First Order Approach to L^p Estimates for the Stokes Operator on Lipschitz domains

Alan McIntosh [*] Sylvie Monniaux [†]

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1 Introduction

At the ISAAC meeting in Macau, the first author discussed the harmonic analysis of first order systems on bounded domains, with particular reference to his current joint research with the second author concerning the L^p theory of Hodge-Dirac operators on Lipschitz domains, with implications for the Stokes' operator on such domains with Hodge boundary conditions. In this article, we present an overview of this material, staying with the three dimensional situation. Full definitions and proofs in higher dimensions can be found in [14]. In other papers with Marius Mitrea, the second author has pursued applications to the Navier-Stokes equation on Lipschitz domains. We will not comment further on that here, except to mention that the non-linear applications depend on having results for the linear Stokes operator in the case $p = 3$ or possibly $p = 3/2$ (the dual exponent to 3).

Harmonic analysis	$\Longrightarrow$	*First order systems* **Hodge-Dirac systems on bounded domains**
		$\Downarrow$
Potential Application **Navier-Stokes equation with Hodge boundary conditions**	$\Longleftarrow$	*Second order equations* **Hodge-Laplacian; Hodge-Stokes Operator**

[*]Mathematical Sciences Institute, Australian National University, Canberra, ACT 2601, Australia - email: alan.mcintosh@anu.edu.au

[†]Aix-Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France - email: sylvie.monniaux@univ-amu.fr

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3 Hodge-Dirac operators

Our aim is to investigate the L^p theory of the first order *Hodge-Dirac operator*

$$D_H = d_\Omega + \delta_{\overline{\Omega}}$$

acting on a *bounded domain* $\Omega \subset \mathbb{R}^3$ satisfying some kind of *Lipschitz condition.*

Here d_Ω is the *exterior derivative* acting on *differential forms* in $L^p(\Omega, \Lambda)$, and $\delta_{\overline{\Omega}}$ is the adjoint operator which includes the *tangential boundary condition*

$$\nu \lrcorner\, u|_{\partial\Omega} = 0$$

i.e. the *normal component* of u at the boundary $\partial\Omega$ is zero, at least on that part of the boundary where it is well-defined. This is effectively half a boundary condition for D_H, which is what is expected for a first order system.

Let us now define our terms.

4 Lipschitz domains

Henceforth Ω denotes a *bounded connected open subset* of $\mathbb{R}^3$, and B denotes the unit ball in $\mathbb{R}^3$. We say that

- Ω is *very weakly Lipschitz* if $\Omega = \cup_{j=1}^N(\rho_j B)$ for some natural number N, where each map $\rho_j : B \to \rho_j B \subset \mathbb{R}^3$ is uniformly locally bilipschitz, and $1 = \sum_{j=1}^N \chi_j$ on Ω, where each $\chi_j : \Omega \to [0,1]$ is a Lipschitz function with $\mathrm{sppt}_\Omega(\chi_j) \subset \rho_j B$;

- Ω is *strongly Lipschitz* if, locally, the boundary $\partial\Omega$ of Ω is a portion of the graph of a Lipschitz function $g : \mathbb{R}^2 \to \mathbb{R}$ (with respect to some rotated coordinate system), with Ω being to one side of the graph;

- Ω is *smooth* if each such function g is smooth.

In the above, $\mathrm{sppt}_\Omega(\chi_j)$ denotes the closure of $\{x \in \Omega\,;\, \chi_j(x) \neq 0\}$ in Ω.

Every strongly Lipschitz domain is weakly Lipschitz (which we shall not discuss further, but refer the reader to [5]) and every weakly Lipschitz domain is very weakly Lipschitz. A weakly Lipschitz domain which is not strongly Lipschitz is the well known two brick domain (consisting of one brick on top of another, pointing in orthogonal directions), and a very weakly Lipschitz domain which is not weakly Lipschitz is the unit ball with the half-disk $\{(x_1,x_2,x_3) \in B\,;\, x_3 = 0, x_1 > 0\}$ removed.

In a strongly Lipschitz domain (and indeed in a weakly Lipschitz domain), there is a well-defined outward-pointing unit normal $\nu(y)$ for almost every $y \in \partial\Omega$. In fact $\nu \in L^\infty(\partial\Omega;\mathbb{R}^3)$. As can be seen from the above example, the unit normal is not necessarily defined on the whole boundary of a very weakly Lipschitz domain.

5 Exterior Algebra

- The *exterior algebra* on $\mathbb{R}^3$ with basis e_1, e_2, e_3 is

$$\begin{aligned}
&\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \Lambda^3 \approx \mathbb{C} \oplus \mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \mathbb{C} \\
&u = u^0\; + u^1\; + u^2\; + u^3 \qquad \text{where} \\
&\Lambda^0 = \mathbb{C} \\
&\Lambda^1 = \mathbb{C}^3 \;: \qquad u^1 = u^1_1 e_1 + u^1_2 e_2 + u^1_3 e_3 \\
&\Lambda^2 \approx \mathbb{C}^3 \;: \qquad u^2 = u^2_{2,3}\, e_2 \wedge e_3 + u^2_{3,1} e_3 \wedge e_1 + u^2_{1,2}\, e_1 \wedge e_2 \\
&\Lambda^3 \approx \mathbb{C} \;: \qquad u^3 = u^3_{1,2,3}\, e_1 \wedge e_2 \wedge e_3 \qquad (e_k \wedge e_j = -e_j \wedge e_k)
\end{aligned}$$

- $L^p(\Omega,\Lambda) = L^p(\Omega,\mathbb{C}) \oplus L^p(\Omega,\mathbb{C}^3) \oplus L^p(\Omega,\mathbb{C}^3) \oplus L^p(\Omega,\mathbb{C})$
- If $a = \sum_j a_j e_j \in \mathbb{R}^3$, $u \in \Lambda^\ell$, then $a \wedge u = \sum_j a_j e_j \wedge u \in \Lambda^{\ell+1}$
- If also $v \in \Lambda^{\ell+1}$ then $a \lrcorner\, v \in \Lambda^\ell$ and $\langle a \wedge u\,,\, v\rangle = \langle u\,,\, a \lrcorner\, v\rangle$
- $du = \nabla \wedge u = \sum_j e_j \wedge \partial_j u, \quad \delta u = -\nabla \lrcorner\, u = -\sum_j e_j \lrcorner\, \partial_j u$
- The *exterior product* $\wedge$ and the *contraction* $\lrcorner$ can be represented by scalar multiplication, dot products and cross products.

6 The de Rham complex on $\Omega \subset \mathbb{R}^3$

Suppose that Ω denotes a bounded open subset of $\mathbb{R}^3$ and $1 < p < \infty$.

The *exterior derivative* d_Ω defined on Ω can be expressed as follows:

$$d_\Omega : 0 \to L^p(\Omega,\mathbb{C}) \xrightarrow{\nabla_\Omega} L^p(\Omega,\mathbb{C}^3) \xrightarrow{\mathrm{curl}_\Omega} L^p(\Omega,\mathbb{C}^3) \xrightarrow{\mathrm{div}_\Omega} L^p(\Omega,\mathbb{C}) \to 0$$

(noting that curl is sometimes written as rot or $\nabla\times$, and div as $\nabla.$).

As an operator, $d_\Omega : \mathcal{D}^p(d_\Omega) \to L^p(\Omega,\Lambda)$ is an unbounded operator with *domain* $\mathcal{D}^p(d_\Omega) = \{u \in L^p(\Omega,\Lambda)\,;\, d_\Omega u \in L^p(\Omega,\Lambda)\}$.

Note that ${d_\Omega}^2 = 0$ because $\mathrm{curl}_\Omega \nabla_\Omega = 0$ and $\mathrm{div}_\Omega \mathrm{curl}_\Omega = 0$, or as we can see directly, ${d_\Omega}^2 u = \sum_{j,k} e_j \wedge e_k \partial_j \partial_k u = 0$ by the skew-symmetry of the wedge product.

Hence the *range* of d_Ω is contained in the *null-space* of d_Ω, i.e. $\mathcal{R}^p(d_\Omega) \subset \mathcal{N}^p(d_\Omega)$ where $\mathcal{R}^p(d_\Omega) = \{v \in L^p(\Omega,\Lambda)\,;\, v = d_\Omega u \text{ for some } u \in \mathcal{D}^p(d_\Omega)\}$ and $\mathcal{N}^p(d_\Omega) = \{u \in \mathcal{D}^p(d_\Omega)\,;\, d_\Omega u = 0\}$.

