Stokes Problems in Irregular Domains with Various Boundary Conditions

Sylvie Monniaux and Zhongwei Shen

Contents

S. Monniaux (✉)
Aix-Marseille Université, CNRS, Centrale Marseille, I2M, Marseille, France
e-mail: sylvie.monniaux@univ-amu.fr

Z. Shen
University of Kentucky, Lexington, KY, USA
e-mail: zshen2@uky.edu

Y. Giga, A. Novotny (eds.), *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids*, DOI 10.1007/978-3-319-10151-4_4-1

Abstract

Different boundary conditions for the Navier-Stokes equations in bounded Lipschitz domains in $\mathbb{R}^3$, such as Dirichlet, Neumann, Hodge, or Robin boundary conditions, are presented here. The situation is a little different from the case of smooth domains. The analysis of the problem involves a good comprehension of the behavior near the boundary. The linear Stokes operator associated to the various boundary conditions is first studied. Then a classical fixed-point theorem is used to show how the properties of the operator lead to local solutions or global solutions for small initial data.

1 Introduction

The aim of this chapter is to describe how to find solutions of the Navier-Stokes equations

$$\begin{cases} \partial_t u - \Delta u + \nabla \pi + (u \cdot \nabla) u = 0 \ \text{ in } (0,T) \times \Omega, \\ \operatorname{div} u = 0 \ \text{ in } (0,T) \times \Omega, \\ u(0) = u_0 \text{ in } \Omega, \end{cases} \tag{NS}$$

in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ and a time interval $(0,T)$ ($T \leq \infty$), for initial data u_0 in a critical space, with one of the following boundary conditions on $\partial\Omega$:

1. Dirichlet boundary conditions:

$$u = 0, \tag{Dbc}$$

   also called "no-slip" boundary conditions, which can be also decomposed as a nonpenetration condition $\nu \cdot u = 0$ and a tangential part $\nu \times u = 0$ which model the fact that the fluid does not slip at the boundary; this is commonly used for a boundary between a fluid and a rigid surface;
2. Neumann boundary conditions:

$$[\lambda(\nabla u) + (\nabla u)^\top]\nu - \pi\nu = 0, \quad \lambda \in (-1, 1], \tag{Nbc}$$

   which can be rewritten as $T_\lambda(u,\pi)\nu = 0$ where $T_\lambda(u,\pi) := \lambda(\nabla u) + (\nabla u)^\top - \pi\,\mathrm{Id}$; if $\lambda = 0$, (Nbc) becomes $\partial_\nu u = \pi\nu$; if $\lambda = 1$, $T_1(u,\pi)$ is the Cauchy's stress tensor so that (Nbc) can be viewed, for instance, as an absence of stress on the interface separating two media in the case of a free boundary; (Nbc) can be decomposed into its normal and tangential parts and can be rewritten in the following form:

$$(1+\lambda)\,\nu \cdot \partial_\nu u = \pi, \quad \big[\big(\lambda(\nabla u) + (\nabla u)^\top\big)\nu\big]_{\tan} = 0; \tag{1}$$

3. Hodge boundary conditions:

$$\nu \cdot u = 0, \quad \nu \times \operatorname{curl} u = 0, \tag{Hbc}$$

also called "absolute" boundary conditions (see [53, Section 9] or "perfect wall" condition (see [1]); they have been studied in, e.g., [4] and [24]; they are related to the more traditionally used "Navier's slip" boundary condition:

$$\nu \cdot u = 0, \quad \big[(\nabla u)^\top + (\nabla u)\nu\big]_{\tan} = 0. \tag{2}$$

See discussion below (see also a detailed discussion in [35, Section 2]).

4. Robin boundary conditions:

$$\nu \cdot u = 0, \quad \nu \times \operatorname{curl} u = \alpha\, u, \quad \alpha > 0; \tag{Rbc}$$

since $\nu \cdot u = 0$, u is a tangential vector field at the boundary, so it makes sense to compare it to the tangential part of the vorticity: it describes the fact that the fluid slips with a friction proportional to the vorticity. Remark that (Hbc) is recovered if $\alpha = 0$ and (Dbc) if $\alpha = \infty$.

In the boundary conditions above, $\nu(x)$ denotes the unit exterior normal vector at a point $x \in \partial\Omega$ (defined almost everywhere when $\partial\Omega$ is a Lipschitz boundary).

As explained in [35, Section 2 and Section 6], the Hodge boundary conditions (Hbc) are close to the Navier's slip boundary conditions (2). Indeed, if Ω is assumed to be smooth enough, say of class $\mathscr{C}^2$, under the condition $\nu \cdot u = 0$, the following holds:

$$\big[(\nabla u)^\top + (\nabla u)\nu\big]_{\tan} = -\nu \times \operatorname{curl} u + 2\,\mathcal{W} u$$

where $\mathcal{W}$ is the Weingarten map (also called the shape operator, see [45, Chapter 5]) on $\partial\Omega$ acting on tangential fields (see also [17, Section 3]). In particular, the term $\mathcal{W}u$ is a zero-order term, depending linearly on the velocity field u and is equal to 0 on flat portions of the boundary.

The strategy in this chapter to solve the Navier-Stokes equations with one of the boundary conditions described above is to find a functional setting in which the Fujita-Kato scheme applies, such as in their fundamental paper [20]. In all situations, the idea is to study the linear problem to prove enough regularizing properties of the Stokes semigroup so that the nonlinear problem can be treated via a fixed-point method. For the last two types of boundary conditions (Hbc) and (Rbc), the Navier-Stokes system is rewritten as follows:

$$\begin{cases} \partial_t u - \Delta u + \nabla \pi - u \times \operatorname{curl} u = 0 \;\text{ in } (0,T)\times\Omega, \\ \operatorname{div} u = 0 \;\text{ in } (0,T)\times\Omega, \\ u(0) = u_0 \text{ in } \Omega. \end{cases} \tag{NS'}$$

This is motivated by the form of the boundary conditions and the fact that, for a smooth enough vector field u,

$$(u \cdot \nabla)u = \tfrac{1}{2}\nabla|u|^2 - u \times \operatorname{curl} u,$$

so that (NS) becomes (NS') with the pressure π replaced by the so-called dynamical pressure $\pi + \frac{1}{2}|u|^2$ (see, e.g., [24] or [4]).

In this chapter, $\Omega \subset \mathbb{R}^3$ is a bounded, simply connected, Lipschitz domain. The chapter is organized as follows. In Sect. 2, the Dirichlet-Stokes operator is defined in the L^2 setting and then in the L^p theory. Existence of a local solution of the system $\{$(NS), (Dbc)$\}$ for initial values in a critical space in the L^2-Stokes scale is then shown. In Sect. 3, the previous proofs are adapted in the case of Neumann boundary conditions, i.e., for the system $\{$(NS), (Nbc)$\}$. In Sect. 4, the system $\{$(NS'), (Hbc)$\}$ is studied for initial conditions in the critical space $\{u \in L^3(\Omega;\mathbb{R}^3); \operatorname{div} u = 0 \text{ in } \Omega, \nu \cdot u = 0 \text{ on } \partial\Omega\}$, whereas in Sect. 5, the system $\{$(NS'), (Rbc)$\}$ is considered in a $\mathscr{C}^1$ domain.

2 Dirichlet Boundary Conditions

For a more complete exposition of the results in this section, as well as an extension to more general domains, the reader can refer to [34, 41] and [51]. The case where Ω is smooth was solved by Fujita and Kato in [20]. In [15], the case of bounded Lipschitz domains Ω was studied for initial data not in a critical space.

2.1 The Linear Dirichlet-Stokes Operator

2.1.1 The L^2 Theory

The following remarks about L^2 vector fields on Ω will be used throughout this chapter.

Remark 1. For $\Omega \subset \mathbb{R}^3$ a bounded Lipschitz domain, let $u \in L^2(\Omega;\mathbb{R}^3)$ such that $\operatorname{div} u \in L^2(\Omega;\mathbb{R})$. Then $\nu \cdot u$ can be defined on $\partial\Omega$ in the following weak sense in $H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R})$: for $\phi \in H^1(\Omega;\mathbb{R})$,

$$\langle u, \nabla\phi\rangle_\Omega + \langle \operatorname{div} u, \phi\rangle_\Omega = \langle \nu \cdot u, \varphi\rangle_{\partial\Omega}, \qquad (3)$$

where $\varphi = \operatorname{Tr}_{|\partial\Omega}\phi$, the right-hand side of (3) depends only on φ on $\partial\Omega$ and not on the choice of ϕ, its extension to Ω. The notation $\langle\cdot,\cdot\rangle_E$ stands for the L^2-scalar product on E.

The following Hodge decomposition holds on vector fields: $L^2(\Omega;\mathbb{R}^3)$ is equal to the orthogonal direct sum $H_D \overset{\perp}{\oplus} G$, where

$$H_D = \{u \in L^2(\Omega;\mathbb{R}^3); \operatorname{div} u = 0 \text{ in } \Omega, \nu \cdot u = 0 \text{ on } \partial\Omega\} \qquad (4)$$

and $G = \nabla H^1(\Omega;\mathbb{R})$. This follows from the following theorem due to Georges de Rham [12, Chap. IV §22, Theorem 17']; see also [55, Chap.I §1.4, Proposition 1.1].

Theorem 1 (de Rham). *Let T be a distribution in $\mathscr{C}_c^\infty(\Omega;\mathbb{R}^3)'$ such that $\langle T,\phi\rangle = 0$ for all $\phi \in \mathscr{C}_c^\infty(\Omega;\mathbb{R}^3)$ with $\operatorname{div}\phi = 0$ in Ω. Then there exists a distribution $S \in \mathscr{C}_c^\infty(\Omega;\mathbb{R})'$ such that $T = \nabla S$. Conversely, if $T = \nabla S$ with $S \in \mathscr{C}_c^\infty(\Omega;\mathbb{R})'$, then $\langle T,\phi\rangle = 0$ for all $\phi \in \mathscr{C}_c^\infty(\Omega;\mathbb{R}^3)$ with $\operatorname{div}\phi = 0$ in Ω.*

Remark 2. In the case of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, the space H_D coincides with the closure in $L^2(\Omega;\mathbb{R}^3)$ of the space of vector fields $u \in \mathscr{C}_c^\infty(\Omega;\mathbb{R}^3)$ with $\operatorname{div} u = 0$ in Ω.

