aligned}$$

which proves that

$${}_{H^{-1/2}}\langle \beta u - \nu\times\operatorname{curl} u, v\rangle_{H^{1/2}} = 0 \quad \text{for all } v \in V. \tag{3.13}$$

Let $\varphi \in H^{1/2}(\partial\Omega, \mathbb{R}^3) \cap L^2_{\tan}(\partial\Omega, \mathbb{R}^3)$ be arbitrary. By Lemma 2.4, we can find $v \in V$ such that $v_{|\partial\Omega} = \varphi$ on $\partial\Omega$. Therefore, (3.13) implies that for all $\varphi \in H^{1/2}(\partial\Omega, \mathbb{R}^3) \cap L^2_{\tan}(\partial\Omega, \mathbb{R}^3)$

$${}_{H^{-1/2}}\langle \beta u - \nu\times\operatorname{curl} u, \varphi\rangle_{H^{1/2}} = 0, \tag{3.14}$$

With $w \in L^2(\Omega, \mathbb{R}^3)$ such that $\operatorname{curl} w \in L^2(\Omega, \mathbb{R}^3)$ satisfying $\beta u = \nu\times w$ in $H^{-1/2}(\partial\Omega, \mathbb{R}^3)$ as in Lemma 2.3, it follows from (3.14) that $w_1 := w - \operatorname{curl} u$ satisfies

$$\langle \operatorname{curl} w_1, v\rangle_\Omega - \langle w_1, \operatorname{curl} v\rangle_\Omega = 0 \quad \text{for all } v \in W_T. \tag{3.15}$$

Let now $v \in H^1(\Omega, \mathbb{R}^3)$ and denote for $\varepsilon > 0$, $v_\varepsilon = (1 + \varepsilon B_0)^{-1} v$ (recall that the operator B_0 has been defined in (2.10)). It is clear that $v_\varepsilon \in W_T$ for all $\varepsilon > 0$ and

$$v_\varepsilon \longrightarrow v \quad \text{in } L^2(\Omega, \mathbb{R}^3) \quad \text{as } \varepsilon \to 0.$$

Moreover, thanks to (2.14), we have that

$$\operatorname{curl} v_\varepsilon = (1 + \varepsilon B_1)^{-1} \operatorname{curl} v \longrightarrow v \quad \text{in } L^2(\Omega, \mathbb{R}^3) \quad \text{as } \varepsilon \to 0.$$

Applying (3.15) to v_ε and taking the limit as $\varepsilon \to 0$, we obtain

$$0 = \langle \operatorname{curl} w_1, v_\varepsilon\rangle_\Omega - \langle w_1, \operatorname{curl} v_\varepsilon\rangle_\Omega \longrightarrow \langle \operatorname{curl} w_1, v\rangle_\Omega - \langle w_1, \operatorname{curl} v\rangle_\Omega \quad \text{as } \varepsilon \to 0.$$

It follows then that $\nu \times w_1 = 0$ in $H^{-1/2}(\partial\Omega, \mathbb{R}^3)$ and therefore

$$\beta u - \nu\times\operatorname{curl} u = 0 \quad \text{in} \quad H^{-1/2}(\partial\Omega, \mathbb{R}^3).$$

Finally, the fact that $-A_\beta$ generates an analytic semigroup of contractions follows from the fact that A_β is a non-negative self-adjoint operator. The equality $D(A_\beta^{\frac{1}{2}}) = V$ is a standard result for symmetric bilinear closed forms (see [17] and [14]). □

Corollary 3.3. *If $u \in D(A_\beta)$ then $\operatorname{curl} u \in L^3(\Omega, \mathbb{R}^3)$ and there exists a constant C_Ω independent of u such that*

$$\|\operatorname{curl} u\|_3 \le C_\Omega \Big(\|A_\beta u\|_H + (\|\beta\|_\infty + 1)\|u\|_V \Big).$$