If Ω is *very weakly Lipschitz*, then $\overline{\mathcal{R}^p(d_\Omega)} = \mathcal{R}^p(d_\Omega)$ and the codimension of $\mathcal{R}^p(d_\Omega)$ in $\mathcal{N}^p(d_\Omega)$ is finite dimensional. We return to these facts in Section 20.

7 The dual de Rham complex

With Ω and p as above, let $q = p'$ (i.e. $\frac{1}{p} + \frac{1}{q} = 1$).

The dual of the exterior derivative $d_\Omega : \mathcal{D}^q(d_\Omega) \to L^q(\Omega,\Lambda)$:

$$d_\Omega : 0 \to L^q(\Omega,\mathbb{C}) \xrightarrow{\nabla_\Omega} L^q(\Omega,\mathbb{C}^3) \xrightarrow{\mathrm{curl}_\Omega} L^q(\Omega,\mathbb{C}^3) \xrightarrow{\mathrm{div}_\Omega} L^q(\Omega,\mathbb{C}) \to 0$$

is $\delta_{\overline{\Omega}} : \mathcal{D}^p(\delta_{\overline{\Omega}}) \to L^p(\Omega,\Lambda)$:

$$0 \leftarrow L^p(\Omega,\mathbb{C}) \xleftarrow{-\mathrm{div}_{\overline{\Omega}}} L^p(\Omega,\mathbb{C}^3) \xleftarrow{\mathrm{curl}_{\overline{\Omega}}} L^p(\Omega,\mathbb{C}^3) \xleftarrow{-\nabla_{\overline{\Omega}}} L^p(\Omega,\mathbb{C}) \leftarrow 0 : \delta_{\overline{\Omega}}$$

where the domain $\mathcal{D}^p(\delta_{\overline{\Omega}})$ is the completion of $C_c^\infty(\Omega,\Lambda)$ in the graph norm $\|u\|_p + \|\delta_{\overline{\Omega}} u\|_p$.

Again, ${\delta_{\overline{\Omega}}}^2 = 0$, i.e. $\mathcal{R}^p(\delta_{\overline{\Omega}}) \subset \mathcal{N}^p(\delta_{\overline{\Omega}})$.

If Ω is *very weakly Lipschitz*, then $\overline{\mathcal{R}^p(\delta_{\overline{\Omega}})} = \mathcal{R}^p(\delta_{\overline{\Omega}})$, with finite codimension in $\mathcal{N}^p(\delta_{\overline{\Omega}})$.

If Ω is *strongly Lipschitz*, then the *normal component* of $u \in \mathcal{D}^p(\delta_{\overline{\Omega}})$ at the boundary is *zero*, i.e.

$$\mathcal{D}^p(\delta_{\overline{\Omega}}) = \{u \in L^p(\Omega,\Lambda)\,;\, \delta_\Omega u \in L^p(\Omega,\Lambda), \nu \lrcorner u|_{\partial\Omega} = 0\}\ .$$

Remark 7.1. *The condition $\nu \lrcorner u|_{\partial\Omega} = 0$ is to be understood in the following sense: for $u \in L^p(\Omega,\Lambda)$ such that $\delta_\Omega u \in L^p(\Omega,\Lambda)$ in a strongly Lipschitz domain, the normal component at the boundary $\nu \lrcorner u|_{\partial\Omega}$ is defined as a functional on traces of differential forms $v \in W^{1,p'}(\Omega,\Lambda)$ (where $\frac{1}{p'} + \frac{1}{p} = 1$) by the integration by parts formula:*

$$\langle \nu \lrcorner u, v\rangle_{\partial\Omega} = \langle u, dv\rangle_\Omega - \langle \delta u, v\rangle_\Omega.$$

Since $Tr_{|\partial\Omega}\big(W^{1,p'}(\Omega,\Lambda)\big) \subseteq B^{p',p'}_{1/p}(\partial\Omega,\Lambda)$, we obtain that $\nu \lrcorner u \in B^{p,p}_{-1/p}(\partial\Omega,\Lambda)$. For more details, we refer to [16, §2.3].

Remark 7.2. *Some care needs to be taken when consulting references, in that different authors use different sign conventions for δ and Δ.*

Remark 7.3. *The definitions and results concerning very weakly Lipschitz domains in $\mathbb{R}^3$ can be adapted to domains in a Riemannian manifold with very little effort.*

8 Hypothesis

For the rest of this article, Ω denotes a *very weakly Lipschitz* **domain in $\mathbb{R}^3$.**

9 The Hodge-Dirac operator $D_H = d_\Omega + \delta_{\overline{\Omega}}$ in $L^2(\Omega, \Lambda)$

First we consider the case $p = 2$. Then the exterior derivative d_Ω and adjoint interior derivative $\delta_{\overline{\Omega}}$ are unbounded operators in $L^2(\Omega, \Lambda)$ which satisfy

- ${d_\Omega}^2 = 0\ , \quad {\delta_{\overline{\Omega}}}^2 = 0\ , \quad {d_\Omega}^* = \delta_{\overline{\Omega}}\ , \quad {\delta_{\overline{\Omega}}}^* = d_\Omega\,.$

In $L^2(\Omega, \Lambda)$, define the *Hodge-Dirac operator with tangential boundary condition* $D_H := d_\Omega + \delta_{\overline{\Omega}}$ with $\mathcal{D}^2(D_H) = \mathcal{D}^2(d_\Omega) \cap \mathcal{D}^2(\delta_{\overline{\Omega}})$. It is straightforward to check the following properties (using the properties of d_Ω and $\delta_{\overline{\Omega}}$ just described):

- The *Hodge-Dirac operator* $D_H = d_\Omega + \delta_{\overline{\Omega}}$ is self-adjoint in $L^2(\Omega, \Lambda)$;
- $\mathcal{N}^2(D_H) = \mathcal{N}^2(d_\Omega) \cap \mathcal{N}^2(\delta_{\overline{\Omega}})$ is finite-dimensional;
- The *Hodge decomposition* of $L^2(\Omega, \Lambda)$ takes the form

$$\begin{array}{rcl}
L^2(\Omega, \Lambda) &=& \mathcal{N}^2(d_\Omega) \overset{\perp}{\oplus} \mathcal{R}^2(\delta_{\overline{\Omega}}) \\
 & & \quad\cup \qquad\qquad \cap \\
L^2(\Omega, \Lambda) &=& \mathcal{R}^2(d_\Omega) \overset{\perp}{\oplus} \mathcal{N}^2(\delta_{\overline{\Omega}}) \qquad\qquad \text{and so} \\
L^2(\Omega, \Lambda) &=& \mathcal{R}^2(d_\Omega) \overset{\perp}{\oplus} \mathcal{R}^2(\delta_{\overline{\Omega}}) \overset{\perp}{\oplus} \mathcal{N}^2(D_H)\ .
\end{array}$$

- In particular, on restricting to the space of square integrable vector fields, $L^2(\Omega, \Lambda^1) = L^2(\Omega, \mathbb{C}^3)$, we have

$$\begin{array}{rcl}
L^2(\Omega, \Lambda^1) &=& \mathcal{N}^2(\mathrm{curl}_\Omega) \overset{\perp}{\oplus} \mathcal{R}^2(\mathrm{curl}_{\overline{\Omega}}) \\
 & & \quad\cup \qquad\qquad \cap \\
L^2(\Omega, \Lambda^1) &=& \mathcal{R}^2(\nabla_\Omega) \overset{\perp}{\oplus} \boxed{\mathcal{N}^2(\mathrm{div}_{\overline{\Omega}}) = \mathcal{H}^2} \qquad\qquad \text{and so} \\
L^2(\Omega, \Lambda^1) &=& \mathcal{R}^2(\nabla_\Omega) \overset{\perp}{\oplus} \mathcal{R}^2(\mathrm{curl}_{\overline{\Omega}}) \overset{\perp}{\oplus} \mathcal{N}^2(D_H)
\end{array}$$

  where $\mathcal{H}^2 := \mathcal{N}^2(\mathrm{div}_{\overline{\Omega}}) \subset L^2(\Omega, \Lambda^1)$.

In the case when Ω is strongly Lipschitz, $\mathcal{H}^2$ is the space of divergence-free square integrable vector fields which satisify the tangential boundary condition $\nu.u|_{\partial\Omega} = 0$.

10 The Hodge-Laplacian $-\Delta_H = {D_H}^2$

In $L^2(\Omega,\Lambda)$, define the *Hodge-Laplacian* $-\Delta_H := {D_H}^2 = d_\Omega\delta_{\overline{\Omega}} + \delta_{\overline{\Omega}} d_\Omega$ with $\mathcal{D}^2(\Delta_H) = \mathcal{D}^2(d_\Omega\delta_{\overline{\Omega}}) \cap \mathcal{D}^2(\delta_{\overline{\Omega}} d_\Omega)$. This is called the Hodge-Laplacian with absolute or generalised boundary conditions. We remark that Δ_H has the sign convention $\Delta_H u = {\partial_1}^2 u + {\partial_2}^2 u + {\partial_3}^2 u$, $u \in \mathcal{D}^2(\Delta_H)$.