Denote by $J : H_D \hookrightarrow L^2(\Omega;\mathbb{R}^3)$ the canonical embedding and $\mathbb{P} : L^2(\Omega;\mathbb{R}^3) \to H_D$ the orthogonal projection, called either *Leray* or *Helmholtz* projection. It is clear that $\mathbb{P}J = \mathrm{Id}_{H_D}$. Define now the space $V_D = H_0^1(\Omega;\mathbb{R}^3) \cap H_D$: it is a closed subspace of $H_0^1(\Omega;\mathbb{R}^3)$. The embedding J restricted to V_D maps V_D to $H_0^1(\Omega;\mathbb{R}^3)$: denote it by $J_0 : V_D \hookrightarrow H_0^1(\Omega;\mathbb{R}^3)$. Its adjoint $J_0' = \mathbb{P}_1 : H^{-1}(\Omega;\mathbb{R}^3) \to V_D'$ is then an extension of the orthogonal projection $\mathbb{P}$. The space H_D is endowed with the norm $u \mapsto \|u\|_2$ and V_D with the norm $u \mapsto \|\nabla u\|_2$.

The definition of the Dirichlet-Stokes operator then follows.

Definition 1. The Dirichlet-Stokes operator is defined as being the associated operator of the bilinear form:

$$a : V_D \times V_D \to \mathbb{R}, \quad a(u,v) = \sum_{i=1}^{3} \langle \partial_i J_0 u, \partial_i J_0 v\rangle.$$

Proposition 1. *The Dirichlet-Stokes operator A_D is the part in H_D of the bounded operator $A_{0,D} : V_D \to V_D'$ defined by $A_{0,D}u : V_D \to \mathbb{R}$, $(A_{0,D}u)(v) = a(u,v)$, and satisfies*

$$\mathsf{D}(A_D) = \left\{u \in V_D; \mathbb{P}_1(-\Delta_D^\Omega)J_0 u \in H_D\right\},$$

$$A_D u = \mathbb{P}_1(-\Delta_D^\Omega)J_0 u \quad u \in \mathsf{D}(A_D),$$

where Δ_D^Ω denotes the weak vector-valued Dirichlet-Laplacian in $L^2(\Omega;\mathbb{R}^3)$. The operator A_D is self-adjoint, invertible, $-A_D$ generates an analytic semigroup of contractions on H_D, $\mathsf{D}(A_D^{\frac{1}{2}}) = V_D$, and for all $u \in \mathsf{D}(A_D)$, there exists $\pi \in L^2(\Omega;\mathbb{R})$ such that

$$JA_D u = -\Delta J_0 u + \nabla\pi \tag{5}$$

and $\mathsf{D}(A_D)$ admits the following description:

$$\mathsf{D}(A_D) = \big\{u \in V_D; \exists\, \pi \in L^2(\Omega;\mathbb{R}) : -\Delta J_0 u + \nabla\pi \in H_D\big\}.$$

Proof. By definition, for $u \in \mathsf{D}(A_D)$ and for all $v \in V_D$,

$$\begin{aligned}
\langle A_D u, v\rangle = a(u,v) &= \sum_{j=1}^{n} \langle \partial_j J_0 u, \partial_j J_0 v\rangle \\
&= -\sum_{j=1}^{n} {}_{H^{-1}}\langle \partial_j^2 J_0 u, J_0 v\rangle_{H_0^1} = {}_{H^{-1}}\langle (-\Delta) J_0 u, J_0 v\rangle_{H_0^1} \\
&= {}_{V_D'}\langle \mathbb{P}_1(-\Delta) J_0 u, v\rangle_{V_D}.
\end{aligned}$$

The third equality comes from the definition of weak derivatives in L^2; the fourth equality comes from the fact that $\sum_{j=1}^n \partial_j^2 = \Delta$. The last equality is due to the fact that $J_0' = \mathbb{P}_1$. Therefore, $A_D u$ and $\mathbb{P}_1(-\Delta)J_0 u$ are two linear forms which coincide on V_D; they are then equal, which proves that $A_{0,D} = \mathbb{P}_1(-\Delta)J_0 : V_D \to V_D'$. Moreover, the fact that $u \in D(A_D)$ implies that $A_D u$ is a linear form on H_D, so that the linear form $\mathbb{P}_1(-\Delta)J_0 u$, originally defined on V_D, extends to a linear form on H_D (since V_D is dense in H_D by de Rham's theorem). The fact that A_D is self-adjoint and $-A_D$ generates an analytic semigroup of contractions comes from the properties of the form a: a is bilinear, symmetric, sectorial of angle 0, and coercive on $V_D \times V_D$. The property that $\mathsf{D}(A_D^{\frac{1}{2}}) = V_D$ is due to the fact that A_D is self-adjoint, applying a result by J.L. Lions [29, Théorème 5.3].

To prove the last assertions of this proposition, let $u \in \mathsf{D}(A_D)$. Then $A_D u \in H_D$ and $\mathbb{P}_1 J(A_D u) = \mathbb{P} J(A_D u) = u$. Moreover, if $u \in \mathsf{D}(A_D)$, u belongs, in particular, to V_D. Therefore, $J_0 u \in H_0^1(\Omega;\mathbb{R}^3)$ and $(-\Delta)J_0 u \in H^{-1}(\Omega;\mathbb{R}^3)$. The following identities take place in V_D':

$$\mathbb{P}_1\big(J(A_D u) - (-\Delta)J_0 u\big) = \mathbb{P}_1 J(A_D u) - \mathbb{P}_1(-\Delta)J_0 u = A_D u - A_D u = 0.$$

By de Rham's theorem, this implies that there exists $p \in \mathscr{C}_c^\infty(\Omega;\mathbb{R})'$ such that $J(A_D u) - (-\Delta)\tilde{J} u = \nabla p$: $\nabla p \in H^{-1}(\Omega;\mathbb{R}^3)$, which implies that $p \in L^2(\Omega;\mathbb{R})$. $\square$

The relations between the spaces and the operators described above are summarized in the following commutative diagram:

$$\begin{tikzcd}
V_D \arrow[r, hook, "J_0"] \arrow[d, hook, "d"'] \arrow[dd, bend right=60, "A_{0,D}"'] & H_0^1 \arrow[d, hook, "d"] \arrow[dd, bend left=60, "(-\Delta_D^\Omega)"] \\
H_D \arrow[r, hook, shift left, "J"] \arrow[d, hook, "d"'] & L^2 \arrow[l, shift left, "\mathbb{P}=J'"] \arrow[d, hook, "d"] \\
V_D' & H^{-1} \arrow[l, "\mathbb{P}_1=J_0'"]
\end{tikzcd}$$

In the case of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, the following property of $\mathsf{D}(A_D^{\frac{3}{4}})$ also holds; see [34, Corollary 5.5].

Proposition 2. *The domain of $A_D^{\frac{3}{4}}$ is continuously embedded into $W_0^{1,3}(\Omega;\mathbb{R}^3)$.*

It has been proved by R. Brown and Z. Shen [7] that the domain of A_D is embedded into $W_0^{1,p}(\Omega;\mathbb{R}^3) \cap W^{\frac{3}{2},2}(\Omega,\mathbb{R}^3)$ for some $p > 3$. The proof Proposition 2 uses the well-posedness result for the Poisson problem of the Stokes system [16, Theorem 5.6], similar to the corresponding result proved in [26] for the Laplacian.

2.1.2 The L^p Theory

P. Deuring provided in [14] an example of a domain with one conical singularity such that the Dirichlet-Stokes semigroup does not extend to an analytic semigroup in L^p for p large, away from 2. M.E. Taylor in [54], however, conjectured that this should be true for p in an interval containing $\left[\frac{3}{2}, 3\right]$, which was indeed proved 12 years later by the second author in [51].

Let $\mathscr{C}_{c,\sigma}^\infty(\Omega)$ denote the space of vector fields $u \in \mathscr{C}_c^\infty(\Omega;\mathbb{R}^3)$ with $\operatorname{div} u = 0$ in Ω and

$$L_\sigma^p(\Omega) = \text{ the closure of } \mathscr{C}_{c,\sigma}^\infty(\Omega) \text{ in } L^p(\Omega;\mathbb{R}^3). \tag{6}$$

Note that if Ω is Lipschitz and $p = 2$, $L_\sigma^2(\Omega) = H_D$. In view of Proposition 1, the Dirichlet-Stokes operator in the L^p setting for $1 < p < \infty$ is defined by

$$A_{D,p} = -\Delta u + \nabla \pi, \tag{7}$$

with the domain

$$\begin{aligned} \mathsf{D}(A_{D,p}) = \Big\{ u \in W_0^{1,p}(\Omega;\mathbb{R}^3);\ \operatorname{div} u = 0 \text{ in } \Omega \quad \text{and} \\ -\Delta u + \nabla \pi \in L_\sigma^p(\Omega) \text{ for some } \pi \in L^p(\Omega) \Big\}. \end{aligned} \tag{8}$$

Since $\mathscr{C}^\infty_{c,\sigma}(\Omega) \subset \mathsf{D}(A_{D,p})$, the operator $A_{D,p}$ is densely defined in $L^p_\sigma(\Omega)$ and $A_{D,p}(u) = \mathbb{P}(-\Delta)u$ for $u \in \mathscr{C}^\infty_{c,\sigma}(\Omega)$. If $p = 2$, $A_{D,p}$ agrees with the Dirichlet-Stokes operator A_D defined in the previous subsection.

The following theorem was proved in [51].

Theorem 2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Then there exists $\varepsilon > 0$, depending only on the Lipschitz character of Ω, such that $-A_{D,p}$ generates a bounded analytic semigroup in $L^p_\sigma(\Omega)$ for $(3/2) - \varepsilon < p < 3 + \varepsilon$.*

It was in fact proved in [51] that if Ω is a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$, then $-A_{D,p}$ generates a bounded analytic semigroup in $L^p_\sigma(\Omega)$ for

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon, \tag{9}$$

where $\varepsilon > 0$ depends only on d and the Lipschitz character of Ω. This was done by establishing the following resolvent estimate in L^p:

$$\|(A_{D,p} + \lambda)^{-1} f\|_{L^p(\Omega;\mathbb{C}^d)} \leq C_p\, |\lambda|^{-1} \|f\|_{L^p(\Omega;\mathbb{C}^d)} \tag{10}$$

for any $f \in \mathscr{C}^\infty_c(\Omega;\mathbb{C}^d)$ with $\operatorname{div} f = 0$ in Ω, where p satisfies (9),

$$\lambda \in \Sigma_\theta := \big\{z \in \mathbb{C} :\, \lambda \neq 0 \text{ and } |\arg(z)| < \pi - \theta\big\},$$

and $\theta \in (0, \pi/2)$. The constant C_p in (10) depends only on d, θ, p, and Ω. It has long been known that if Ω is a bounded $\mathscr{C}^2$ domain in $\mathbb{R}^d$, the resolvent estimate (10) holds for $\lambda \in \Sigma_\theta$ and $1 < p < \infty$ (see [21]). Consequently, the operator $A_{D,p}$ generates a bounded analytic semigroup in L^p for any $1 < p < \infty$, if Ω is $\mathscr{C}^2$. The case of nonsmooth domains is much more delicate. As mentioned earlier, P. Deuring constructed a three-dimensional Lipschitz domain for which the L^p resolvent estimate (10) fails for p sufficiently large. This was somewhat unexpected. Indeed it was proved in [48] that the L^p resolvent estimate holds for $1 < p < \infty$ in bounded Lipschitz domains in $\mathbb{R}^3$ for any second-order elliptic systems with constant coefficients satisfying the Legendre-Hadamard conditions (the range is $\frac{2d}{d+3} - \varepsilon < p < \frac{2d}{d-3} + \varepsilon$ for $d \geq 4$). It is worth mentioning that it is not known whether the range of p in Theorem 2 is sharp.