Proof. Let $u \in D(A_\beta)$. By Theorem 3.2, $\operatorname{curl} u \in L^2(\Omega, \mathbb{R}^3)$, $\operatorname{curl}\operatorname{curl} u \in L^2(\Omega, \mathbb{R}^3)$ and $\nu\times\operatorname{curl} u = \beta u \in L^2(\partial\Omega, \mathbb{R}^3)$. Therefore, by Proposition 2.1, $\operatorname{curl} u \in H^{1/2}(\Omega, \mathbb{R}^3)$ with the estimate

$$\begin{aligned}&\|\operatorname{curl} u\|_{H^{1/2}(\Omega,\mathbb{R}^3)}\\ &\le C\left(\|\operatorname{curl} u\|_{L^2(\Omega,\mathbb{R}^3)} + \|\operatorname{curl}\operatorname{curl} u\|_{L^2(\Omega,\mathbb{R}^3)} + \|\beta u\|_{L^2(\partial\Omega,\mathbb{R}^3)}\right)\\ &\le C\left((\|\beta\|_\infty + 1)\|u\|_V + \|\operatorname{curl}\operatorname{curl} u\|_{L^2(\Omega,\mathbb{R}^3)}\right).\end{aligned}$$

This latter estimate together with the following Sobolev embedding valid in dimension 3

$$H^{1/2}(\Omega, \mathbb{R}^3) \hookrightarrow L^3(\Omega, \mathbb{R}^3)$$

proves the corollary. □

4. Maximal Regularity for Non-autonomous Equations

Our aim in this section is to show maximal regularity for the Stokes problem. We first recall some recent results on maximal regularity for evolution equations associated with time-dependent sesquilinear forms.

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{V}$ be another Hilbert space with dense and continuous embedding in $\mathcal{H}$. Consider a family of sesquilinear forms $(\mathfrak{a}(t))_{0\le t\le\tau}$ such that $D(\mathfrak{a}(t)) = \mathcal{V}$ for all t. We suppose that $(\mathfrak{a}(t))_{0\le t\le\tau}$ is uniformly bounded in the sense that there exists a constant M independent of t such that

$$|\mathfrak{a}(t;u,v)| \le M\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}} \tag{4.1}$$

for all $u, v \in \mathcal{V}$. Here $\|v\|_{\mathcal{V}}$ denotes the norm of $\mathcal{V}$. We also suppose that $(\mathfrak{a}(t))_{0\le t\le\tau}$ is quasi-coercive, i.e., there exists $\delta > 0$ and $\mu \in \mathbb{R}$ such that for all $u \in \mathcal{V}$

$$\delta\|u\|^2_{\mathcal{V}} \le \mathfrak{a}(t;u,u) + \mu\|u\|^2_{\mathcal{H}}. \tag{4.2}$$

For each fixed t, the form $\mathfrak{a}(t)$ is closed. Denote by $\mathcal{A}(t) : \mathcal{V} \to \mathcal{V}'$ the operator associated with $\mathfrak{a}(t)$ in the sense that

$$\mathfrak{a}(t;u,v) = {}_{\mathcal{V}'}\langle \mathcal{A}(t)u, v\rangle_{\mathcal{V}},\ \forall\ u,v \in \mathcal{V}.$$

The operator associated with $\mathfrak{a}(t)$ on $\mathcal{H}$ is the part of $\mathcal{A}(t)$. That is,

$$D(A(t)) = \big\{u \in \mathcal{V}, \mathcal{A}(t)u \in \mathcal{H}\big\}, \quad \mathcal{A}(t)u = A(t)u.$$

We recall now the famous Lions' maximal regularity result in $\mathcal{V}'$:

Theorem 4.1. *Under the above assumptions, for all $f \in L^2(0,\tau;\mathcal{V}')$ and all $u_0 \in \mathcal{H}$, there exists a unique $u \in L^2(0,\tau;\mathcal{V}) \cap H^1(0,\tau;\mathcal{V}')$ solution of*

$$u'(t) + \mathcal{A}(t)u(t) \;=\; f(t), \quad u(0) \;=\; u_0. \tag{P}$$

Moreover, $u \in \mathscr{C}([0,\tau],\mathcal{H})$.

One says that (P) has L^p maximal regularity in $\mathcal{H}$ if for every $f \in L^p(0,\tau;\mathcal{H})$ there exists a unique $u \in W^{1,p}(0,\tau;\mathcal{H})$ which satisfies the problem in the L^p-sense. Note that one has in addition that $t \mapsto A(t)u(t)$ is in $L^p(0,\tau;\mathcal{H})$.
Maximal regularity for non-autonomous equations in $\mathcal{H}$ has been investigated recently in the context of operators associated with forms as we described above. The following is a particular case of a result proved in [12].