It is straightforward to check the following properties:

- The *Hodge-Laplacian* $-\Delta_H = d_\Omega\delta_{\overline{\Omega}} + \delta_{\overline{\Omega}} d_\Omega$ is non-negative self-adjoint in $L^2(\Omega,\Lambda)$;
- $\mathcal{N}^2(\Delta_H) = \mathcal{N}^2(D_H) = \mathcal{N}^2(d_\Omega) \cap \mathcal{N}^2(\delta_{\overline{\Omega}})$;
- The Hodge-Laplacian preserves each of the spaces $L^2(\Omega,\Lambda^k)$, $0 \le k \le 3$, and so splits as a direct sum of its restrictions to these spaces, as can be seen from the expression $-\Delta_H = d_\Omega\delta_{\overline{\Omega}} + \delta_{\overline{\Omega}} d_\Omega$ with:

$$d_\Omega : 0 \underset{\leftarrow}{\rightarrow} L^2(\Omega,\mathbb{C}) \underset{\underset{-\operatorname{div}_{\overline{\Omega}}}{\leftarrow}}{\overset{\nabla_\Omega}{\rightarrow}} \underset{\underset{\boxed{\mathcal{H}^2}}{\cup}}{L^2(\Omega,\mathbb{C}^3)} \underset{\underset{\operatorname{curl}_{\overline{\Omega}}}{\leftarrow}}{\overset{\operatorname{curl}_\Omega}{\rightarrow}} L^2(\Omega,\mathbb{C}^3) \underset{\underset{-\nabla_{\overline{\Omega}}}{\leftarrow}}{\overset{\operatorname{div}_\Omega}{\rightarrow}} L^2(\Omega,\mathbb{C}) \underset{\leftarrow}{\rightarrow} 0 : \delta_{\overline{\Omega}}$$

$$-\Delta_H = \underset{(= -\Delta_{\text{Neumann}})}{-\operatorname{div}_{\overline{\Omega}}\nabla_\Omega} \oplus (-\nabla_\Omega \operatorname{div}_{\overline{\Omega}} + \operatorname{curl}_{\overline{\Omega}}\operatorname{curl}_\Omega) \oplus (\operatorname{curl}_\Omega\operatorname{curl}_{\overline{\Omega}} - \nabla_{\overline{\Omega}}\operatorname{div}_\Omega) \oplus \underset{(= -\Delta_{\text{Dirichlet}})}{-\operatorname{div}_\Omega \nabla_{\overline{\Omega}}}$$

Indeed it also preserves each component of the Hodge decomposition, in particular $\mathcal{H}^2 = L^2(\Omega,\Lambda^1) \cap \mathcal{N}^2(\delta_{\overline{\Omega}}) = \mathcal{N}^2(\operatorname{div}_{\overline{\Omega}})$.

11 The Hodge-Stokes operator $S_H = -\Delta_H|_{\mathcal{H}^2}$

In $\mathcal{H}^2$, define the *Stokes operator with Hodge boundary conditions* by $S_H u = -\Delta_H u = \operatorname{curl}_{\overline{\Omega}}\operatorname{curl}_\Omega u$, $u \in \mathcal{H}^2$ (i.e. $\operatorname{div}_{\overline{\Omega}} u = 0$) with $\mathcal{D}^2(S_H) = \{u \in L^2(\Omega,\Lambda^1)\,;\, \operatorname{div}_{\overline{\Omega}} u = 0, \operatorname{curl}_{\overline{\Omega}}\operatorname{curl}_\Omega u \in L^2(\Omega,\Lambda^1)\}$. It is straightforward to check the following properties:

- The *Hodge-Stokes* operator $S_H = \operatorname{curl}_{\overline{\Omega}}\operatorname{curl}_\Omega$ is non-negative self-adjoint in $\mathcal{H}^2$;
- $\mathcal{N}^2(S_H) = \mathcal{N}^2(D_H) \cap L^2(\Omega,\Lambda^1) = \mathcal{N}^2(\operatorname{curl}_\Omega) \cap \mathcal{N}^2(\operatorname{div}_{\overline{\Omega}})$ is finite-dimensional;

If Ω is *strongly Lipschitz* and $u \in \mathcal{D}^2(S_H)$, then the *tangential boundary conditions* $\nu.u|_{\partial\Omega} = 0$; $\nu \times \operatorname{curl} u|_{\partial\Omega} = 0$ hold. See, e.g., [17, §3].

12 L^2 results for D_H, Δ_H and S_H

To summarise, we have the following properties:

- $L^2(\Omega,\Lambda) = \mathcal{R}^2(d_\Omega) \overset{\perp}{\oplus} \mathcal{R}^2(\delta_{\overline{\Omega}}) \overset{\perp}{\oplus} \mathcal{N}^2(D_H)$;
- Hodge-Dirac operator $D_H = d_\Omega + \delta_{\overline{\Omega}}$ is self-adjoint in $L^2(\Omega,\Lambda)$;
- Hodge-Laplacian $-\Delta_H = {D_H}^2 = d_\Omega\delta_{\overline{\Omega}} + \delta_{\overline{\Omega}}d_\Omega$ is non-negative self-adjoint in $L^2(\Omega,\Lambda)$;
- Hodge-Stokes operator $S_H = -\Delta_H|_{\mathcal{H}^2}$ is non-negative self-adjoint in $\mathcal{H}^2 = \mathcal{N}^2(\mathrm{div}_{\overline{\Omega}})$.

So D_H, Δ_H, S_H all have *resolvent bounds*, e.g.

$$\begin{aligned}
\|(I+itD_H)^{-1}u\|_2 &\le \|u\|_2 \quad \forall u \in L^2(\Omega,\Lambda)\ , \quad \forall\, t \in \mathbb{R}\setminus\{0\}\\
\|(I-t^2\Delta_H)^{-1}u\|_2 &\le \|u\|_2 \quad \forall u \in L^2(\Omega,\Lambda)\ , \quad \forall\, t > 0\\
\|(I+t^2S_H)^{-1}u\|_2 &\le \|u\|_2 \qquad \forall u \in \mathcal{H}^2 \qquad , \quad \forall\, t > 0
\end{aligned}$$

and all have *functional calculi* of self-adjoint operators, in particular

$$\|D_Hu\|_2 = \|\,\mathrm{sgn}(D_H)\sqrt{-\Delta_H}\,u\|_2 = \|\sqrt{-\Delta_H}\,u\|_2 \quad \forall u \in \mathcal{D}^2(D_H) = \mathcal{D}^2(\sqrt{-\Delta_H})\ .$$

13 L^p questions for D_H, Δ_H and S_H, $1 < p < \infty$

Whether or not the L^p versions of these properties hold, depends on Ω and p. Of course, we no longer have orthogonality of the Hodge decomposition, and the constants in the resolvent bounds and the functional calculi may depend on p. Allowing for this, when Ω is smooth, all of the properties hold for all $p \in (1,\infty)$.

In our situation, namely when Ω is a very weakly Lipschitz domain, we list the main properties and then discuss their relationship with one another, and conditions under which they hold.

(H_p) D_H has an L^p Hodge decomposition: $L^p(\Omega,\Lambda) = \mathcal{R}^p(d_\Omega) \oplus \mathcal{R}^p(\delta_{\overline{\Omega}}) \oplus \mathcal{N}^p(D_H)$;

(R_p) D_H is bisectorial in L^p, in particular $\|(I+itD_H)^{-1}u\|_p \le C\|u\|_p \quad \forall\, t \in \mathbb{R}\setminus\{0\}$;

(F_p) D_H has a bounded $H^\infty(S^o_\mu)$ functional calculus in $L^p(\Omega,\Lambda)$ for all $\mu > 0$: $\|f(D_H)u\|_p \le C_\mu\|f\|_\infty\|u\|_p\ \forall f \in H^\infty(S^o_\mu)$, in particular, $\|D_Hu\|_p \approx \|\sqrt{-\Delta_H}\,u\|_p$.

Here $S^o_\mu = \{z \in \mathbb{C};\, |\arg z| < \mu \text{ or } |\arg(-z)| < \mu\}$, $0 < \mu < \pi/2$.