The approach used in [51] to the proof of (10) is described below. Consider the operator T_λ on $L^2(\Omega;\mathbb{C}^d)$, defined by $T_\lambda(f) = \lambda u$, where $\lambda \in \Sigma_\theta$ and $u \in H^1_0(\Omega;\mathbb{C}^d)$ are the unique solution to the Stokes system:

$$\begin{cases} -\Delta u + \nabla \pi + \lambda u = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{11}$$

Note that T_λ is bounded on $L^2(\Omega;\mathbb{C}^d)$ and $\|T_\lambda\|_{L^2\to L^2} \le C$. To show that T_λ is bounded on $L^p(\Omega;\mathbb{C}^d)$ and $\|T_\lambda\|_{L^p\to L^p} \le C$ for $2 < p < \frac{2d}{d-1} + \varepsilon$, a real variable argument is used, which may be regarded as a refined (and dual) version of the celebrated Calderón-Zygmund lemma. According to this argument, which originated from [8] and was further developed in [49,50], one only needs to establish the weak reverse Hölder estimate:

$$\left(\fint_{B(x_0,r)\cap\Omega} |u|^{p_d}\right)^{1/p_d} \le C\left(\fint_{B(x_0,2r)\cap\Omega} |u|^2\right)^{1/2} \tag{12}$$

for $p_d = \frac{2d}{d-1}$, whenever $u \in H^1_0(\Omega;\mathbb{C}^d)$ is a (local) solution of the Stokes system:

$$\begin{cases} -\Delta u + \nabla\pi + \lambda u = 0, \\ \operatorname{div} u = 0 \end{cases} \tag{13}$$

in $B(x_0,3r)\cap\Omega$ for some $x_0 \in \overline{\Omega}$ and $0 < r < c\,\operatorname{diam}(\Omega)$. The extra ε in the range of p is due to the self-improvement property of the weak reverse Hölder inequalities (see, e.g., [25]).

To prove the estimate (12), the Dirichlet problem for the Stokes system (13) is considered in a bounded Lipschitz domain Ω in $\mathbb{R}^d$, with boundary data $u = f$ on $\partial\Omega$, where $f \in L^2(\partial\Omega;\mathbb{C}^d)$ and $\int_{\partial\Omega} f\cdot\nu = 0$. The goal is to show that

$$\|(u)^*\|_{L^2(\partial\Omega)} \le C\,\|f\|_{L^2(\partial\Omega)}, \tag{14}$$

where $(u)^*$ denotes the nontangential maximal function of u and is defined by

$$(u)^*(Q) := \sup\Big\{|u(x)| : x \in \Omega \text{ and } |x-Q| < C_0\,\operatorname{dist}(x,\partial\Omega)\Big\}$$

for any $Q \in \partial\Omega$ ($C_0 > 1$ is a large fixed constant depending on d and Ω). This, together with the inequality

$$\left(\int_\Omega |u|^{p_d}\right)^{1/p_d} \le C\left(\int_{\partial\Omega} |(u)^*|^2\right)^{1/2},$$

which holds for any continuous function u in Ω, leads to

$$\left(\int_\Omega |u|^{p_d}\right)^{1/p_d} \le C\left(\int_{\partial\Omega} |u|^2\right)^{1/2}. \tag{15}$$

The desired estimate (12) follows by applying (15) in the domain $B(x_0,tr)\cap\Omega$ for $t \in (1,2)$ and then integrating the resulting inequality with respect to t over $(1,2)$.

Finally, the nontangential maximal function estimate (14) is established by the method of layer potentials. The case $\lambda = 0$ was studied in [11, 18], where the L^2 Dirichlet problem as well as the Neumann type boundary value problems with boundary data in L^2 for the system $-\Delta u + \nabla\pi = 0$ and $\operatorname{div} u = 0$ in a Lipschitz domain Ω was solved by the method of layer potentials, using the Rellich-type estimates:

$$\Big\| \frac{\partial u}{\partial \rho} \Big\|_{L^2(\partial\Omega)} \approx \|\nabla_{\tan} u\|_{L^2(\partial\Omega)}.$$

Here $\frac{\partial u}{\partial \rho}$ is a conormal derivative and $\nabla_{\tan} u$ denotes the tangential derivative of u on $\partial\Omega$. The reader is referred to the book [27] by C. Kenig for references on related work on L^p boundary value problems for elliptic and parabolic equations in nonsmooth domains. In an effort to solve the L^2 initial boundary value problems for the nonstationary Stokes equations $\partial_t u - \Delta u + \nabla\pi = 0$ and $\operatorname{div} u = 0$ in a Lipschitz cylinder $(0, T) \times \Omega$, the Stokes system (13) for $\lambda = i\tau$ with $\tau \in \mathbb{R}$ was considered by the second author in [47]. One of the key observations in [47] is that if $\lambda = i\tau$ and $\tau \in \mathbb{R}$ is large, the Rellich estimates for the system (13) involve two extra terms $|\tau|^{1/2}\|u\|_{L^2(\partial\Omega)}$ and $|\tau|\|u\cdot\nu\|_{H^{-1}(\partial\Omega)}$, where $H^{-1}(\partial\Omega)$ denotes the dual of $H^1(\partial\Omega)$. While the first term $|\tau|^{1/2}\|u\|_{L^2(\partial\Omega)}$ was expected in view of the Rellich estimates for the Helmholtz equation $-\Delta + i\tau$ in [6], the second term $|\tau|\|u\cdot\nu\|_{H^{-1}(\partial\Omega)}$ was not. Let

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial \nu} - \pi\nu.$$

By following the general approach in [47], it was proved in [51] that if (u, π) is a suitable solution of (13) in Ω, then

$$\|\frac{\partial u}{\partial \rho}\|_{L^2(\partial\Omega)} \approx \|\nabla_{\tan} u\|_{L^2(\partial\Omega)} + |\lambda|^{1/2}\|u\|_{L^2(\partial\Omega)} + |\lambda|\|u\cdot\nu\|_{H^{-1}(\partial\Omega)} \tag{16}$$

holds uniformly in λ for $\lambda \in \Sigma_\theta$ with $|\lambda| \geq c > 0$. As in the case of Laplace's equation [56], the estimate (14) follows from (16) by the method of layer potentials. The reader is referred to [51] for the details.

2.2 The Nonlinear Dirichlet-Navier-Stokes Equations

The system $\{$(NS), (Dbc)$\}$ is invariant under the scaling $u_\lambda(t, x) = \lambda\, u(\lambda^2 t, \lambda x)$, $(\lambda^2 t, \lambda x) \in (0, T) \times \Omega$ $(\lambda > 0)$: if u is a solution of $\{$(NS), (Dbc)$\}$ in $(0, T) \times \Omega$ for the initial value u_0, then u_λ is a solution of $\{$(NS), (Dbc)$\}$ in $\big(0, \frac{T}{\lambda^2}\big) \times \frac{1}{\lambda}\,\Omega$ for the initial value $x \mapsto \lambda u_0(\lambda x)$.

The goal here is to find the so-called mild solutions of the system $\{$(NS), (Dbc)$\}$ for initial values u_0 in a critical space, in the same spirit as in [20].

Lemma 1. *The space $\mathsf{D}(A_D^{\frac{1}{4}})$ is a critical space for the Navier-Stokes equations.*

Proof. The space $\mathsf{D}(A_D^{\frac{1}{4}})$ is invariant under the scaling $u_\lambda(x) = \lambda u_0(\lambda x)$ for $x \in \frac{1}{\lambda}\Omega$, $\lambda > 0$. Indeed, it suffices to check that $\|u_\lambda\|_2 = \lambda^{-\frac{1}{2}}\|u\|_2$ and $\|\nabla u_\lambda\|_2 = \lambda^{\frac{1}{2}}\|\nabla u\|_2$ and apply the fact that $\mathsf{D}(A_D^{\frac{1}{4}})$ is the interpolation space (with coefficient $\frac{1}{2}$) between H_D, closed subspace of $L^2(\Omega;\mathbb{R}^3)$, and $V_D = \mathsf{D}(A_D^{\frac{1}{2}})$, closed subspace of $H_0^1(\Omega;\mathbb{R}^3)$. □

For $T > 0$, define the space $\mathscr{E}_T$ by

$$\mathscr{E}_T = \Big\{u \in \mathscr{C}_b([0,T];\mathsf{D}(A_D^{\frac{1}{4}})); u(t) \in \mathsf{D}(A_D^{\frac{3}{4}}), u'(t) \in \mathsf{D}(A_D^{\frac{1}{4}}) \text{ for all } t \in (0,T] \\ \text{and } \sup_{t\in(0,T)} \|t^{\frac{1}{2}}A_D^{\frac{3}{4}}u(t)\|_2 + \sup_{t\in(0,T)} \|tA_D^{\frac{1}{4}}u'(t)\|_2 < \infty\Big\}$$

endowed with the norm

$$\|u\|_{\mathscr{E}_T} = \sup_{t\in(0,T)} \|A_D^{\frac{1}{4}}u(t)\|_2 + \sup_{t\in(0,T)} \|t^{\frac{1}{2}}A_D^{\frac{3}{4}}u(t)\|_2 + \sup_{t\in(0,T)} \|tA_D^{\frac{1}{4}}u'(t)\|_2.$$