Theorem 4.2. *Let $(\mathfrak{a}(t))_{0\le t\le\tau}$ be a family of sesquilinear forms satisfying the previous conditions (4.1) and (4.2). Suppose in addition that $t \mapsto \mathfrak{a}(t)$ is piecewise $\alpha-$Hölder continuous for some $\alpha > 1/2$ in the sense that there exist $t_0 = 0 < t_1 < \cdots < t_k = \tau$ and constants M_i such that the restriction of $t \mapsto \mathfrak{a}(t;\cdot,\cdot)$ to (t_i, t_{i+1}) satisfies*

$$|\mathfrak{a}(t;u,v) - \mathfrak{a}(s;u,v)| \le M_i|t-s|^\alpha\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}} \quad \text{for all } u,v \in \mathcal{V}. \tag{4.3}$$

Then the Cauchy problem (P) has L^2-maximal regularity for all initial value $u_0 \in D((\mu + A(0))^{1/2})$.

Note that if the form $\mathfrak{a}(0)$ is symmetric then $D((\mu + A(0))^{1/2}) = \mathcal{V}$. Recall also that if the L^2-maximal regularity holds for (P) then the solution u satisfies the a priori estimate

$$\|u\|_{H^1(0,\tau;\mathcal{H})} + \|A(t)u(t)\|_{L^2(0,\tau;\mathcal{H})} \le C\big(\|f\|_{L^2(0,\tau;\mathcal{H})} + \|u_0\|_{\mathcal{V}}\big). \tag{4.4}$$

Interpolating between Theorems 4.1 and 4.2 we obtain also

Corollary 4.3. *Under the assumptions of Theorem 4.2, we have that for all $u_0 \in D((\mu + A(0))^{1/4}) =: \mathcal{V}_{1/2}$ and for all $f \in L^2(0,\tau;\mathcal{V}'_{1/2})$ there exists a unique $u \in H^1(0,\tau;\mathcal{V}'_{1/2})$ such that $t \mapsto \mathcal{A}(t)u(t) \in L^2(0,\tau;\mathcal{V}'_{1/2})$, solution of the evolution problem (P).*

Now we turn back to the Robin–Stokes operator A_β. As previously, Ω denotes a bounded domain of $\mathbb{R}^3$ which is either $\mathscr{C}^{1,1}$ or convex. Let $\mathcal{H} := H$ defined by (2.19), that is

$$H := \{u \in L^2(\Omega, \mathbb{R}^3); \operatorname{div} u = 0 \text{ in } \Omega, \nu \cdot u = 0 \text{ on } \partial\Omega\}$$

and $\mathfrak{a}_\beta$ the form defined by (3.1). We assume in addition to (1.1), (1.2) and (1.3) that $t \mapsto \beta(t, x)$ is piecewise Hölder continuous of order $\alpha > 1/2$ in the sense of (1.4).

The family of forms $\mathfrak{a}_\beta = \mathfrak{a}_{\beta(t,\cdot)}$, $0 \le t \le \tau$, satisfies the assumptions of Theorem 4.2. In order to check (4.3) we write for $u, v \in V$ and $t, s \in (t_i, t_{i+1})$

$$\begin{aligned}|\mathfrak{a}_{\beta(t,\cdot)}(u, v) - \mathfrak{a}_{\beta(s,\cdot)}(u, v)| &= \langle(\beta(t, \cdot) - \beta(s, \cdot))u, v\rangle_{\partial\Omega} \\ &\le \sup_{x\in\partial\Omega} \|\beta(t, x) - \beta(s, x)\|_{\mathscr{M}_3} \|\mathrm{Tr}_{|\partial\Omega} u\|_{L^2(\partial\Omega,\mathbb{R}^3)} \|\mathrm{Tr}_{|\partial\Omega} v\|_{L^2(\partial\Omega,\mathbb{R}^3)} \\ &\le CM_i|t - s|^\alpha \|u\|_V \|v\|_V.\end{aligned}$$

The last inequality follows from (1.4) and Proposition 2.1. Therefore we conclude that L^2-maximal regularity holds for the Robin–Stokes operator A_β on the Hilbert space H.