Let us note that:

- $(F_p) \implies (H_p)$: Exercise.
- $(F_p) \implies (R_p) \implies$ Hodge-Laplacian is sectorial in $L^p(\Omega,\Lambda)$, in particular Δ_H has the L^p resolvent bounds

$$\|(I - t^2\Delta_H)^{-1}u\|_p = \|(I + itD_H)^{-1}(I - itD_H)^{-1}u\|_p \leq C^2\|u\|_p \quad \forall\, t > 0\ .$$

- $(F_p) \implies$ Hodge-Laplacian has a bounded $H^\infty(S^o_\mu)$ functional calculus $\forall \mu > 0$

$\implies$ *maximal regularity* results for the parabolic equation (see §14)

$$\begin{aligned} \partial_t F(t,\,.\,) - \Delta_H F(t,\,.\,) = h(t,\,.\,) \in L^q((0,T);L^p(\Omega,\Lambda))\ ,t>0 \\ F(0,\,.\,) = 0\ . \end{aligned}$$

14 Background on bisectorial operators and holomophic functional calculus

If the reader maintains attention on the resolvent bounds stated for the Hodge-Dirac operator, the Hodge-Laplacian and the Hodge-Stokes operator, then this material is not needed. But we will briefly describe the above-mentioned concepts for those who are interested.

Let $0 \leq \omega < \mu < \frac{\pi}{2}$. Define *closed and open sectors and double sectors* in the complex plane by

$$\begin{aligned} S_{\omega+} &:= \{z \in \mathbb{C}\ :\ |\arg z| \leq \omega\} \cup \{0\}\,, \qquad S_{\omega-} := -S_{\omega+}\,, \\ S^o_{\mu+} &:= \{z \in \mathbb{C}\ :\ z \neq 0,\ |\arg z| < \mu\}\,, \qquad S^o_{\mu-} := -S^o_{\mu+}\,, \\ S_\omega &:= S_{\omega+} \cup S_{\omega-}\,, \qquad S^o_\mu := S^o_{\mu+} \cup S^o_{\mu-}\ . \end{aligned}$$

Let $0 \leq \omega < \frac{\pi}{2}$. A closed operator D acting on a closed subspace $\mathcal{X}^p$ of $L^p(\Omega,\Lambda)$ is called *bisectorial with angle* ω if its spectrum $\sigma(D) \subset S_\omega$, and for all $\theta \in (\omega, \frac{\pi}{2})$ there exists $C_\theta > 0$ such that

$$\|\lambda(\lambda I - D)^{-1}u\|_p \leq C_\theta\|u\|_p \quad \forall \lambda \in \mathbb{C} \setminus S_\theta\ , \forall u \in \mathcal{X}^p\ .$$

In (R_p), we really mean that D_H is *bisectorial with angle* 0, and present the particular resolvent bounds for $\lambda = i/t$ with t real.

Let $0 \leq \omega < \pi$. A closed operator D acting on $\mathcal{X}^p$ is called *sectorial with angle* ω if $\sigma(D) \subset S_{\omega+}$, and for all $\theta \in (\omega, \pi)$ there exists $C_\theta > 0$ such that

$$\|\lambda(\lambda I - D)^{-1}u\|_p \leq C_\theta\|u\|_p \quad \forall \lambda \in \mathbb{C} \setminus S_{\theta+}\ , \forall u \in \mathcal{X}^p\ .$$

For the Hodge-Laplacian, we really mean *sectorial with angle* 0, and present the particular resolvent bounds for $\lambda = -1/t^2$ with $t > 0$.

Denote by $H^\infty(S^o_\mu)$ the space of all bounded holomorphic functions on S^o_μ, and by $\Psi(S^o_\mu)$ the subspace of those functions ψ which satisfy $|\psi(z)| \le C\min\{|z|^\alpha, |z|^{-\alpha}\}$ for some $\alpha > 0$. Similarly define $H^\infty(S^o_{\mu+})$ and $\Psi(S^o_{\mu+})$.

For D bisectorial with angle ω in $\mathcal{X}^p$ and $\psi \in \Psi(S^o_\mu)$, $\omega < \mu < \frac{\pi}{2}$ (or sectorial with angle ω and $\psi \in \Psi(S^o_{\mu+})$, $\omega < \mu < \pi$) define $\psi(D)$ through the Cauchy integral

$$\psi(D)u = \frac{1}{2\pi i}\int_\gamma \psi(z)(zI - D)^{-1}u\,dz, \qquad u \in \mathcal{X}^p\ ,$$

where γ denotes the boundary of S_θ (or $S_{\theta+}$) for some $\theta \in (\omega, \mu)$, oriented counter-clockwise. Then D is said to have a *bounded holomorphic functional calculus with angle* μ, or a *bounded* $H^\infty(S^o_\mu)$ (or $H^\infty(S^o_{\mu+})$) *functional calculus* in $\mathcal{X}^p$ if there exists $C > 0$ such that

$$\|\psi(D)u\|_p \le C_p\|\psi\|_\infty\|u\|_p \qquad \forall u \in \mathcal{X}^p\ , \forall\psi \in \Psi(S^o_\mu)\ (\text{or } \Psi(S^o_{\mu+}))\ .$$

For such an operator, the functional calculus extends to all $f \in H^\infty(S^o_\mu)$ (or $H^\infty(S^o_{\mu+})$) on defining

$$f(D)u = \lim_{n\to\infty}\psi_n(D)u, \qquad u \in \mathcal{X}^p,$$

where the functions $\psi_n \in \Psi(S^o_\mu)$ are uniformly bounded and tend locally uniformly to f. (We are implicitly taking $f(0) = 0$ here.)

We list some properties.

- If D is bisectorial of angle $\omega < \pi/2$, then D^2 is sectorial of angle $2\omega < \pi$.
- If D has a bounded $H^\infty(S^o_\mu)$ functional calculus, then D^2 has a bounded $H^\infty(S^o_{2\mu+})$ functional calculus.
- If D is a bisectorial operator with a bounded holomorphic functional calculus in $\mathcal{X}^p$, then $\|\operatorname{sgn}(D)u\|_p \le C_p\|u\|_p$ for all $u \in \mathcal{X}^p$ where

  $$\operatorname{sgn}(z) = \begin{cases} -1 & z \in S^o_{\mu-} \\ 0 & z = 0 \\ +1 & z \in S^o_{\mu+} \end{cases}$$

  and so D has Riesz transform bounds in $\mathcal{X}^p$:

  $$\begin{aligned} \|Du\|_p &= \|\operatorname{sgn}(D)\sqrt{D^2}\,u\|_p \le C_p\|\sqrt{D^2}\,u\|_p \\ \|\sqrt{D^2}\,u\|_p &= \|\operatorname{sgn}(D)Du\|_p \le C_p\|Du\|_p\ , \quad u \in \mathcal{D}(D) = \mathcal{D}(\sqrt{D^2})\ . \end{aligned}$$

- If S is a sectorial operator with a bounded holomorphic functional calculus of angle $< \pi/2$ in $\mathcal{X}^p$, and $1 < q < \infty$, $0 < T \le \infty$, then the parabolic equation

$$\begin{aligned} \partial_t F(t,\,.\,) + SF(t,\,.\,) &= h(t,\,.\,) \in L^q((0,T);\mathcal{X}^p)\ ,t>0 \\ F(0,\,.\,) &= 0 \end{aligned}$$

has maximal regularity in the sense that

$$\Big\{\int_0^T \|F(t,\,.\,)\|_p{}^q\,dt\Big\}^{1/q} + \Big\{\int_0^T \|SF(t,\,.\,)\|_p{}^q\,dt\Big\}^{1/q} \le C_{p,q}\Big\{\int_0^T \|h(t,\,.\,)\|_p{}^q\,dt\Big\}^{1/q}.$$

For further details on the above material, see [13, 8, 2] or the lecture notes [1, 12].

Solution to Exercise. Show that (H_p) is a consequence of $\|D_H u\|_p \approx \|\sqrt{-\Delta_H}\,u\|_p$. We need $\|D_H u\|_p \approx \|d_\Omega u\|_p + \|\delta_{\overline{\Omega}} u\|_p$, or equivalently $\|d_\Omega u\|_p \lesssim \|D_H u\|_p$.