The fact that $\mathscr{E}_T$ is a Banach space is straightforward. Assume now that $u \in \mathscr{E}_T$ and that (J_0u, p) (with $p \in L^2(\Omega;\mathbb{R})$) satisfy $\{$(NS), (Dbc)$\}$ in $H^{-1}(\Omega;\mathbb{R}^3)$: indeed, every term ∇p, $\partial_t J_0 u$, $-\Delta J_0 u$, and $(J_0u\cdot\nabla)J_0u$ independently belongs to $H^{-1}(\Omega;\mathbb{R}^3)$. Apply $\mathbb{P}_1$ to the equations and obtain

$$u'(t) + A_D u(t) = -\mathbb{P}_1\big((J_0u\cdot\nabla)J_0u\big)$$

since $\mathbb{P}_1\nabla p = 0$ and $\mathbb{P}_1(-\Delta)J_0u = A_{0,D}u$. The problem $\{$(NS), (Dbc)$\}$ is then reduced to the abstract Cauchy problem:

$$\begin{aligned} u'(t) + A_{0,D}u(t) &= -\mathbb{P}_1\big((J_0u\cdot\nabla)J_0u\big) \\ u(0) &= u_0, \quad u \in \mathscr{E}_T, \end{aligned} \tag{17}$$

for which a mild solution is given by the Duhamel formula:

$$u = \alpha + \phi(u,u), \tag{18}$$

where $\alpha(t) = e^{-tA_D}u_0$ and

$$\phi(u,v)(t) = \int_0^t e^{-(t-s)A_D}\Big(-\tfrac{1}{2}\mathbb{P}_1\big((J_0u(s)\cdot\nabla)J_0v(s) + (J_0v(s)\cdot\nabla)J_0u(s)\big)\Big)\,\mathrm{d}s. \tag{19}$$

The strategy to find $u \in \mathscr{E}_T$ satisfying $u = \alpha + \phi(u,u)$ is to apply a fixed-point theorem. For that, $\mathscr{E}_T$ needs to be a “good” space for the problem, i.e., $\alpha \in \mathscr{E}_T$ and

$\phi(u,u) \in \mathscr{E}_T$. The fact that $\alpha \in \mathscr{E}_T$ follows directly from the properties of the Stokes operator A_D and the semigroup $(e^{-tA_D})_{t\geq 0}$.

Proposition 3. *The mapping $\phi : \mathscr{E}_T \times \mathscr{E}_T \to \mathscr{E}_T$ is bilinear, continuous, and symmetric.*

Proof. The fact that ϕ is bilinear and symmetric is immediate, once it is proved that it is well-defined. For $u, v \in \mathscr{E}_T$, let

$$f(t) = -\tfrac{1}{2}\mathbb{P}_1\big((J_0u(t)\cdot\nabla)J_0v(t) + (J_0v(t)\cdot\nabla)J_0u(t)\big), \quad t \in (0,T). \tag{20}$$

By the definition of $\mathscr{E}_T$ and Sobolev embeddings, it is easy to see that

$$(J_0u(t)\cdot\nabla)J_0v(t) + (J_0v(t)\cdot\nabla)J_0u(t) \in L^2(\Omega;\mathbb{R}^3)$$

and

$$\big\|(J_0u(t)\cdot\nabla)J_0v(t) + (J_0v(t)\cdot\nabla)J_0u(t)\big\|_2 \leq C\, t^{-\frac{3}{4}}\, \|u\|_{\mathscr{E}_T}\|v\|_{\mathscr{E}_T}$$

where C is a constant independent from t, which gives the following estimate:

$$\big\|f(t)\big\|_2 \leq C\, t^{-\frac{3}{4}}\, \|u\|_{\mathscr{E}_T}\|v\|_{\mathscr{E}_T} \tag{21}$$

Therefore,

$$\begin{aligned}\|A_D^{\frac{1}{4}}\phi(u,v)(t)\|_2 \leq &\int_0^t \|A_D^{\frac{1}{4}}e^{-(t-s)A_D}\|_{\mathscr{L}(H_D)} C\, s^{-\frac{3}{4}}\, \|u\|_{\mathscr{E}_T}\|v\|_{\mathscr{E}_T}\,\mathrm{d}s\\ \leq &C\Big(\int_0^t (t-s)^{-\frac{1}{4}}s^{-\frac{3}{4}}\,\mathrm{d}s\Big)\, \|u\|_{\mathscr{E}_T}\|v\|_{\mathscr{E}_T},\end{aligned}$$

and since $\int_0^t (t-s)^{-\frac{1}{4}}s^{-\frac{3}{4}}\,\mathrm{d}s = \int_0^1 (1-s)^{-\frac{1}{4}}s^{-\frac{3}{4}}\,\mathrm{d}s$, the following estimate is finally obtained:

$$\|A_D^{\frac{1}{4}}\phi(u,v)(t)\|_2 \leq C\, \|u\|_{\mathscr{E}_T}\|v\|_{\mathscr{E}_T}. \tag{22}$$

The proof of the continuity of $t \mapsto A_D^{\frac{1}{4}}\phi(u,v)(t)$ on H_D is straightforward once the estimate (22) is established. The proof of the fact that

$$\|\sqrt{t}A_D^{\frac{3}{4}}\phi(u,v)(t)\|_2 \leq C\, \|u\|_{\mathscr{E}_T}\|v\|_{\mathscr{E}_T} \tag{23}$$

is proved the same way, replacing $A_D^{\frac{1}{4}}$ by $A_D^{\frac{3}{4}}$ and using the fact that

$$\|A_D^{\frac{3}{4}} e^{-(t-s)A_D}\|_{\mathscr{L}(H_D)} \leq C\,(t-s)^{-\frac{3}{4}}$$

and

$$\int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{3}{4}}\,\mathrm{d}s = t^{-\frac{1}{2}} \int_0^1 (1-s)^{-\frac{3}{4}} s^{-\frac{3}{4}}\,\mathrm{d}s.$$

It remains to prove the estimate on the derivative with respect to t of $\phi(u,v)$. Rewrite f as defined in (20) as follows:

$$f(s) = -\tfrac{1}{2}\mathbb{P}_1 \nabla \cdot \big(J_0 u(s) \otimes J_0 v(s) + J_0 v(s) \otimes J_0 u(s)\big) \tag{24}$$

where $u \otimes v$ denotes the matrix $(u_i v_j)_{1\leq i,j\leq 3}$ and the differential operator $\nabla\cdot$ acts on matrices $M = (m_{i,j})_{1\leq i,j\leq 3}$ the following way:

$$\nabla \cdot M = \Big(\sum_{i=1}^{3} \partial_i m_{i,j}\Big)_{1\leq j\leq 3}.$$

For $u, v \in \mathscr{E}_T$ and $s \in (0,T)$,

$$\begin{aligned} f'(s) =& -\tfrac{1}{2}\mathbb{P}_1 \nabla \cdot \Big(J u'(s) \otimes J_0 v(s) + J_0 u(s) \otimes J v'(s) \\ &+ J v'(s) \otimes J_0 u(s) + J_0 v(s) \otimes J u'(s)\Big) \end{aligned}$$

For all $s \in (0,T)$,

$$\begin{aligned} s^{\frac{5}{4}} \|J u'(s) \otimes J_0 v(s)\|_2 &\leq \|s J u'(s)\|_3 \|s^{\frac{1}{4}} J_0 v(s)\|_6 \\ &\leq \|s A_D^{\frac{1}{4}} u'(s)\|_2 \|s^{\frac{1}{4}} A_D^{\frac{1}{2}} v(s)\|_2 \\ &\leq \|u\|_{\mathscr{E}_T} \|v\|_{\mathscr{E}_T}, \end{aligned}$$

where the first inequality comes from the fact that $L^3 \cdot L^6 \hookrightarrow L^2$, the second inequality comes from the Sobolev embeddings $\mathsf{D}(A_D^{\frac{1}{4}}) \hookrightarrow L^3(\Omega;\mathbb{R}^3)$ and $\mathsf{D}(A_D^{\frac{1}{2}}) \hookrightarrow L^6(\Omega;\mathbb{R}^3)$, and the third inequality follows directly from the definition of the space $\mathscr{E}_T$. Of course the same occurs for the other three terms $J_0 u(s) \otimes J v'(s)$, $J v'(s) \otimes J_0 u(s)$, and $J_0 v(s) \otimes J u'(s)$. Therefore, since $A_D^{-\frac{1}{2}}$ maps V_d' to H_D,

$$\sup_{0<s<T} \|s^{\frac{5}{4}} A_D^{-\frac{1}{2}} f'(s)\|_2 \leq c\,\|u\|_{\mathscr{E}_T} \|v\|_{\mathscr{E}_T}. \tag{25}$$

It is straightforward that

$$\phi(u,v)(t) = \int_0^{\frac{t}{2}} e^{-sA_D} f(t-s)ds + \int_0^{\frac{t}{2}} e^{-(t-s)A_D} f(s)\,\mathrm{d}s \quad t \in (0,T),$$

and therefore

$$\phi(u,v)'(t) = e^{-\frac{t}{2}A_D} f\left(\tfrac{t}{2}\right) + \int_0^{\frac{t}{2}} A_D^{\frac{1}{2}} e^{-sA_D} A_{0,D}^{-\frac{1}{2}} f'(t-s)\,\mathrm{d}s$$
$$+ \int_0^{\frac{t}{2}} -A_D e^{-(t-s)A_D} f(s)\,\mathrm{d}s,$$

which yields

$$\|A_D^{\frac{1}{4}} \phi(u,v)'(t)\|_2 \le \frac{c}{t^{\frac{1}{4}}} \left\| f\left(\tfrac{t}{2}\right)\right\|_2 + c\left(\int_0^{\frac{t}{2}} \frac{1}{s^{\frac{3}{4}}} \frac{1}{(t-s)^{\frac{5}{4}}}\,\mathrm{d}s\right) \|u\|_{\mathscr{E}_T} \|v\|_{\mathscr{E}_T}$$
$$+ c\left(\int_0^{\frac{t}{2}} \frac{1}{(t-s)^{\frac{5}{4}}} \frac{1}{s^{\frac{3}{4}}}\,\mathrm{d}s\right) \|u\|_{\mathscr{E}_T} \|v\|_{\mathscr{E}_T}$$
$$\le \frac{c}{t}\left(1 + \int_0^{\frac{1}{2}} \frac{\mathrm{d}\sigma}{(1-\sigma)^{\frac{5}{4}} \sigma^{\frac{3}{4}}}\right) \|u\|_{\mathscr{E}_T} \|v\|_{\mathscr{E}_T},$$

where the estimates (21), (25) and the fact that $-A_D$ generates a bounded analytic semigroup (so that $\|A_D^{\alpha} e^{-tA_D}\|_{\mathscr{L}(H_D)} \le C\, t^{-\alpha}$) were used. This last inequality together with (22) and (23) ensures that $\phi(u,v) \in \mathscr{E}_T$ whenever $u, v \in \mathscr{E}_T$. □

This section is concluded by applying Picard's fixed-point theorem (see, e.g., [28, Theorem 13.2] or [42, Theorem A.1]) to obtain the following existence result for the system $\{$(NS), (Dbc)$\}$.

