Convergence of eigenvalues

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It is a folklore theorem that uniform resolvent convergence of unbounded positive self-adjoint operators with compact resolvents implies that the successive eigenvalues converge. The aim of this note is to give a three line proof that is based on the max-min theorem.

Let H be a Hilbert space and B a positive self-adjoint compact operator with infinite spectrum. We denote the non-zero eigenvalues by $\mu_1 \geq \mu_2 \geq \ldots$, repeated with multiplicity. Then the max-min theorem of Courant gives

$$\mu_k = \max_{\substack{W \subset H \\ \dim W = k}} \min_{\substack{x \in W \\ \|x\|_H = 1}} (Bx, x)_H. \tag{1}$$

This follows easily from the spectral theorem. See [Bré] Problem 37.4. The alluded folklore theorem is as follows.

Theorem. *Let H be a Hilbert space and $A_\infty, A_1, A_2, \ldots$ be unbounded positive self-adjoint operators with compact resolvents. Suppose that $\lim_{n\to\infty}(I + A_n)^{-1} = (I + A_\infty)^{-1}$ in $\mathcal{L}(H)$. For all $n \in \mathbb{N} \cup \{\infty\}$ let $\lambda_1^{(n)} \leq \lambda_2^{(n)} \leq \ldots$ be the eigenvalues of A_n, repeated with multiplicity. Let $k \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \lambda_k^{(n)} = \lambda_k^{(\infty)}.$$

Proof. Let $\varepsilon > 0$. By the resolvent convergence there exists an $N \in \mathbb{N}$ such that

$$((I + A_\infty)^{-1}x, x)_H - \varepsilon \, \|x\|_H^2 \leq ((I + A_n)^{-1}x, x)_H \leq ((I + A_\infty)^{-1}x, x)_H + \varepsilon \, \|x\|_H^2$$

for all $x \in H$ and $n \in \mathbb{N}$ with $n \geq N$. Hence $\frac{1}{1+\lambda_k^{(\infty)}} - \varepsilon \leq \frac{1}{1+\lambda_k^{(n)}} \leq \frac{1}{1+\lambda_k^{(\infty)}} + \varepsilon$ by (1). $\square$

Remark. The above theorem and proof is also valid for self-adjoint graphs.

Reference

[Bré] Brézis, H., *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

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A much longer proof is as follows.

Proof. Let $\varepsilon > 0$. By the resolvent convergence there exists an $N \in \mathbb{N}$ such that

$$((I + A_\infty)^{-1}x, x)_H - \varepsilon \|x\|_H^2 \leq ((I + A_n)^{-1}x, x)_H \leq ((I + A_\infty)^{-1}x, x)_H + \varepsilon \|x\|_H^2$$

for all $x \in H$ and $n \in \mathbb{N}$ with $n \geq N$. The max-min theorem applied to $(I+A_n)^{-1}$ and $(I+A_\infty)^{-1}$ gives

$$\frac{1}{1 + \lambda_k^{(\infty)}} - \varepsilon \leq \frac{1}{1 + \lambda_k^{(n)}} \leq \frac{1}{1 + \lambda_k^{(\infty)}} + \varepsilon$$

for all $n \in \mathbb{N}$ with $n \geq N$. The theorem follows. □