Theorem 4.4. *Under the above assumptions, for every $u_0 \in V$ and every $f \in L^2(0, \tau; H)$ there exists a unique $u \in H^1(0, \tau; H)$ such that $u(t) \in D(A_{\beta(t)})$ for almost all $t \in [0, \tau]$ and*

$$\begin{cases} \partial_t u(t, \cdot) + A_{\beta(t)} u(t, \cdot) = f(t) \\ u(0) = u_0. \end{cases} \tag{4.5}$$

In addition there exists a constant C_{MR} independent of t, f and u_0 such that

$$\|u\|_{H^1(0,\tau;H)} + \|A_{\beta(t)} u(t)\|_{L^2(0,\tau;H)} \le C_{MR}(\|f\|_{L^2(0,\tau;H)} + \|u_0\|_V). \tag{4.6}$$

Note that if (1.4) holds with $\alpha = 1$ then we can apply the results from [2] and obtain the previous theorem with the additional information that the solution $u \in \mathscr{C}([0, \tau]; V)$. In particular, $u \in L^\infty(0, \tau; V)$. This latter property is not covered by the results in [12] when (1.4) holds for some $\alpha > 1/2$. However, in the recent paper [3, Theorem 4.4], it has been proven that this is true in our particular situation for the operators $A_{\beta(t,\cdot)}$. We give a proof here in the more general setting of forms for which (1.4) holds for some $\alpha > 1/2$.

As in the beginning of this section, let $(\mathfrak{a}(t))_{0\le t\le\tau}$ be a family of symmetric forms on a Hilbert space $\mathcal{H}$ which satisfy (4.1) and (4.2). Suppose that $t \mapsto \mathfrak{a}(t)$ is piecewise $\alpha-$Hölder continuous for some $\alpha > 1/2$ (see Theorem 4.2). We define the space of maximal regularity

$$\begin{aligned}E :=\{&u \in H^1(0, \tau; \mathcal{H}), u(t) \in D(A_{\beta(t)}) \text{ a.e.}, \\ &t \mapsto A(t)u(t) \in L^2(0, \tau; \mathcal{H}) \text{ and } u(0) \in \mathcal{V}\}.\end{aligned} \tag{4.7}$$

The space E is endowed with the natural norm

$$\|u\|_E := \|u(\cdot)\|_{H^1(0,\tau;\mathcal{H})} + \|A(\cdot)u(\cdot)\|_{L^2(0,\tau;\mathcal{H})} + \|u(0)\|_{\mathcal{V}}.$$

Clearly, $(E, \|\cdot\|_E)$ is a Banach space. Note that if $u(\cdot) \in H^1(0, \tau; \mathcal{H})$ then $u \in \mathscr{C}([0, \tau]; \mathcal{H})$ and hence $u(0)$, needed in the definition of E, is well defined.

Proposition 4.5. *The space E is continuously embedded into $L^\infty(0, \tau; \mathcal{V})$.*

Proof. First by adding a positive constant to $A(t)$, it is clear that we may suppose without loss of generality that (4.2) holds with $\mu = 0$.
Let $u \in E$ and set $f := \partial_t u + A(\cdot)u(\cdot) \in L^2(0, \tau, \mathcal{H})$. As in [12], taking the derivative of $s \mapsto v(s) := e^{-(t-s)\mathcal{A}(t)}u(s)$ for $0 < s \le t < \tau$ and then integrating from 0 to t it follows that

$$u(t) = \int_0^t e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s)\,\mathrm{d}s$$

$$+ e^{-tA(t)}u(0) + \int_0^t e^{-(t-s)A(t)} f(s)\,\mathrm{d}s. \tag{4.8}$$

We estimate the norm in $\mathcal{V}$ of each term. Recall that $-\mathcal{A}(t)$ generates a bounded holomorphic semigroup in $\mathcal{V}'$ (see [28, Chapter 1]) with bound independent of $t \in [0,\tau]$ thanks to (4.1) and (4.2). In particular, there exist a constant C such that for all $s > 0$ and $t \in [0,\tau]$

$$\|e^{-s\mathcal{A}(t)}\|_{\mathscr{L}(\mathcal{V}',\mathcal{V})} \le \delta^{-1}\|\mathcal{A}(t)e^{-s\mathcal{A}(t)}\|_{\mathscr{L}(\mathcal{V}')} \le \frac{C}{s}. \tag{4.9}$$