Theorem 3 (Existence). *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $u_0 \in \mathsf{D}(A_D^{\frac{1}{4}})$. Let α and ϕ be defined as above:*

(i) If $\|A_D^{\frac{1}{4}} u_0\|_2$ is small enough, then there exists a unique $u \in \mathscr{E}_\infty$ solution of $u = \alpha + \phi(u,u)$.
(ii) For all $u_0 \in \mathsf{D}(A_D^{\frac{1}{4}})$, there exists $T > 0$ and a unique $u \in \mathscr{E}_T$ solution of $u = \alpha + \phi(u,u)$.

Uniqueness in the larger space $\mathscr{C}_b([0,T); \mathsf{D}(A_D^{\frac{1}{4}}))$ can be obtained, by applying [40, Theorem 1.1]. The argument there is somewhat stronger though, since uniqueness in $\mathscr{C}_b([0,T); L^3)$ is proved, using a maximal regularity result by Z. Shen [47, Theorem 5.1.2].

Theorem 4 (Uniqueness). *Let $u, v \in \mathscr{C}_b([0,T); \mathsf{D}(A_D^{\frac{1}{4}}))$ both be mild solutions of the system $\{$(NS), (Dbc)$\}$, i.e., they both satisfy (18). Then $u = v$ on $[0,T)$.*

Before proving this theorem, the following lemma is shown, similar to [39, Proposition 2].

Lemma 2. *Let $p \in (1,\infty)$ and $\tau \in (0,T]$: ϕ defined by (19) maps $L^p(0,\tau;\mathsf{D}(A_D^{\frac{1}{4}})) \times L^\infty(0,\tau;\mathsf{D}(A_D^{\frac{1}{4}}))$ to $L^p(0,\tau;\mathsf{D}(A_D^{\frac{1}{4}}))$. Moreover, there exists a constant $C_p > 0$ independent of τ such that*

$$\|\phi(u,v)\|_{L^p(0,\tau;\mathsf{D}(A_D^{1/4}))} \le C_p \,\|u\|_{L^p(0,\tau;\mathsf{D}(A_D^{1/4}))} \|v\|_{L^\infty(0,\tau;\mathsf{D}(A_D^{1/4}))}. \tag{26}$$

If $v \in L^\infty(0,\tau;V_D)$, the following improved estimate holds

$$\|\phi(u,v)\|_{L^p(0,\tau;\mathsf{D}(A_D^{\frac{1}{4}}))} \le K_p\, \tau^{\frac{1}{4}} \,\|u\|_{L^p(0,\tau;\mathsf{D}(A_D^{1/4}))} \|v\|_{L^\infty(0,\tau;V_D)}, \tag{27}$$

where $K_p > 0$ is a constant independent of τ.

Proof. First, let $\mathcal{M}$ be the maximal regularity operator on H_D: for all $\varphi \in L^p(0,\tau;H_D)$, $\mathcal{M}\varphi$ is defined by

$$\mathcal{M}\varphi(t) := \int_0^t A_D e^{-(t-s)A_D}\varphi(s)\,\mathrm{d}s, \quad t \in (0,\tau).$$

Since H_D is a Hilbert space and $-A_D$ generates an analytic semigroup in H_D, the operator $\mathcal{M}$ is bounded on $L^p(0,\tau;H_D)$ for all $p \in (1,\infty)$ and all $\tau > 0$; see, e.g., [13]. Moreover, $\|\mathcal{M}\|_{\mathscr{L}(L^p(0,\tau;H_D))}$ is independent of τ. Then

$$A_D^{\frac{1}{4}}\phi(u,v) = \mathcal{M}\big(A_D^{-\frac{3}{4}} f\big)$$

where f is defined by (24). For $u \in L^p(0,\tau;\mathsf{D}(A_D^{\frac{1}{4}})$ and $v \in L^\infty(0,\tau;\mathsf{D}(A_D^{\frac{1}{4}})$, by Sobolev embeddings, $Ju \otimes Jv + Jv \otimes Ju \in L^p(0,\tau;L^{3/2}(\Omega;\mathbb{R}^3))$, with the estimate

$$\|Ju \otimes Jv + Jv \otimes Ju\|_{L^p(0,\tau;L^{3/2}(\Omega;\mathbb{R}^3))} \le C\,\|u\|_{L^p(0,\tau;\mathsf{D}(A_D^{1/4}))}\|v\|_{L^\infty(0,\tau;\mathsf{D}(A_D^{1/4}))},$$

where the constant C depends only on the constant of the embedding $\mathsf{D}(A_D^{\frac{1}{4}}) \hookrightarrow L^3(\Omega;\mathbb{R}^3)$. This implies that $f \in L^p\big(0,\tau;\mathbb{P}_1(W^{-1,3/2})\big)$. Since $\mathsf{D}(A_D^{\frac{3}{4}}) \hookrightarrow W_0^{1,3}(\Omega;\mathbb{R}^3)$ (see Proposition 2), the embedding $\mathbb{P}_1\big(W^{-1,3/2}(\Omega;\mathbb{R}^3)\big) \hookrightarrow \big(\mathsf{D}(A_D^{\frac{3}{4}})\big)'$ holds and therefore $A_D^{-\frac{3}{4}} f \in L^p(0,\tau;H_D)$ with

$$\|A_D^{-\frac{3}{4}} f\|_{L^p(0,\tau;H_D)} \le C\, \|u\|_{L^p(0,\tau;\mathsf{D}(A_D^{1/4}))} \|v\|_{L^\infty(0,\tau;\mathsf{D}(A_D^{1/4}))}.$$

Using the L^p maximal regularity result in H_D gives (26).

To prove (27), let $u \in L^p(0,\tau;\mathsf{D}(A_D^{\frac{1}{4}}))$ and $v \in L^\infty(0,\tau;V_D)$. Using the embeddings $\mathsf{D}(A_D^{\frac{1}{4}}) \hookrightarrow L^3(\Omega;\mathbb{R}^3)$ and $V_D \hookrightarrow L^6(\Omega;\mathbb{R}^3)$,

$$\|Ju \otimes Jv + Jv \otimes Ju\|_{L^p(0,\tau;L^2(\Omega,\mathbb{R}^3))} \le C\, \|u\|_{L^p(0,\tau;\mathsf{D}(A_D^{1/4}))} \|v\|_{L^\infty(0,\tau;V_D)}.$$

As before, this implies that $f \in L^p(0,\tau;V_D')$, and therefore

$$A_D^{\frac{1}{4}}\phi(u,v)(t) = \int_0^t A_D^{\frac{3}{4}} e^{(t-s)A_D}\big(A_D^{-\frac{1}{2}} f(s)\big)\,\mathrm{d}s, \quad t \in (0,\tau).$$

Using the analyticity of the semigroup $(e^{-tA_D})_{t\ge 0}$ in H_D and Young's inequality,

$$\|A_D^{\frac{1}{4}}\phi(u,v)\|_{L^p(0,\tau;H_D)} \le C\, \|t \mapsto t^{-\frac{3}{4}}\|_{L^1(0,\tau)} \|u\|_{L^p(0,\tau;\mathsf{D}(A_D^{1/4}))} \|v\|_{L^\infty(0,\tau;V_D)}.$$

$\square$

Proof of Theorem 4. The proof is inspired by the method described in [39] (see also [2, Section 8]). Let $p \in (1,\infty)$, $\varepsilon > 0$ to be chosen later and $w := u - v \in \mathscr{C}_b(0,T;\mathsf{D}(A_D^{\frac{1}{4}})) \subset L^p(0,T;\mathsf{D}(A_D^{\frac{1}{4}}))$: w satisfies

$$\begin{aligned} w &= \phi(u,w) + \phi(w,v) = \phi(w,u+v-2\alpha) + 2\phi(w,\alpha) \\ &= \phi(w,u+v-2\alpha) + 2\phi(w,\alpha-\alpha_\varepsilon) + 2\phi(w,\alpha_\varepsilon) \end{aligned}$$

where $\alpha_\varepsilon(t) = e^{-tA_D}u_{0,\varepsilon}$, with $u_{0,\varepsilon} \in V_D$ satisfying the estimate $\|u_{0,\varepsilon} - u_0\|_{\mathsf{D}(A_D^{1/4})} \le \varepsilon$. Using Lemma 2, w is estimated in $L^p(0,\tau;\mathsf{D}(A^{\frac{1}{4}}))$ as follows:

$$\begin{aligned} &\|w\|_{L^p(0,\tau;\mathsf{D}(A^{1/4}))} \\ &\quad \le \|w\|_{L^p(0,\tau;\mathsf{D}(A^{1/4}))}\Big(C_p(\|u+v-2\alpha\|_{L^\infty(0,\tau;\mathsf{D}(A_D^{1/4}))} + \varepsilon) + K_p\, \tau^{\frac{1}{4}}\|u_{0,\varepsilon}\|_{V_D}\Big) \\ &\quad \le \kappa_p\big(\varepsilon + g_\varepsilon(\tau)\big)\|w\|_{L^p(0,\tau;\mathsf{D}(A^{1/4}))}, \end{aligned}$$

where $g_\varepsilon(\tau) = \|u+v-2\alpha\|_{L^\infty(0,\tau;\mathsf{D}(A_D^{1/4}))} + \tau^{\frac{1}{4}}\|u_{0,\varepsilon}\|_{V_D} \xrightarrow[t\to 0]{} 0$. This shows that choosing $\varepsilon > 0$ small enough, there exists $\tau > 0$ such that $\|w\|_{L^p(0,\tau;\mathsf{D}(A^{1/4}))} \le \frac{1}{2}\,\|w\|_{L^p(0,\tau;\mathsf{D}(A^{1/4}))}$; in other terms, $w = 0$ on $[0,\tau)$ (recall that w is continuous on $[0,T)$). If $\tau = T$, then it was proved that $u = v$ on $[0,T)$. If $\tau < T$, by continuity, $w(\tau) = 0$ also holds. The previous reasoning can be iterated on intervals of the form $[k\tau, (k+1)\tau)$ to prove ultimately that $w = 0$ on $[0, T)$ (remark again that all constants C_p, K_p, κ_p appearing in the estimates above are independent of τ). $\square$

3 Neumann Boundary Conditions

In this section, the system $\{$(NS), (Nbc)$\}$ is studied. The results proved in [37] will be only surveyed, the method to prove existence of solutions being similar to what has been done in Sect. 2.