Write $u = \sum_{k=0}^3 u^k$, $u^k \in L^p(\Omega,\Lambda^k)$, then

$$\begin{aligned} \|d_\Omega u\|_p &\approx \sum_{\ell=0}^3 \|(d_\Omega u)^\ell\|_p = \sum_{k=0}^3 \|d_\Omega(u^k)\|_p \le \sum_{k=0}^3 \|D_H(u^k)\|_p \approx \sum_{k=0}^3 \|\sqrt{-\Delta_H}(u^k)\|_p \\ &= \sum_{k=0}^3 \|(\sqrt{-\Delta_H}\,u)^k\|_p \approx \|\sqrt{-\Delta_H}\,u\|_p \approx \|D_H u\|_p\ . \end{aligned}$$

(The bound $\|d_\Omega(u^k)\|_p \le \|d_\Omega(u^k) + \delta_{\overline{\Omega}}(u^k)\|_p$ holds because $d_\Omega(u^k) \in L^p(\Omega,\Lambda^{k+1})$ and $\delta_{\overline{\Omega}}(u^k) \in L^p(\Omega,\Lambda^{k-1})$.) The idea for this result comes from [3, §5]. $\square$

15 L^p Hodge decomposition

It is a consequence of the interpolation properties of the spaces $\mathcal{R}^p(d_\Omega)$ and $\mathcal{R}^p(\delta_{\overline{\Omega}})$ (see Remark 20.2) that property (H_p) is stable in p in the following sense.

Theorem 15.1. *There exist Hodge exponents p_H, $p^H = p_H{}'$ with $1 \le p_H < 2 < p^H \le \infty$ such that the Hodge decomposition (H_p)*

$$L^p(\Omega,\Lambda) = \mathcal{R}^p(d_\Omega) \oplus \mathcal{R}^p(\delta_{\overline{\Omega}}) \oplus \mathcal{N}^p(D_H)$$

holds in the L^p norm if and only if $p_H < p < p^H$.

This is proved in [14, §4], following a similar proof in [11, §3.2].

It is well known that, when Ω has *smooth* boundary, then $p_H = 1$ and $p^H = \infty$. See, e.g., [18, Theorem 2.4.2 and 2.4.14] for the general case of smooth compact Riemannian manifolds with boundary.

If Ω is a *strongly Lipschitz* domain in $\mathbb{R}^3$, then $p_H < 3/2 < 3 < p^H$. See, e.g., [15, Theorem 1.1]. In [14] we reprove this result, with the new techniques having the advantage of providing a new result in higher dimensions, namely that $p_H < 2n/(n+1) < 2n/(n-1) < p^H$ when Ω is a bounded strongly Lipschitz domain in $\mathbb{R}^n$. In fact we show that D_H has a bounded holomorphic functional calculus in $L^p(\Omega,\Lambda)$ for some $p < 2n/(n+1)$ (and hence, by duality, in $L^{p'}(\Omega,\Lambda)$), and apply the Exercise in §13.

16 L^p results for D_H, Δ_H and S_H, $p_H < p < p^H$

In [14], we prove that for all p in the Hodge range, the Hodge-Dirac operator has a bounded holomorphic functional calculus. We do not include a proof here, but say a little more in §24.

Theorem 16.1. *Suppose that Ω is a very weakly Lipschitz domain in $\mathbb{R}^3$, and that $p_H < p < p^H$, i.e. (H_p) $L^p(\Omega,\Lambda) = \mathcal{R}^p(d_\Omega) \oplus \mathcal{R}^p(\delta_{\overline{\Omega}}) \oplus \mathcal{N}^p(D_H)$. Then*

(R_p) *The Hodge-Dirac operator D_H is bisectorial in $L^p(\Omega,\Lambda)$, in particular $\|(I+itD_H)^{-1}u\|_p \le C\|u\|_p \quad \forall\, t \in \mathbb{R}\setminus\{0\},\ \forall u \in L^p(\Omega,\Lambda)$;*

(F_p) *D_H has a bounded $H^\infty(S^o_\mu)$ functional calculus in $L^p(\Omega,\Lambda)$ for all $\mu > 0$, in particular, $\|D_H u\|_p \approx \|\sqrt{-\Delta_H}\,u\|_p$ for all $u \in \mathcal{D}^p(D_H) = \mathcal{D}^p(\sqrt{-\Delta_H})$.*

Corollary 16.2. *(i) The Hodge-Laplacian $-\Delta_H = {D_H}^2 = d_\Omega\delta_{\overline{\Omega}} + \delta_{\overline{\Omega}} d_\Omega$ is L^p sectorial with a bounded holomorphic functional calculus, in particular,*
$\|(I-t^2\Delta_H)^{-1}u\|_p \le C^2\|u\|_p \quad \forall\, t>0,\ \forall u \in L^p(\Omega,\Lambda)$.
(ii) The Hodge-Stokes operator $S_H = -\Delta_H|_{\mathcal{H}^p}$ is sectorial with a bounded holomorphic functional calculus in $\mathcal{H}^p := \{u \in L^p(\Omega,\Lambda^1)\,;\, \mathrm{div}_{\overline{\Omega}}\, u = 0\}$, in particular,
$\|(I+t^2S_H)^{-1}u\|_p \le C^2\|u\|_p \quad \forall\, t>0,\ \forall u \in \mathcal{H}^p$.

In the case of a bounded strongly Lipschitz domain, it was shown in [17] that $-\Delta_H$ and S_H are L^p sectorial for p in an open interval containing $[\frac{3}{2}, 3]$ in dimension 3. To our knowledge, the fact that they have a functional calculus is new, due to [14]. It was proved in [10] that for the same range of p the Riesz transforms $d_\Omega(-\Delta_H)^{-\frac{1}{2}}$ and $\delta_{\overline{\Omega}}(-\Delta_H)^{-\frac{1}{2}}$ are bounded in $L^p(\Omega,\Lambda)$, again in the case of a bounded strongly Lipschitz domain.

Again: If Ω is a *very weakly Lipschitz* domain in $\mathbb{R}^3$, and $p_H < p < p^H$, then D_H, Δ_H, S_H all have L^p resolvent bounds,

$$\begin{aligned}
\|(I+itD_H)^{-1}u\|_p &\le C\|u\|_p \quad \forall u \in L^p(\Omega,\Lambda),\ \forall\, t \in \mathbb{R}\setminus\{0\}\\
\|(I-t^2\Delta_H)^{-1}u\|_p &\le C^2\|u\|_p \quad \forall u \in L^p(\Omega,\Lambda),\ \forall\, t>0\\
\|(I+t^2S_H)^{-1}u\|_p &\le C^2\|u\|_p \quad \forall u \in \mathcal{H}^p,\ \forall\, t>0
\end{aligned}$$

and all have corresponding holomorphic functional calculi.

In fact, D_H can NOT have a functional calculus in $L^p(\Omega, \Lambda)$ for p outside the interval (p_H, p^H), as shown in the Exercise in §13.

But S_H CAN, and DOES, at least for $\max\{1, p_{HS}\} < p \leq p_H$ where p_{HS} is the Sobolev exponent below p_H i.e. $\frac{1}{p_{HS}} = \frac{1}{p_H} + \frac{1}{3}$.

Note: (i) Since $p_H < 2$, it is easily computed that $p_{HS} < 6/5$.

(ii) If Ω is *strongly Lipschitz*, then $p_H < 3/2$, and so $p_{HS} < 1$.

17 L^p result for Hodge-Stokes operator S_H, $p_{HS} < p < p^H$

Theorem 17.1. *Suppose Ω is a very weakly Lipschitz domain in $\mathbb{R}^3$, and $\max\{1, p_{HS}\} < p < p^H$. Then the Hodge-Stokes operator $S_H = -\Delta_H|_{\mathcal{H}^p}$ is sectorial with a bounded holomorphic functional calculus in $\mathcal{H}^p = \{u \in L^p(\Omega, \Lambda^1)\,;\, \mathrm{div}_{\overline{\Omega}}\, u = 0\}$. In particular,*

$$\|(I + t^2 S_H)^{-1} u\|_p \leq C^2 \|u\|_p \ , \quad \forall u \in \mathcal{H}^p, \ \forall t > 0 \ .$$

Corollary 17.2. *Suppose Ω is a strongly Lipschitz domain in $\mathbb{R}^3$, and $1 < p < p^H$. Then S_H is sectorial with a bounded holomorphic functional calculus in $\mathcal{H}^p$.*

These results are proved in [14]. Here we will not look further into functional calculi, but will indicate how to apply the fact that the Hodge-Dirac operator has L^q resolvent bounds when $p_H < q < p^H$, to derive L^p resolvent bounds for the Hodge-Stokes operator when $p_{HS} < p \leq p_H$.

The proofs depend on the theory of regularized Poincaré and Bogovskiĭ potential operators as developed in [16] and [7] for the case when Ω is starlike or strongly Lipschitz. Here we start with the special case of the unit ball $B \subset \mathbb{R}^3$, and then derive what we need for very weakly Lipschitz domains.