Therefore,

$$\begin{aligned}
&\Big\|\int_0^t e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\,\mathrm{d}s\Big\|_{\mathcal{V}} \\
&\quad\le \int_0^t \frac{C}{t-s}\,\|(\mathcal{A}(t)-\mathcal{A}(s))u(s)\|_{\mathcal{V}'}\,\mathrm{d}s \\
&\quad\le \int_0^t \frac{C\omega(t-s)}{t-s}\,\|u(s)\|_{\mathcal{V}}\,\mathrm{d}s
\end{aligned}$$

where $r \mapsto \omega(r)$ is piecewise α-Hölder continuous on $[0,\tau]$ with $\alpha > 1/2$ by assumption. By the Cauchy–Schwarz inequality we conclude that

$$\Big\|\int_0^t e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\,\mathrm{d}s\Big\|_{\mathcal{V}} \le C'\Big(\int_0^t \|u(s)\|_{\mathcal{V}}^2\,\mathrm{d}s\Big)^{1/2}. \tag{4.10}$$

The second term is easily estimated since the semigroup $(e^{-sA(t)})_{s\ge 0}$ is uniformly bounded on $\mathcal{V}$ (see again [28, Chapter 1]). Thus

$$\|e^{-tA(t)}u(0)\|_{\mathcal{V}} \le C\|u(0)\|_{\mathcal{V}} \quad \text{for all } t \ge 0. \tag{4.11}$$

It remains to estimate the third term. Set $v(s) := \int_0^s e^{-(s-r)A(t)}f(r)\,\mathrm{d}r$, $s \ge 0$. The function v satisfies

$$\partial_s v + A(t)v = f, \quad v(0) = 0.$$

Fix $\varepsilon > 0$. Since $A(t)^{1/2}e^{-\varepsilon A(t)}$ is a bounded operator on $\mathcal{H}$ we have that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|A(t)^{1/2}e^{-\varepsilon A(t)}v(s)\|_{\mathcal{H}}^2 &= (A(t)^{1/2}e^{-\varepsilon A(t)}v'(s), A(t)^{1/2}e^{-\varepsilon A(t)}v(s)) \\
&= (-A(t)v(s) + f(s), A(t)e^{-2\varepsilon A(t)}v(s)).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|A(t)^{1/2}e^{-\varepsilon A(t)}v(s)\|_{\mathcal{H}}^2 + \|A(t)e^{-\varepsilon A(t)}v(s)\|_{\mathcal{H}}^2 \\
&\quad= (f(s), A(t)e^{-2\varepsilon A(t)}v(s)) \\
&\quad\le \frac{1}{2}\,\|f(s)\|_{\mathcal{H}}^2 + \frac{1}{2}\,\|A(t)e^{-2\varepsilon A(t)}v(s)\|_{\mathcal{H}}^2.
\end{aligned}$$

Next we integrate from 0 to t and then letting $\varepsilon \to 0$ it follows that

$$\Big\|A(t)^{1/2}\int_0^t e^{-(t-r)A(t)}f(r)\,\mathrm{d}r\Big\|_{\mathcal{V}}^2 \le \|f\|_{L^2(0,\tau,\mathcal{H})}^2.$$

From the coercivity assumption (4.2) with $\mu = 0$, it follows that

$$\Big\|\int_0^t e^{-(t-r)A(t)} f(r)\,\mathrm{d}r\Big\|_{\mathcal{V}}^2 \le \delta^{-1}\|f\|^2_{L^2(0,\tau,\mathcal{H})}. \tag{4.12}$$

We obtain from (4.8) and the forgoing estimates (4.10)–(4.12) that for some constant $C_0 > 0$

$$\|u(t)\|^2_{\mathcal{V}} \le C_0\Big[\int_0^t \|u(s)\|^2_{\mathcal{V}}\,\mathrm{d}s + \|u(0)\|^2_{\mathcal{V}} + \|f\|^2_{L^2(0,\tau,\mathcal{H})}\Big].$$

It follows from Gronwall's lemma that

$$\|u(t)\|^2_{\mathcal{V}} \le C_0\, e^{C_0\tau}\Big[\|u(0)\|^2_{\mathcal{V}} + \|f\|^2_{L^2(0,\tau,\mathcal{H})}\Big].$$

Replacing $f(t)$ by its expression $f(t) = \partial_t u(t) + A(t)u(t)$, the conclusion of the proposition follows. □

Combining now Corollary 4.3 with Proposition 4.5 we obtain the analog of Theorem 4.4.