3.1 The Linear Neumann-Stokes Operator

Before defining the Neumann-Stokes operator, the following integration by parts formula will be useful.

Lemma 3. *Let $\lambda \in \mathbb{R}$, $u, w : \Omega \to \mathbb{R}^3$, $\pi, \rho : \Omega \to \mathbb{R}$ be sufficiently nice functions defined on the Lipschitz domain $\Omega \subset \mathbb{R}^3$. Let $L_\lambda u = \Delta u + \lambda \nabla(\operatorname{div} u)$ and define the conormal derivative:*

$$\partial_\nu^\lambda(u, \pi) = \big(\lambda \nabla u + (\nabla u)^\top\big)\nu - \pi\nu \quad \text{on } \partial\Omega. \tag{28}$$

Then the following integration by parts formula holds:

$$\int_\Omega (L_\lambda u - \nabla\pi)\cdot w \,\mathrm{d}x = -\int_\Omega \big[I_\lambda(\nabla u, \nabla w) - \pi \operatorname{div} w\big]\,\mathrm{d}x + \int_{\partial\Omega} \partial_\nu^\lambda(u, \pi)\cdot w \,\mathrm{d}\sigma \tag{29}$$

$$= \int_\Omega (L_\lambda w - \nabla\rho)\cdot u \,\mathrm{d}x + \int_\Omega \big[\pi \operatorname{div} w - \rho \operatorname{div} u\big]\,\mathrm{d}x$$

$$+ \int_{\partial\Omega} \big[\partial_\nu^\lambda(u, \pi)\cdot w - \partial_\nu^\lambda(w, \rho)\cdot u\big]\,\mathrm{d}\sigma, \tag{30}$$

where

$$I_\lambda(\xi, \zeta) = \sum_{i,j=1}^{3} (\xi_{i,j}\zeta_{i,j} + \lambda\xi_{i,j}\zeta_{j,i}), \quad \text{for } \xi = (\xi_{i,j})_{1\le i,j\le 3} \text{ and } \zeta = (\zeta_{i,j})_{1\le i,j\le 3}.$$

Recall that $\nabla u = (\partial_i u_j)_{1\le i,j\le 3}$.

The space $L^2(\Omega; \mathbb{R}^3)$ admits the following Hodge decomposition, dual to the one shown in Sect. 2: $H_N \overset{\perp}{\oplus} G_0$, where $G_0 := \{\nabla\pi; \pi \in H_0^1(\Omega; \mathbb{R})\}$ and

$$H_N := \{u \in L^2(\Omega;\mathbb{R}^3); \operatorname{div} u = 0\}. \tag{31}$$

Following the steps of the previous section, define $V_N = H^1(\Omega;\mathbb{R}^3) \cap H_N$ and $J_N : H_N \hookrightarrow L^2(\Omega;\mathbb{R}^3)$ the canonical embedding, $\mathbb{P}_N = J_N' : L^2(\Omega;\mathbb{R}^3) \to H_N$ the orthogonal projection, and $\tilde{J}_N : V_N \hookrightarrow H^1(\Omega;\mathbb{R}^3)$ the restriction of J_N on V_N and $\tilde{J}_N' = \tilde{\mathbb{P}}_N : (H^1(\Omega;\mathbb{R}^3))' \to V_N'$, extension of $\mathbb{P}_N$ to $(H^1(\Omega;\mathbb{R}^3))'$. The Neumann-Stokes operator is defined as follows.

Definition 2. Let $\lambda \in \mathbb{R}$. The Neumann-Stokes operator A_λ is defined as being the associated operator of the bilinear form:

$$a_\lambda : V_N \times V_N \to \mathbb{R}, \quad a_\lambda(u,v) = \int_\Omega I_\lambda(\nabla \tilde{J}_N u, \nabla \tilde{J}_N v)\, dx$$

In the case where $\lambda \in (-1,1]$, the bilinear form a_λ is continuous, symmetric, coercive, and sectorial. So its associated operator is self-adjoint, invertible and the negative generator of an analytic semigroup of contractions on H_N.

The following proposition is a consequence of the integration by parts formula (29), [37, Theorem 6.8] and [29, Théorème 5.3].

Proposition 4. *Let $\lambda \in (-1,1]$. The Neumann-Stokes operator A_λ is the part in H_N of the bounded operator $A_{0,\lambda} : V_N \to V_N'$ defined by $(A_{0,\lambda}u)(v) = a_\lambda(u,v)$. The operator A_λ is self-adjoint, invertible, $-A_\lambda$ generates an analytic semigroup of contractions on H_N, $\mathsf{D}(A_\lambda^{\frac12}) = V_N$ and for all $u \in \mathsf{D}(A_\lambda)$, there exists $\pi \in L^2(\Omega;\mathbb{R})$ such that*

$$J_N A_\lambda u = -\Delta \tilde{J}_N u + \nabla \pi \tag{32}$$

and $\mathsf{D}(A_\lambda)$ admits the following description:

$$\mathsf{D}(A_\lambda) = \{u \in V_N; \exists\, \pi \in L^2(\Omega;\mathbb{R}) : f = -\Delta \tilde{J}_N u + \nabla \pi \in H_N \text{ and } \partial_\nu^\lambda(u,\pi)_f = 0\},$$

where $\partial_\nu^\lambda(u,\pi)_f$ is defined in a weak sense for all $f \in (H^1(\Omega;\mathbb{R}^3))'$ by

$$\langle \partial_\nu^\lambda(u,\pi)_f, \psi\rangle_{\partial\Omega} = {}_{(H^1)'}\langle f, \Psi\rangle_{H^1} + \int_\Omega I_\lambda(\nabla \tilde{J}_n u, \nabla \Psi)\, \mathrm{d}x - {}_{L^2}\langle \pi, \operatorname{div}\Psi\rangle_{L^2}$$

for $\Psi \in H^1(\Omega)$ and $\psi = \mathrm{Tr}_{\partial\Omega}\Psi$.

Remark 3. If $f \in (H^1(\Omega;\mathbb{R}^3))'$, the quantity $\partial_\nu^\lambda(u,\pi)_f$ exists on $\partial\Omega$ in the Besov space $B^{2,2}_{-\frac12}(\partial\Omega;\mathbb{R}^3) = H^{-\frac12}(\partial\Omega,\mathbb{R}^3)$ according to [37, Proposition 3.6].

Thanks to [37, Sections 9 & 10], a good description of the domain of fractional powers of the Neumann-Stokes operator A_λ can be given. In particular, in [37, Corollary 10.6], it was established that

$$\mathsf{D}(A_\lambda^{\frac{3}{4}}) \text{ is continuously embedded into } W^{1,3}(\Omega;\mathbb{R}^3). \tag{33}$$

3.2 The Nonlinear Neumann-Navier-Stokes Equations

The results in Sect. 3.1 allow to prove a result similar to Theorem 3 for the Navier-Stokes system with Neumann boundary conditions $\{$(NS), (Nbc)$\}$. As in the previous section, it is not difficult to see that $\mathsf{D}(A_\lambda^{\frac{1}{4}}) \hookrightarrow L^3(\Omega;\mathbb{R}^3)$ is a critical space for the system. For $T \in (0,\infty]$, following the definition of $\mathscr{E}_T$ in Sect. 2, define

$$\mathscr{F}_T = \Big\{u \in \mathscr{C}_b([0,T];\mathsf{D}(A_\lambda^{\frac{1}{4}}));\, u(t) \in \mathsf{D}(A_\lambda^{\frac{3}{4}}),\, u'(t) \in \mathsf{D}(A_\lambda^{\frac{1}{4}}) \text{ for all } t \in (0,T]$$
$$\text{and } \sup_{t\in(0,T)} \|t^{\frac{1}{2}} A_\lambda^{\frac{3}{4}} u(t)\|_2 + \sup_{t\in(0,T)} \|t A_\lambda^{\frac{1}{4}} u'(t)\|_2 < \infty\Big\}$$

endowed with the norm

$$\|u\|_{\mathscr{F}_T} = \sup_{t\in(0,T)} \|A_\lambda^{\frac{1}{4}} u(t)\|_2 + \sup_{t\in(0,T)} \|t^{\frac{1}{2}} A_\lambda^{\frac{3}{4}} u(t)\|_2 + \sup_{t\in(0,T)} \|t A_\lambda^{\frac{1}{4}} u'(t)\|_2.$$

The same tools as in 2.2 apply, so the following result can be proved (see [37, Theorem 11.3]).

Theorem 5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $u_0 \in \mathsf{D}(A_\lambda^{\frac{1}{4}})$. Let β and ψ be defined by*

$$\beta(t) = e^{-tA_\lambda} u_0, \quad t \geq 0,$$

and for $u, v \in \mathscr{F}_T$ and $t \in (0,T)$,

$$\psi(u,v)(t) = \int_0^t e^{-(t-s)A_\lambda}(-\tfrac{1}{2}\mathbb{P}_N)\big((J_N u(s)\cdot\nabla)\tilde{J}_N v(s) + J_N v(s)\cdot\nabla)\tilde{J}_N u(s)\big)\,\mathrm{d}s :$$

(i) If $\|A_\lambda^{\frac{1}{4}} u_0\|_2$ is small enough, then there exists a unique $u \in \mathscr{F}_\infty$ solution of $u = \beta + \psi(u,u)$.

(ii) For all $u_0 \in \mathsf{D}(A_\lambda^{\frac{1}{4}})$, there exists $T > 0$ and a unique $u \in \mathscr{F}_T$ solution of $u = \beta + \psi(u,u)$.

A comment here may be necessary to link the solution u obtained in Theorem 5 and a solution of the system $\{(\text{NS}), (\text{Nbc})\}$. If $u \in \mathcal{F}_T$, then $u' \in H_N$ and $(J_N u \cdot \nabla)\tilde{J}_N u \in L^2(\Omega; \mathbb{R}^n)$. Moreover, if u satisfies the equation $u = \beta + \psi(u, u)$, then u is a mild solution of

$$A_\lambda u = -u' - \mathbb{P}_N\big((J_N u \cdot \nabla)\tilde{J}_N u\big) \in H_N.$$

Going further,

$$J_N \mathbb{P}_N\big((J_N u \cdot \nabla)\tilde{J}_N u\big) = (J_N u \cdot \nabla)\tilde{J}_N u - \nabla q$$

where $q \in H_0^1(\Omega; \mathbb{R})$ satisfies

$$\Delta q = \operatorname{div}(J_N u \cdot \nabla)\tilde{J}_N u) \in H^{-1}(\Omega; \mathbb{R}^n).$$

Therefore, by definition of A_λ, there exists $\pi \in L^2(\Omega, \mathbb{R})$ such that

$$-\Delta \tilde{J}_n u + \nabla \pi = J_N(A_\lambda u) = -J_N u' - (J_N u \cdot \nabla)\tilde{J}_N u + \nabla q$$

and at the boundary, (u, π) satisfies (Nbc) in the weak sense as in Proposition 4. Since $q \in H_0^1(\Omega; \mathbb{R})$, $(u, \pi - q)$ satisfies also (Nbc). This proves that $(u, \pi - q)$ is a solution of the system $\{(\text{NS}), (\text{Nbc})\}$.