18 Potential operator on the unit ball

Let

- $B = B(0,1)$, the unit ball in $\mathbb{R}^3$, centred at the origin;
- $\theta \in C_c^\infty(\frac{1}{2}B, \mathbb{R})$ with $\int \theta = 1$;
- $R_B : L^p(B, \Lambda) \to W^{1,p}(B, \Lambda)$, the *regularized Poincaré potential operator* defined by $R_B u = \sum_{k=1}^3 R_B u^k$,

$$R_B u^k(x) = \int_B \theta(a)(x-a) \lrcorner \int_0^1 t^{k-1} u^k(a + t(x-a))\, dt\, da \qquad (k = 1, 2, 3) \ ,$$

$u = \sum_{k=0}^3 u^k \in L^p(B, \Lambda) = \oplus_{k=0}^3 L^p(B, \Lambda^k)$.

$$
\begin{array}{ccccccccc}
 & u^0 & & u^1 & & u^2 & & u^3 & \\
 & \in & & \in & & \in & & \in & \\
d_B : 0 \rightleftarrows & L^p(B,\mathbb{C}) & \underset{R_B}{\overset{\nabla_B}{\rightleftarrows}} & L^p(B,\mathbb{C}^3) & \underset{R_B}{\overset{\mathrm{curl}_B}{\rightleftarrows}} & L^p(B,\mathbb{C}^3) & \underset{R_B}{\overset{\mathrm{div}_B}{\rightleftarrows}} & L^p(B,\mathbb{C}) & \rightleftarrows 0
\end{array}
$$

Then $R_B : L^p(B,\Lambda) \to W^{1,p}(B,\Lambda)$ is bounded, $R_B : L^p(B,\Lambda) \to L^p(B,\Lambda)$ is compact, and

$$
d_B R_B u + R_B d_B u + \Big(\int \theta u^0\Big) 1 = u \qquad \forall\, u \in L^p(B,\Lambda)
$$

(where 1 denotes the constant function $1 \in L^p(\Omega,\Lambda^0)$). We write this as

$$
d_B R_B u + R_B d_B u + K_B u = u
$$

where $K_B u = (\int \theta u^0)1$ and note that $K_B : L^p(B,\Lambda) \to L^\infty(B,\Lambda^0)$ is bounded, and $K_B : L^p(B,\Lambda) \to L^p(B,\Lambda^0)$ is compact. The operator K_B compensates for the fact that the above sequence for d_B misses out on being exact, due to the gradient map ∇_B having a one dimensional null-space consisting of constant functions in $L^p(B,\Lambda^0)$.

Moreover, if $1 < p = q_S < q < \infty$, where $p = q_S$ is the Sobolev exponent below q, i.e.

$$
\tfrac{1}{p} = \tfrac{1}{q} + \tfrac{1}{3}
$$

then the potential map $R_B : L^p(B,\Lambda) \to L^q(B,\Lambda)$ is bounded.

19 Potential operator on bilipschitz transformation of the unit ball

Suppose $\rho : B \to \rho B \subset \mathbb{R}^3$ is a uniformly locally bilipschitz transformation. Then the pull-back $\rho^* : L^p(\rho B,\Lambda) \to L^p(B,\Lambda)$ is bounded, and

$$
d_{\rho B} = (\rho^*)^{-1} d_B \rho^* ;
$$

recall that $(\rho^* u)(x) = (\underline{\rho}_x)^* u(\rho(x))$ where $\underline{\rho}_x$ is the Jacobian matrix of ρ at x.

Define $R_{\rho B} : L^p(\rho B,\Lambda) \to L^q(\rho B,\Lambda)$ and $K_{\rho B} : L^p(\rho B,\Lambda) \to L^\infty(\rho B,\Lambda)$ by

$$
R_{\rho B} = (\rho^*)^{-1} R_B \rho^* \qquad \text{and} \qquad K_{\rho B} = (\rho^*)^{-1} K_B \rho^*
$$

so that

$$
d_{\rho B} R_{\rho B} u + R_{\rho B} d_{\rho B} u + K_{\rho B} u = u \ .
$$

$$
d_{\rho B} : 0 \rightleftarrows L^p(\rho B,\mathbb{C}) \underset{R_{\rho B}}{\overset{\nabla_{\rho B}}{\rightleftarrows}} L^p(\rho B,\mathbb{C}^3) \underset{R_{\rho B}}{\overset{\mathrm{curl}_{\rho B}}{\rightleftarrows}} L^p(\rho B,\mathbb{C}^3) \underset{R_{\rho B}}{\overset{\mathrm{div}_{\rho B}}{\rightleftarrows}} L^p(\rho B,\mathbb{C}) \rightleftarrows 0
$$

The operators $R_{\rho B}$ and $K_{\rho B}$ have the same boundedness and compactness properties as R_B and K_B.

20 Potential operators on *very weakly Lipschitz* domains

- $1 < p < q < \infty \qquad (\frac{1}{p} = \frac{1}{q} + \frac{1}{3})$.
- Ω is *very weakly Lipschitz*, i.e. $\Omega = \cup_{j=1}^{N}(\rho_j B)$ where each $\rho_j : B \to \rho_j B \subset \mathbb{R}^3$ is uniformly locally bilipschitz, and
- $1 = \sum_{j=1}^{N} \chi_j$ on Ω, where each $\chi_j : \Omega \to [0,1]$ is a Lipschitz function with $\mathrm{sppt}_\Omega(\chi_j) \subset \rho_j B$.
- Define $R_\Omega = \sum_{j=1}^{N} \chi_j R_{\rho_j B}$ and $K_\Omega u = \sum_{j=1}^{N}(\chi_j K_{\rho_j B} u - (\nabla\chi_j) \wedge R_{\rho_j B} u)$.

$$\begin{array}{ccccccccccc} & & u^0 & & u^1 & & u^2 & & u^3 & & \\ & & \in & & \in & & \in & & \in & & \\ d_\Omega : 0 & \rightleftarrows & L^p(\Omega,\mathbb{C}) & \underset{R_\Omega}{\overset{\nabla_\Omega}{\rightleftarrows}} & L^p(\Omega,\mathbb{C}^3) & \underset{R_\Omega}{\overset{\mathrm{curl}_\Omega}{\rightleftarrows}} & L^p(\Omega,\mathbb{C}^3) & \underset{R_\Omega}{\overset{\mathrm{div}_\Omega}{\rightleftarrows}} & L^p(\Omega,\mathbb{C}) & \rightleftarrows & 0 \end{array}$$

It is straightforward to apply the properties mentioned in the previous two sections to prove the following result.

Theorem 20.1. *The exterior derivative* d_Ω *has a potential map* $R_\Omega : L^p(\Omega,\Lambda) \to L^q(\Omega,\Lambda)$ *satisfying*

$$d_\Omega R_\Omega u + R_\Omega d_\Omega u + K_\Omega u = u \qquad \forall u \in L^p(\Omega,\Lambda) ,$$

where $K_\Omega : L^p(\Omega,\Lambda) \to L^q(\Omega,\Lambda)$. *Moreover* K_Ω *and* R_Ω *are compact operators in* $L^p(\Omega,\Lambda)$.

Remark 20.2. *Although we will not use this fact in the coming sections, we remark than* R_Ω *can be modified in such a way that* $d_\Omega R_\Omega u = u$ *for all* $u \in \mathcal{R}^p(d_\Omega)$.

Using this modification, we have that $d_\Omega R_\Omega : L^p(\Omega,\Lambda) \to \mathcal{R}^p(d_\Omega)$ *is a bounded projection for all* p, $1 < p < \infty$, *and as a corollary, the spaces* $\mathcal{R}^p(d_\Omega)$ $(1 < p < \infty)$ *are closed subspaces of* $L^p(\Omega,\Lambda)$ *which interpolate by the complex method.*

In this case, R_Ω is a true potential operator. For example, if u^1 is a gradient vector field, then $w_0 = R_\Omega u^1 \in L^q(\Omega,\mathbb{C})$ is its potential, because $\nabla_\Omega w_0 = d_\Omega R_\Omega u^1 = u^1$.

Remark 20.3. *With a modified* R_Ω *as in Remark 20.2, define* $\mathcal{Z}^p = K_\Omega(\mathcal{N}^p(d_\Omega))$. *Then* $\mathcal{N}^p(d_\Omega) = \mathcal{R}^p(d_\Omega) \oplus \mathcal{Z}^p$ *with decomposition* $u = d_\Omega R_\Omega u + K_\Omega u$ *for all* $u \in \mathcal{N}^p(d_\Omega)$. *So the spaces in the decomposition are closed, and* $\mathcal{Z}^p$ *is finite dimensional, on account of the compactness of* K_Ω. *Thus* $\mathcal{R}^p(d_\Omega)$ *has finite codimension in* $\mathcal{N}^p(d_\Omega)$, *as claimed in Section 6.*

In the following section $T_{\overline{\Omega}}$ could be similarly modified to give $u = \delta_{\overline{\Omega}} T_{\overline{\Omega}} u$ for all $u \in \mathcal{R}^p(\delta_{\overline{\Omega}})$.