Theorem 4.6. *Under the assumptions of Theorem 4.4, for every $u_0 \in V_{\frac12}$ and every $f \in L^2(0,\tau;V'_{\frac12})$ there exists a unique $u \in H^1(0,\tau;V'_{\frac12})$ such that $t \mapsto \mathcal{A}_{\beta(t)}u(t) \in V'_{\frac12}$ for almost all $t \in [0,\tau]$ solution of (4.5).*

In addition there exists a constant C'_{MR} independent of t, f and u_0 such that

$$\|u\|_{H^1(0,\tau;V'_{\frac12})} + \|\mathcal{A}_{\beta(t)}u(t)\|_{L^2(0,\tau;V'_{\frac12})} \le C'_{MR}\big(\|f\|_{L^2(0,\tau;V'_{\frac12})} + \|u_0\|_{V_{\frac12}}\big), \tag{4.13}$$

where $V_{\frac12} := [H,V]_{\frac12}$ denotes the interpolation space between H and V, which coincides with $D(A^{1/4}_{\beta(t)})$.

Moreover, the space of solutions described above continuously embeds into $L^\infty(0,\tau;V_{\frac12})$.

5. The Navier–Stokes System with Robin Boundary Conditions

As in the previous sections, Ω denotes a bounded $\mathscr{C}^{1,1}$ or convex domain of $\mathbb{R}^3$ and $\beta : [0,\tau]\times\partial\Omega \to \mathscr{M}_3(\mathbb{R})$ satisfies (1.1)–(1.4) for some $\alpha > \frac12$. Recall from Sect. 3 that

$$H = \big\{u \in L^2(\Omega,\mathbb{R}^3); \operatorname{div} u = 0 \text{ in } \Omega, \nu\cdot u = 0 \text{ on } \partial\Omega\big\}$$

and

$$V = \big\{u \in L^2(\Omega,\mathbb{R}^3); \operatorname{div} u = 0 \text{ in } \Omega, \operatorname{curl} u \in L^2(\Omega,\mathbb{R}^3) \text{ and } \nu\cdot u = 0 \text{ on } \partial\Omega\big\}.$$

The latter space is the domain of the bilinear symmetric form which gives rise to the Robin–Stokes operator A_β defined in Sect. 3.

We consider the Navier–Stokes system with Robin-type boundary conditions on the time interval $[0,\tau]$

$$\begin{cases} \partial_t u - \Delta u + \nabla\pi - u\times\operatorname{curl} u = 0 & \text{in } [0,\tau]\times\Omega \\ \operatorname{div} u = 0 & \text{in } [0,\tau]\times\Omega \\ \nu\cdot u \;=\; 0, \quad \nu\times\operatorname{curl} u = \beta u & \text{on } [0,\tau]\times\partial\Omega \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{NS}$$

Our main result in this section is the following existence, uniqueness and regularity result for (NS).

Theorem 5.1. *There exists $\epsilon > 0$ such that for every $u_0 \in V$ with $\|u_0\|_V \le \epsilon$, there exists a unique $u \in H^1(0,\tau;H)$ with $t \mapsto A_{\beta(t)}u(t) \in L^2(0,\tau;H)$ and $\pi \in L^2(0,\tau;H^1(\Omega))$ such that (u,π) satisfies (NS) for a.e. $(t,x) \in [0,\tau]\times\Omega$. In addition there exists a constant C independent of u and π such that*

$$\|u\|_{H^1(0,\tau;H)} + \|-\Delta u\|_{L^2(0,\tau;L^2(\Omega,\mathbb{R}^3))} + \|\nabla\pi\|_{L^2(0,\tau;L^2(\Omega,\mathbb{R}^3))} \le C\epsilon. \tag{5.1}$$