The uniqueness is true in a larger space than $\mathscr{F}_T$: for each $u_0 \in \mathsf{D}(A^{\frac{1}{4}})$, there is at most one $u \in \mathscr{C}_b([0, T); \mathsf{D}(A^{\frac{1}{4}}))$, mild solution of the system $\{(\text{NS}), (\text{Nbc})\}$. For a more precise statement, see [37, Theorem 11.8].

4 Hodge Boundary Conditions

Most of the results presented here are proved thoroughly in [36] for the linear theory and [35] for the nonlinear system. The linear Hodge-Laplacian on L^p-spaces is first studied and then the Hodge-Stokes operator before applying the properties of this operator to prove the existence of mild solutions of the Hodge-Navier-Stokes system in L^3. Some recent developments/improvements can be found in [30].

4.1 The Hodge-Laplacian and the Hodge-Stokes Operators

We denote by H the space $L^2(\Omega; \mathbb{R}^3)$. Let

$$W_T := \big\{u \in H; \operatorname{curl} u \in H, \operatorname{div} u \in L^2(\Omega; \mathbb{R}) \text{ and } \nu \cdot u = 0 \text{ on } \partial\Omega\big\},$$

$$\text{and} \quad W_N := \big\{u \in H; \operatorname{curl} u \in H, \operatorname{div} u \in L^2(\Omega; \mathbb{R}) \text{ and } \nu \times u = 0 \text{ on } \partial\Omega\big\},$$

(subscript T is for "tangential" and N for "normal") both endowed with the scalar product

$$\langle\langle u, v\rangle\rangle_W := \langle \operatorname{curl} u, \operatorname{curl} v\rangle_\Omega + \langle \operatorname{div} u, \operatorname{div} v\rangle_\Omega + \langle u, v\rangle_\Omega,$$

where $\langle\cdot,\cdot\rangle_E$ denotes the $L^2(E)$-pairing.

Remark 4. As in Remark 1 for a bounded Lipschitz domain Ω and a vector field $w \in H$ satisfying $\operatorname{curl} w \in H$, define $\nu \times w$ on $\partial\Omega$ in the following weak sense in $H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)$: for $\phi \in H^1(\Omega;\mathbb{R}^3)$,

$$\langle \operatorname{curl} w, \phi\rangle_\Omega - \langle w, \operatorname{curl}\phi\rangle_\Omega = \langle \nu \times w, \varphi\rangle_{\partial\Omega} \tag{34}$$

where $\varphi = \operatorname{Tr}_{|\partial\Omega}\phi$, the right-hand side of (34) depends only on φ on $\partial\Omega$ and not on the choice of ϕ, its extension to Ω.

Remark 5. In the case of smooth bounded domains, i.e., with a $\mathscr{C}^{1,1}$ boundary or convex, the spaces W_T and W_N are contained in $H^1(\Omega;\mathbb{R}^3)$ (see, e.g., [3, Theorems 2.9, 2.12, and 2.17]).

This is not the case if Ω is only Lipschitz. The Sobolev embedding associated to the spaces $W_{T,N}$ is as follows: $W_{T,N} \hookrightarrow H^{\frac{1}{2}}(\Omega;\mathbb{R}^3)$ with the estimate

$$\|u\|_{H^{1/2}} \le C\left[\|u\|_2 + \|\operatorname{curl} u\|_2 + \|\operatorname{div} u\|_2\right], \quad u \in W_{T,N}; \tag{35}$$

see, for instance, [9] or [32, Theorem 11.2] where it was proved moreover that

$$\text{if } u \in W_{T,N}, \text{ then } u \text{ has an } L^2 \text{ trace at the boundary } \partial\Omega:$$

$$u_{|\partial\Omega} = (\nu\cdot u)\nu + (\nu \times u)\times \nu \in L^2(\partial\Omega;\mathbb{R}^3), \tag{36}$$

$$\text{and } \|u_{|\partial\Omega}\|_{L^2(\partial\Omega;\mathbb{R}^3)} \le C\ \left[\|u\|_2 + \|\operatorname{curl} u\|_2 + \|\operatorname{div} u\|_2\right]. \tag{37}$$

Remark 6. If Ω is of class $\mathscr{C}^1$, the previous result applies also if $u \in L^p(\Omega;\mathbb{R}^3)$ with $\operatorname{curl} u \in L^p(\Omega;\mathbb{R}^3)$, $\operatorname{div} u \in L^p(\Omega;\mathbb{R})$, and $\nu\cdot u = 0$ on $\partial\Omega$ (or $\nu \times u = 0$ on $\partial\Omega$) if $p \in (1,\infty)$ (see [32, Theorem 11.2], where it was proved that if Ω is only Lipschitz, it is also true for p in a range around 2).

Remark 7. The Helmholtz projection $\mathbb{P} : L^2(\Omega;\mathbb{R}^3) \to H_D$ defined in Sect. 2 (after Remark 2) maps also W_T to the space $\{u \in W_T; \operatorname{div} u = 0\} =: \mathcal{V}_T$.

The projection $\mathbb{P}_N : L^2(\Omega;\mathbb{R}^3) \to H_N$ defined in Sect. 3 (before Definition 2) maps also W_N to the space $\{u \in W_N; \operatorname{div} u = 0\} =: \mathcal{V}_N$.

On $W_T \times W_T$, we define the following form:

$$b_T : W_T \times W_T \to \mathbb{R}, \quad b_T(u,v) = \langle \operatorname{curl} u, \operatorname{curl} v\rangle + \langle \operatorname{div} u, \operatorname{div} v\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes either the scalar or the vector-valued L^2-pairing. Similarly, we define

$$b_N : W_N \times W_N \to \mathbb{R}, \quad b_N(u,v) = \langle \operatorname{curl} u, \operatorname{curl} v\rangle + \langle \operatorname{div} u, \operatorname{div} v\rangle.$$

Proposition 5. *The Hodge-Laplacian operators B_T and B_N, defined as the associated operators in H of the forms b_T and b_N, satisfy*

$$\mathsf{D}(B_{T,N}) = \left\{ u \in W_{T,N}; \nabla \operatorname{div} u \in H, \operatorname{curl}\operatorname{curl} u \in H \textit{ and } \left| \begin{array}{l} \nu \times \operatorname{curl} u \\ (\operatorname{div} u)\nu \end{array} \right. = 0 \textit{ on } \partial\Omega \right\}$$

$$B_{T,N} u = -\Delta u, \quad u \in \mathsf{D}(B_{T,N}). \tag{38}$$

Proof. Let $u \in W_{T,N}$ and $v \in H^1_0(\Omega;\mathbb{R}^3) \subset W_{T,N}$. Then

$$b_{T,N}(u,v) = {}_{H^{-1}}\langle -\nabla \operatorname{div} u + \operatorname{curl}\operatorname{curl} u, v\rangle_{H^1_0} = {}_{H^{-1}}\langle -\Delta u, v\rangle_{H^1_0}$$

so that $B_{T,N} u = -\Delta u$ in $H^{-1}(\Omega;\mathbb{R}^3)$.

The proof of Proposition 5 is described now in the case of b_T defined on $W_T \times W_T$. The case of b_N defined on $W_N \times W_N$ can be proved with the same arguments (using $\mathbb{P}_N$ instead of $\mathbb{P}$ in what follows). Let D be the space

$$D := \left\{ u \in W_T; \nabla \operatorname{div} u \in H, \operatorname{curl}\operatorname{curl} u \in H \text{ and } \nu \times \operatorname{curl} u = 0 \text{ on } \partial\Omega \right\}.$$

If $u \in D$, then $B_T u = -\Delta u \in H$ and therefore $u \in \mathsf{D}(B_T)$.

Conversely, assume that $u \in \mathsf{D}(B_T)$. Then $(\mathrm{Id} - \mathbb{P})B_T u \in H$ satisfies for all $v \in W_T$

$$\begin{aligned} \langle (\mathrm{Id} - \mathbb{P})B_T u, v\rangle &= \langle B_T u, (\mathrm{Id} - \mathbb{P})v\rangle = b_T(u,v) - b_T(u, \mathbb{P}v) \\ &= \langle \operatorname{div} u, \operatorname{div} v\rangle = {}_{W_T'}\langle -\nabla \operatorname{div} u, v\rangle_{W_T}, \end{aligned}$$

so that $-\nabla \operatorname{div} u = (\mathrm{Id} - \mathbb{P})B_T u \in H$. Then $\operatorname{curl}\operatorname{curl} u = B_T u + \nabla \operatorname{div} u \in H$. It remains to prove that $\nu \times \operatorname{curl} u = 0$ on $\partial\Omega$. Remark that it makes sense to consider the tangential part of $w := \operatorname{curl} u$ on the boundary $\partial\Omega$ since it was just proved that $\operatorname{curl} w \in H$, and, therefore, thanks to (34), $\nu \times w \in H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)$. For all $\varphi \in H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^3) \cap L^2_{\tan}(\partial\Omega;\mathbb{R}^3)$, there exists $\phi \in H^1(\Omega;\mathbb{R}^3)$ such that $\phi_{|\partial\Omega} = \varphi$. In that case, $\phi \in W_T$, and therefore

$$\begin{aligned} \langle -\nabla \operatorname{div} u + \operatorname{curl}\operatorname{curl} u, \phi\rangle &= \langle B_T u, \phi\rangle = b_T(u,\phi) \\ &= \langle \operatorname{div} u, \operatorname{div} \phi\rangle + \langle \operatorname{curl} u, \operatorname{curl} \phi\rangle \end{aligned}$$

$$= \langle -\nabla \mathrm{div}\, u + \mathrm{curl}\,\mathrm{curl}\, u, \phi\rangle - {}_{H^{-1/2}(\partial\Omega)}\langle \nu \times \mathrm{curl}\, u, \varphi\rangle_{H^{1/2}(\partial\Omega)}.$$

It proves that ${}_{H^{-1/2}(\partial\Omega)}\langle \nu \times \mathrm{curl}\, u, \varphi\rangle_{H^{1/2}(\partial\Omega)} = 0$ for all $\varphi \in H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^3) \cap L^2_{\tan}(\partial\Omega;\mathbb{R}^3)$, and then $\nu \times \mathrm{curl}\, u = 0$ on $\partial\Omega$. □

Since the forms $b_{T,N}$ are continuous, bilinear, symmetric, coercive, and sectorial, the operators $-B_{T,N}$ generate analytic semigroups of contractions on H; $B_{T,N}$ is self-adjoint and $\mathsf{D}(B_{T,N}^{1/2}) = W_{T,N}$. The following property will be useful in the next section; it links B_T and B_N, as shown in [43, Proposition 2.2].