21 Dual potential operators

- $1 < p < q < \infty \qquad (\frac{1}{p} = \frac{1}{q} + \frac{1}{3})$;
- $T_{\overline{\Omega}} : L^p(\Omega, \Lambda) \to L^q(\Omega, \Lambda)$ is dual to $R_\Omega : L^{q'}(\Omega, \Lambda) \to L^{p'}(\Omega, \Lambda)$;
- $L_{\overline{\Omega}} : L^p(\Omega, \Lambda) \to L^q(\Omega, \Lambda)$ is dual to $K_\Omega : L^{q'}(\Omega, \Lambda) \to L^{p'}(\Omega, \Lambda)$.

Then, dual to the equation $d_\Omega R_\Omega u + R_\Omega d_\Omega u + K_\Omega u = u$, is :

$$\boxed{u = \delta_{\overline{\Omega}} T_{\overline{\Omega}} u + T_{\overline{\Omega}} \delta_{\overline{\Omega}} u + L_{\overline{\Omega}} u}$$

so that $T_{\overline{\Omega}}$ is a potential operator for $\delta_{\overline{\Omega}}$, called the Bogovskiĭ operator :

$$0 \leftrightarrows L^p(\Omega, \mathbb{C}) \underset{T_{\overline{\Omega}}}{\overset{-\nabla_{\overline{\Omega}}}{\leftrightarrows}} L^p(\Omega, \mathbb{C}^3) \underset{T_{\overline{\Omega}}}{\overset{\mathrm{curl}_{\overline{\Omega}}}{\leftrightarrows}} L^p(\Omega, \mathbb{C}^3) \underset{T_{\overline{\Omega}}}{\overset{-\mathrm{div}_{\overline{\Omega}}}{\leftrightarrows}} L^p(\Omega, \mathbb{C}) \leftrightarrows 0 : \delta_{\overline{\Omega}}$$

22 L^p results for Δ_H on $\mathcal{N}^p(\delta_{\overline{\Omega}})$, $p_{H_S} < p < p^H$

Suppose that Ω is a *very weakly Lipschitz* domain. We have stated in Theorem 16.1 that when $p_H < q < p^H$, the Hodge-Dirac operator $D_H = d_\Omega + \delta_{\overline{\Omega}}$ is bisectorial with a bounded holomorphic funtional calculus in $L^q(\Omega, \Lambda)$. Our aim now is to extend this result as follows.

Theorem 22.1. *Suppose that*

- $p_H < q < p^H$;
- $\max\{1, q_S\} \le p \le q$ *where q_S is the lower Sobolev exponent of q, i.e.* $\frac{1}{q_S} = \frac{1}{q} + \frac{1}{3}$.

Then the Hodge-Laplacian $-\Delta_H$ is sectorial with a bounded holomorphic functional calculus in $\mathcal{N}^p(\delta_{\overline{\Omega}}) = \{u \in L^p(\Omega, \Lambda) \,;\, \delta_{\overline{\Omega}} u = 0\}$. In particular,

$$\|(I - t^2 \Delta_H)^{-1} u\|_p \le C^2 \|u\|_p \ , \quad \forall u \in \mathcal{N}^p(\delta_{\overline{\Omega}}),\ \forall t > 0 \ . \tag{1}$$

Similar resolvent bounds also holds on $\mathcal{N}(d_\Omega)$ and hence on $\mathcal{R}(\delta_{\overline{\Omega}})$ and on $\mathcal{R}(d_\Omega)$.

On restricting to $L^p(\Omega, \Lambda^1)$, we obtain Theorem 17.1 as a corollary.

For the results on functional calculi, we refer the reader to [14]. We do not fully prove the resolvent bounds either, but give the spirit of the method by outlining the estimates in the case when $p = q_S$.

23 L^p resolvent bounds for Δ_H on $\mathcal{N}^p(\delta_{\overline{\Omega}})$, $p = q_S$, $p_H < q < p^H$

- Ω is *very weakly Lipschitz* and $p_H < q < p^H$, $p = q_S > 1$.

- The idea is to modify the techniques of Blunck-Kunstmann [6], but there is still quite a bit to do, because we are working on the subspace $\mathcal{N}^p(\delta_{\overline{\Omega}})$. We will not consider the functional calculus here, but will outline a proof of resolvent bounds.

- The easy part: When $t \geq 1$, and $\delta_{\overline{\Omega}} u = 0$, then

$$\begin{aligned}
\|(I - t^2\Delta_H)^{-1}u\|_p &\lesssim \|(I - t^2\Delta_H)^{-1}u\|_q \qquad \text{(because } \Omega \text{ is bounded)}\\
&= \|(I - t^2\Delta_H)^{-1}(\delta_{\overline{\Omega}} T_{\overline{\Omega}} + L_{\overline{\Omega}})u\|_q\\
&\leq t\|\delta_{\overline{\Omega}}(I - t^2\Delta_H)^{-1}T_{\overline{\Omega}} u\|_q + \|(I - t^2\Delta_H)^{-1}L_{\overline{\Omega}} u\|_q\\
&\lesssim \|tD_H(I + t^2{D_H}^2)^{-1}T_{\overline{\Omega}} u\|_q + \|(I + t^2{D_H}^2)^{-1}L_{\overline{\Omega}} u\|_q\\
&\lesssim \|T_{\overline{\Omega}} u\|_q + \|L_{\overline{\Omega}} u\|_q\\
&\lesssim \|u\|_p
\end{aligned}$$

  (using Hodge decomposition in $L^q(\Omega, \Lambda)$ in line 4, and resolvent bounds for D_H in $L^q(\Omega, \Lambda)$ in line 5).

- Henceforth take $0 < t < 1$.

- Cover Ω: Let $\underline{Q}_j^t$ $(j \in J)$ be the cubes in $\mathbb{R}^3$ with side-length t and corners at points in $t\mathbb{Z}^3$, which intersect Ω. Let $Q_j^t = 4\underline{Q}_j^t \cap \Omega$. Then $\Omega = \cup Q_j^t$. Write $1 = \sum_{j\in J} \eta_j^2$ on Ω, where $\eta_j \in C_c^1(4\underline{Q}_j^t, [0, 1])$ and $\|\nabla\eta_j\|_\infty \leq 1/t$. The "cubes" Q_j^t have finite overlap, in fact $\sum_{j\in J} 1_{Q_j^t} \leq 64$. (Here $1_{Q_j^t}$ denotes the function with value 1 on Q_j^t and zero elsewhere on Q.)

- L^q *off-diagonal bounds* in $\operatorname{dist}(Q_j^t, Q_k^t) = \inf\{|x - y|; x \in Q_j^t, y \in Q_k^t\}$ are a consequence of the L^q resolvent bounds. See [14, §5], or adapt the L^2 proofs in [4]. We need the following two bounds.

  For each $N \in \mathbb{N}$, there exists C_N such that, when $\operatorname{sppt}(f) \in Q_k^t$, then

$$\|1_{Q_j^t}(I - t^2\Delta_H)^{-1}f\|_q \leq C_N \left(\tfrac{t}{t+\operatorname{dist}(Q_j^t,Q_k^t)}\right)^N \|f\|_q \qquad \text{and}$$

$$t\|1_{Q_j^t}(I - t^2\Delta_H)^{-1}\delta_{\overline{\Omega}} f\|_q \leq C_N \left(\tfrac{t}{t+\operatorname{dist}(Q_j^t,Q_k^t)}\right)^N \|f\|_q \,.$$

- Decompose $u \in \mathcal{N}^p(\delta_{\overline{\Omega}})$ (using $\delta_{\overline{\Omega}}(\eta_k f) - \eta_k \delta_{\overline{\Omega}} f = (\nabla \eta_k) \lrcorner f$):

$$
\begin{aligned}
u &= \sum_{k\in J} {\eta_k}^2 u = \sum_{k\in J} \eta_k \, I \, \eta_k u \\
&= \sum_{k\in J} (\eta_k \, \delta_{\overline{\Omega}} T_{\overline{\Omega}} \, \eta_k u + \eta_k \, T_{\overline{\Omega}} \, \delta_{\overline{\Omega}} \, \eta_k u + \eta_k \, L_{\overline{\Omega}} \, \eta_k u) \\
&= \sum_{k\in J} (\delta_{\overline{\Omega}}(\eta_k T_{\overline{\Omega}} \, \eta_k u) - (\nabla \eta_k) \lrcorner T_{\overline{\Omega}} \, \eta_k u + \eta_k T_{\overline{\Omega}} (\nabla \eta_k) \lrcorner u + \eta_k L_{\overline{\Omega}} \, \eta_k u) \\
&= \sum_{k\in J} (\delta_{\overline{\Omega}} \, w_k + \tfrac{1}{t} v_k) \qquad \text{where}
\end{aligned}
$$

$$
\begin{aligned}
w_k &= \eta_k T_{\overline{\Omega}} \, \eta_k u \qquad \text{and} \\
v_k &= -(t\nabla \eta_k) \lrcorner T_{\overline{\Omega}} \, \eta_k u + \eta_k T_{\overline{\Omega}} (t \nabla \eta_k) \lrcorner u + t \, \eta_k L_{\overline{\Omega}} \, \eta_k u \ .
\end{aligned}
$$