Proof. Recall the maximal regularity space

$$E = \big\{u \in H^1(0,\tau;H); u(t) \in D(A_{\beta(t)}) \text{ a.e.}, \\ t \mapsto A_{\beta(t)}u(t) \in L^2(0,\tau;H) \text{ and } u(0) \in V\big\}.$$

For all $u \in E$, we have that $u(t) \in D(A_{\beta(t)})$ for a.e. $t \in [0,\tau]$. Then by Corollary 3.3 we obtain

$$\|\operatorname{curl} u(t)\|_3 \le C_\Omega \|A_{\beta(t)}u(t)\|_H + C(\|\beta\|_\infty + 1)\|u(t)\|_V.$$

Using Proposition 4.5 and taking the L^2-norm in time, it follows that

$$\|\operatorname{curl} u\|_{L^2(0,\tau;L^3(\Omega,\mathbb{R}^3))} \le C_\Omega \|u\|_E + C(\|\beta\|_\infty + 1)\|u\|_E = C_1\|u\|_E. \tag{5.2}$$

On the other hand, by (2.7), the classical Sobolev embedding of $H^1(\Omega)$ into $L^6(\Omega)$ in dimension 3 and Proposition 4.5, there exists a constant C_2 such that for every $u \in E$

$$\|u\|_{L^\infty(0,\tau;L^6(\Omega,\mathbb{R}^3))} \le C_2\|u\|_E. \tag{5.3}$$

Let $u_0 \in V$. By Theorem 4.4, there exists a solution $a \in E$ of the problem

$$\partial_t a + A_{\beta(t)}a \;=\; 0 \quad a(0) \;=\; u_0, \tag{5.4}$$

with

$$\|a\|_E \le C_{MR}\|u_0\|_V. \tag{5.5}$$

Let $u, v \in E$ and set $f := \frac{1}{2}\,\mathbb{P}(u \times \operatorname{curl} v + v \times \operatorname{curl} u)$. By (5.2) and (5.3), $f \in L^2(0,\tau;H)$ and

$$\|f\|_{L^2(0,\tau;H)} \le C_1C_2\|u\|_E\|v\|_E. \tag{5.6}$$

Again by Theorem 4.4 there exists w solution of

$$\partial_t w + A_{\beta(t)}w \;=\; f, \quad w(0) \;=\; 0. \tag{5.7}$$

In addition, $w \in E$ and satisfies $\|w\|_E \le C_{MR}\|f\|_{L^2(0,\tau,H)}$.

We define the bilinear application

$$B : E \times E \to E, \quad (u,v) \mapsto w.$$

Then the latter estimate gives

$$\|B(u,v)\|_E = \|w\|_E \le C_{MR}\|f\|_{L^2(0,\tau,H)}. \tag{5.8}$$

Thus we have from (5.6)

$$\|B(u,v)\|_E \le C_{MR}C_1C_2\|u\|_E\|v\|_E. \tag{5.9}$$

We now use Picard's contraction principle (see [15, Theorem 13.2]). Let $\delta > 0$ such that $\delta < \frac{1}{4C_{MR}C_1C_2}$. If $\|a\|_E \le \delta$, the mapping

$$\begin{aligned} T : \overline{B}_E(0,2\delta) &\longrightarrow \overline{B}_E(0,2\delta) \\ v &\longmapsto a + B(v,v) \end{aligned}$$

is a strict contraction. Therefore there exists a unique $u \in \overline{B}_E(0,2\delta)$ satisfying $u = a + B(u,u)$. By (5.5), the condition $\|a\|_E \le \delta$ is satisfied if $\|u_0\|_V \le \epsilon := \frac{\delta}{C_{MR}}$. It remains to prove that u is a solution of (NS) for a.e. $(t,x) \in [0,\tau] \times \Omega$. Since $u = a + B(u,u)$ with a the solution of (5.4) and $w = B(u,u)$ the solution of (5.7) with $v = u$ we obtain

$$\begin{aligned} \partial_t u &= \partial_t a + \partial_t B(u,u) \\ &= -A_\beta a - A_\beta B(u,u) + \mathbb{P}(u \times \operatorname{curl} u) \\ &= -A_\beta u + \mathbb{P}(u \times \operatorname{curl} u). \end{aligned}$$