- On using the $L^p - L^q$ bounds on $T_{\overline{\Omega}}$ and $L_{\overline{\Omega}}$, we obtain

$$
\begin{aligned}
\|w_k\|_q &\lesssim \|\eta_k u\|_p \lesssim \|1_{Q_k^t} u\|_p \quad \text{with } \operatorname{sppt}(w_k) \subset Q_k^t \qquad \text{and} \\
\|v_k\|_q &\lesssim (1+t) \|1_{Q_k^t} u\|_p \lesssim \|1_{Q_k^t} u\|_p \quad \text{with } \operatorname{sppt}(v_k) \subset Q_k^t \ .
\end{aligned}
$$

Here now is the resolvent estimate. Suppose $\delta_{\overline{\Omega}}\, u = 0$. Then

$$\begin{aligned}
\|(I-t^2\Delta_H)^{-1}u\|_p &\leq \Big[\sum_{j\in J}\int_{Q_j^t}|(I-t^2\Delta_H)^{-1}u|^p\Big]^{\frac{1}{p}} \\
&= \Big[\sum_{j\in J}(\|1_{Q_j^t}(I-t^2\Delta_H)^{-1}u\|_p)^p\Big]^{\frac{1}{p}} \\
&\leq \Big[\sum_{j\in J}(\|1_{Q_j^t}(I-t^2\Delta_H)^{-1}u\|_q|Q_j^t|^{\frac{1}{3}})^p\Big]^{\frac{1}{p}} \qquad (\tfrac{1}{p}=\tfrac{1}{q}+\tfrac{1}{3}) \\
&\lesssim \Big[\sum_{j\in J}(\sum_{k\in J}\|1_{Q_j^t}(I-t^2\Delta_H)^{-1}(\delta_{\overline{\Omega}}\, w_k+\tfrac{1}{t}v_k)\|_q t)^p\Big]^{\frac{1}{p}} \\
&\lesssim \Big[\sum_{j\in J}(\sum_{k\in J}\big(\tfrac{t}{t+\mathrm{dist}(Q_j^t,Q_k^t)}\big)^4(\|w_k\|_q+\|v_k\|_q))^p\Big]^{\frac{1}{p}} \qquad (*) \\
&\lesssim \Big[\sum_{j\in J}(\sum_{k\in J}\big(\tfrac{t}{t+\mathrm{dist}(Q_j^t,Q_k^t)}\big)^4\|1_{Q_k^t}u\|_p)^p\Big]^{\frac{1}{p}} \\
&\lesssim \Big(\sup_j\sum_{k\in J}\big(\tfrac{t}{t+\mathrm{dist}(Q_j^t,Q_k^t)}\big)^4\Big)\Big[\sum_{k\in J}\|1_{Q_k^t}u\|_p{}^p\Big]^{\frac{1}{p}} \qquad (**) \\
&\lesssim \Big[\sum_{k\in J}\|1_{Q_k^t}u\|_p{}^p\Big]^{\frac{1}{p}} = \Big[\sum_{k\in J}\int_{Q_k^t}|u|^p\Big]^{\frac{1}{p}} \qquad (***) \\
&= \Big(\int_\Omega\sum_{k\in J}1_{Q_k^t}|u|^p\Big)^{\frac{1}{p}} \lesssim \|u\|_p \qquad (****)
\end{aligned}$$

as claimed.

- In (*) we used the off-diagonal bounds with $N=4$;
- In (**) we used the *Schur estimate* in $\ell^p(J)$, with $A_{j,k}=\big(\frac{t}{t+\mathrm{dist}(Q_j^t,Q_k^t)}\big)^4$ and $\beta_k=\|1_{Q_k^t}u\|_p$:

$$\begin{aligned}
\big[\sum_j|\sum_k A_{j,k}\beta_k|^p\big]^{\frac{1}{p}} &\leq \big(\sup_j\sum_k|A_{j,k}|\big)^{\frac{1}{p'}}\big(\sup_k\sum_j|A_{j,k}|\big)^{\frac{1}{p}}\big(\sum_k|\beta_k|^p\big)^{\frac{1}{p}} \\
&= \big(\sup_j\sum_k|A_{j,k}|\big)\big(\sum_k|\beta_k|^p\big)^{\frac{1}{p}} \quad \text{when } A_{j,k}=A_{k,j}\ ;
\end{aligned}$$

- In (***) we used that, given Q_j^t,

$$\sum_k \left(\tfrac{t}{t+\mathrm{dist}(Q_j^t,Q_k^t)}\right)^4 \lesssim C_0 + \sum_{M=0}^{\infty} \sum_{\{k;2^M t \le \mathrm{dist}(Q_j^t,Q_k^t)<2^{(M+1)}t\}} \tfrac{1}{2^{4M}}$$
$$\lesssim C_0 + \sum_{M=0}^{\infty} 2^{3M} \tfrac{1}{2^{4M}} = C_0 + \sum_{M=0}^{\infty} \tfrac{1}{2^M} \le C \ ;$$

- In (****) we used the finite overlap of the cubes.

This completes the proof of (1) in the case when $p = q_S$. The proof of L^p sectoriality when $q_S \le p < q$ requires minor modification. To show that S_H has a bounded holomorphic functional calculus requires further work, using a Calderón–Zygmund decomposition of Ω. For this, the reader is referred to [14].

24 Remarks on obtaining resolvent bounds in the Hodge range

In the previous section we applied Theorem 16.1. But suppose we just start with the L^2 resolvent bounds. Then a similar procedure to that described above, can be used to obtain resolvent bounds for D_H on $\mathcal{N}^p(\delta_{\overline{\Omega}})$ when $6/5 = 2_S \le p \le 2$ Moreover, use of the potential operators R_Ω will lead to resolvent bounds on $\mathcal{N}^p(d_\Omega)$, also when $6/5 \le p \le 2$. Now, if p is also in the Hodge range, we then obtain resolvent bounds on all of $L^p(\Omega,\Lambda)$, i.e. we obtain resolvent bounds for D_H on $L^p(\Omega,\Lambda)$ when $\max\{6/5, p_H\} < p \le 2$. Repeating this procedure once more if necessary, we obtain resolvent bounds on $L^p(\Omega,\Lambda)$ for $p_H < p \le 2$ (as $(6/5)_S < 1$). A duality argument then gives resolvent bounds when $2 \le p < p^H$. In this way, the statement (R_p) can be proved when $p_H < p < p^H$, as stated in Theorem 16.1. See [14] for details.

We remark that such an iteration method has been used previously in [9] in the study of more general first order systems on $\mathbb{R}^n$. A similar iteration procedure has been used also in [17] and [10].

25 Parabolic equations

As mentioned in §14, operators with a bounded holomorphic functional calculus on a closed subspace $\mathcal{X}^p$ of $L^p(\Omega,\Lambda)$, also satisfy maximal regularity. So, on taking $\mathcal{X}^p = \mathcal{H}^p$, we obtain:

Theorem 25.1. *Suppose that Ω is a very weakly Lipschitz domain in $\mathbb{R}^3$, that $\max\{1, p_{HS}\} < p < p^H$, and that $1 < q < \infty$, $0 < T \le \infty$. Suppose also that*

$$\partial_t F(t,\,.\,) + S_H F(t,\,.\,) = h(t,\,.\,) \in L^q((0,T);\mathcal{H}^p)\ , t > 0$$
$$F(0,\,.\,) = 0\ .$$

Then

$$\left\{\int_0^T \|F(t,\,.\,)\|_p{}^q\,dt\right\}^{1/q} + \left\{\int_0^T \|SF(t,\,.\,)\|_p{}^q\,dt\right\}^{1/q} \leq C_{p,q}\left\{\int_0^T \|h(t,\,.\,)\|_p{}^q\,dt\right\}^{1/q}.$$

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