Since $u \in E$, $t \mapsto A_{\beta(t)}u(t) \in L^2(0,\tau,H)$ and hence by Theorem 3.2,

$$t \mapsto \operatorname{curl}\operatorname{curl} u(t) = -\Delta u(t) \in L^2(0,\tau,L^2(\Omega,\mathbb{R}^3)).$$

Thus, $A_\beta u = -\Delta u + \nabla q$ with $q \in L^2(0,\tau,H^1(\Omega))$. In addition

$$\nu \cdot u = 0 \quad \text{and} \quad \nu \times \operatorname{curl} u = \beta u$$

for a.e. $(t, x) \in (0, \tau) \times \partial\Omega$. By the definition of $\mathbb{P}$ and integrability properties (5.3) (for u) and (5.2) (for $\operatorname{curl} u$), $\mathbb{P}(u \times \operatorname{curl} u) = u \times \operatorname{curl} u + \nabla p$ with $p \in L^2(0, \tau, H^1(\Omega))$. Therefore, if we take $\pi := p + q$ we see that (u, π) satisfy (NS) for a.e. $(t, x) \in [0, \tau] \times \Omega$. $\square$

Remark 5.2. One of the main tools in our method is the maximal regularity for non-autonomous evolution equations. We relied on the results from [12] and this why we had to assume (piecewise) Hölder continuity (1.4) for some $\alpha > 1/2$. Since this work was submitted some progress have been made on maximal regularity when dealing with non-autonomous Robin boundary conditions. Indeed, the results from [3] and [29] show that for such boundary conditions one may assume Hölder continuity with order strictly larger than $1/4$. Based on this, one may weaken slightly the regularity assumption (1.4).

Using Theorem 4.6 we obtain the following existence (and uniqueness) result in the critical space $V_{\frac{1}{2}}$ for small initial conditions. Under our assumptions, $V_{\frac{1}{2}} = D(A_{\beta(t)}^{\frac{1}{4}})$ for all $t \in [0, \tau]$.

Theorem 5.3. *Under the assumptions of this section, there exists $\varepsilon > 0$ such that for every $u_0 \in V_{\frac{1}{2}} = D(A_{\beta(t)}^{\frac{1}{4}})$ with $\|u_0\|_{V_{\frac{1}{2}}} \le \varepsilon$, there exists a unique $u \in H^1(0, \tau; V'_{\frac{1}{2}})$ with $t \mapsto A_{\beta(t)} u(t) \in L^2(0, \tau; V'_{\frac{1}{2}})$ and $\pi \in L^2(0, \tau; H^{1/2}(\Omega))$ such that (u, π) satisfies (NS) in the sense of distributions. In addition there exists a constant C' independent of u and π such that*

$$\|u\|_{H^1(0,\tau;V'_{\frac{1}{2}})} + \|-\Delta u\|_{L^2(0,\tau;H^{-1/2}(\Omega,\mathbb{R}^3))} + \|\nabla \pi\|_{L^2(0,\tau;H^{-1/2}(\Omega,\mathbb{R}^3))} \le C'\varepsilon. \tag{5.10}$$

Proof. The proof goes as the proof of Theorem 5.1, using the space

$$F := \left\{u \in H^1(0, \tau; V'_{\frac{1}{2}}); t \mapsto \mathcal{A}_{\beta(t)} u(t) \in L^2(0, \tau; V'_{\frac{1}{2}}) \text{ and } u(0) \in V_{\frac{1}{2}}\right\}$$

instead of E and Theorem 4.6 instead of Theorem 4.4. We only have to verify that in that case we can make sense of the nonlinearity $u \times \operatorname{curl} u$ in a suitable space. It is immediate to see that $F \hookrightarrow L^4(0, \tau; V)$. Therefore, for $u, v \in F$, we have that $u \times \operatorname{curl} v, v \times \operatorname{curl} u \in L^2(0, \tau; L^{3/2}(\Omega; \mathbb{R}^3))$ and then $f := \frac{1}{2}\mathbb{P}(u \times \operatorname{curl} v + v \times \operatorname{curl} u) \in L^2(0, \tau; V'_{\frac{1}{2}})$ since $J'_0 = \mathbb{P}_1$ maps $L^{3/2}(\Omega; \mathbb{R}^3)$ to $V'_{\frac{1}{2}}$ (see Sect. 3) by interpolation. $\square$

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