Lemma 4. *For $u \in H$ such that* $\mathrm{curl}\, u \in H$*, the following commutator property occurs for all $\varepsilon > 0$:*

$$\mathrm{curl}\,(1+\varepsilon B_T)^{-1}u = (1+\varepsilon B_N)^{-1}\mathrm{curl}\, u. \tag{39}$$

Proof. Let $u \in H$ such that $\mathrm{curl}\, u \in H$. Let $u_\varepsilon = (1+\varepsilon B_T)^{-1}u$ and $w_\varepsilon = (1+\varepsilon B_N)^{-1}\mathrm{curl}\, u$.

`Step 1:` $\mathrm{curl}\, u_\varepsilon \in D(B_N)$.

By (38), it holds $\mathrm{curl}\, u_\varepsilon \in H$, $\mathrm{curl}\,\mathrm{curl}\, u_\varepsilon \in H$, $\mathrm{div}\,(\mathrm{curl}\, u_\varepsilon) = 0 \in H^1(\Omega)$, $\nu \times \mathrm{curl}\, u_\varepsilon = 0$ on $\partial\Omega$, and $\mathrm{div}\,(\mathrm{curl}\, u_\varepsilon) = 0$ on $\partial\Omega$. To prove that $\mathrm{curl}\, u_\varepsilon \in D(B_T)$, it remains to show, thanks to (38), that $\mathrm{curl}\,\mathrm{curl}\,(\mathrm{curl}\, u_\varepsilon) \in H$. This is due to the fact that

$$\mathrm{curl}\,\mathrm{curl}\,(\mathrm{curl}\, u_\varepsilon) = \mathrm{curl}\,(-\Delta u_\varepsilon) \quad \text{in } H^{-1}(\Omega,\mathbb{R}^3).$$

Since

$$-\Delta u_\varepsilon = B_T(1+\varepsilon B_T)^{-1}u = \frac{1}{\varepsilon}\left(u - u_\varepsilon\right)$$

and $\mathrm{curl}\, u_\varepsilon, \mathrm{curl}\, u \in H$, the claim follows.

`Step 2:` $\mathrm{curl}\, u_\varepsilon = w_\varepsilon$.

By Step 1, $\mathrm{curl}\, u_\varepsilon \in D(B_N)$. Moreover, in the sense of distributions,

$$(1+\varepsilon B_N)(\mathrm{curl}\, u_\varepsilon) = \mathrm{curl}\, u_\varepsilon - \varepsilon\Delta \mathrm{curl}\, u_\varepsilon = \mathrm{curl}\,\Big(u_\varepsilon - \varepsilon\Delta u_\varepsilon\Big) = \mathrm{curl}\, u$$

since $u_\varepsilon - \varepsilon\Delta u_\varepsilon = (1+\varepsilon B_T)(1+\varepsilon B_T)^{-1}u = u$. Therefore,

$$\mathrm{curl}\, u_\varepsilon = (1+\varepsilon B_N)^{-1}\mathrm{curl}\, u = w_\varepsilon$$

which proves the claim.

□

To prove that the operators $B_{T,N}$ extend to L^p-spaces, it suffices to prove that their resolvents admit L^2-L^2 off-diagonal estimates. This was proved in, e.g., [36, Section 6] (see also [30]).

Proposition 6. *There exist two constants $C, c > 0$ such that for any open sets $E, F \subset \mathbb{R}^3$ such that* $\operatorname{dist}(E,F) > 0$ *and for all $t > 0$, $f \in H$ and*

$$u = (\mathrm{Id} + t^2 B_{T,N})^{-1}(1\!\!1_F f),$$

it holds

$$\|1\!\!1_E u\|_2 + t\|1\!\!1_E \operatorname{div} u\|_2 + t\|1\!\!1_E \operatorname{curl} u\|_2 \le C e^{-c\frac{\operatorname{dist}(E,F)}{t}} \|1\!\!1_F f\|_2. \tag{40}$$

Proof. Start by choosing a smooth cutoff function $\xi : \mathbb{R}^3 \to \mathbb{R}$ satisfying $\xi = 1$ on E, $\xi = 0$ on F, and $\|\nabla \xi\|_\infty \le \frac{k}{\operatorname{dist}(E,F)}$. Then define $\eta = e^{\delta\xi}$ where $\delta > 0$ is to be chosen later. Next, take the scalar product of the equation:

$$u - t^2 \Delta u = 1\!\!1_F f, \quad u \in \mathsf{D}(B_{T,N})$$

with the function $v = \eta^2 u$. Since $\eta = 1$ on F and $\|u\|_2 \le \|1\!\!1_F f\|_2$, it is easy to check then that

$$\|\eta u\|_2^2 + t^2\|\eta \operatorname{div} u\|_2^2 + t^2\|\eta \operatorname{curl} u\|_2^2$$

$$\le \|1\!\!1_F f\|_2^2 + 2\alpha\|\nabla\xi\|_\infty t^2 \|\eta u\|_2 \big(\|\eta \operatorname{div} u\|_2 + \|\eta \operatorname{curl} u\|_2\big)$$

and therefore, using the estimate on $\|\nabla\xi\|_\infty$ and choosing $\delta = \frac{\operatorname{dist}(E,F)}{4kt}$,

$$\|\eta u\|_2^2 + t^2\|\eta \operatorname{div} u\|_2^2 + t^2\|\eta \operatorname{curl} u\|_2^2 \le 2\|1\!\!1_F f\|_2^2.$$

Using now the fact that $\eta = e^\delta$ on E,

$$\|1\!\!1_E u\|_2 + t\|1\!\!1_E \operatorname{div} u\|_2 + t\|1\!\!1_E \operatorname{curl} u\|_2 \le \sqrt{2} e^{-\frac{\operatorname{dist}(E,F)}{4kt}} \|1\!\!1_F f\|_2,$$

which gives (40) with $C = \sqrt{2}$ and $c = \frac{1}{4k}$. □

With a slight modification of the proof, it can be shown that for all $\theta \in (0,\pi)$, there exist two constants $C, c > 0$ such that for any open sets $E, F \subset \mathbb{R}^3$ such that $\operatorname{dist}(E,F) > 0$, and for all $z \in \Sigma_{\pi-\theta} = \big\{\omega \in \mathbb{C} \setminus \{0\}; |\arg z| < \pi - \theta\big\}$, $f \in H$ and

$$u = (z\mathrm{Id} + B_{T,N})^{-1}(1\!\!1_F f),$$

it holds

$$|z|\|\mathbb{1}_E u\|_2 + |z|^{\frac{1}{2}}\|\mathbb{1}_E \operatorname{div} u\|_2 + |z|^{\frac{1}{2}}\|\mathbb{1}_E \operatorname{curl} u\|_2 \le C\, e^{-c\operatorname{dist}(E,F)|z|^{\frac{1}{2}}}\|\mathbb{1}_F f\|_2. \tag{41}$$

Following [31] and [10] (see also [30]), there exist Bogovskiĭ-type operators R_i, T_i, $i = 1, 2, 3$, and $K_{1,2}$, $L_{1,2}$ such that for all $p \in (1,\infty)$,

$$\begin{aligned}
&R_1 : L^p(\Omega;\mathbb{R}^3) \to W^{1,p}(\Omega;\mathbb{R}), && T_1 : L^p(\Omega;\mathbb{R}^3) \to W_0^{1,p}(\Omega;\mathbb{R}),\\
&R_2 : L^p(\Omega;\mathbb{R}^3) \to W^{1,p}(\Omega;\mathbb{R}^3), && T_2 : L^p(\Omega;\mathbb{R}^3) \to W_0^{1,p}(\Omega;\mathbb{R}^3),\\
&R_3 : L^p(\Omega;\mathbb{R}) \to W^{1,p}(\Omega;\mathbb{R}^3), && T_3 : L^p(\Omega;\mathbb{R}) \to W_0^{1,p}(\Omega;\mathbb{R}^3),\\
&K_{1,2} : L^p(\Omega;\mathbb{R}^3) \to W^{1,p}(\Omega;\mathbb{R}^3), && \text{and } L_{1,2} : L^p(\Omega;\mathbb{R}^3) \to W_0^{1,p}(\Omega;\mathbb{R}^3)
\end{aligned}$$

satisfying

$$\begin{aligned}
R_2\operatorname{curl} u + \nabla R_1 u = u - K_1 u \quad & \forall\, u \in L^p(\Omega;\mathbb{R}^3) \text{ with } \operatorname{curl} u \in L^p(\Omega;\mathbb{R})\\
& \text{and } \operatorname{curl} K_1 u = 0 \text{ if } \operatorname{curl} u = 0, \qquad (42)\\
R_3\operatorname{div} u + \operatorname{curl} R_2 u = u - K_2 u, \quad & \forall\, u \in L^p(\Omega;\mathbb{R}^3) \text{ with } \operatorname{div} u \in L^p(\Omega;\mathbb{R})\\
& \text{and } \operatorname{div} K_2 u = 0 \text{ if } \operatorname{div} u = 0, \qquad (43)\\
T_2\operatorname{curl} u + \nabla T_1 u = u - L_1 u, \quad & \forall\, u \in L^p(\Omega;\mathbb{R}^3) \text{ with } \operatorname{curl} u \in L^p(\Omega;\mathbb{R}),\\
& \nu \times u = 0 \text{ on } \partial\Omega \text{ and } \operatorname{curl} L_1 u = 0 \text{ if } \operatorname{curl} u = 0, \qquad (44)\\
T_3\operatorname{div} u + \operatorname{curl} T_2 u = u - L_2 u, \quad & \forall\, u \in L^p(\Omega;\mathbb{R}^3) \text{ with } \operatorname{div} u \in L^p(\Omega;\mathbb{R}),\\
& \nu \cdot u = 0 \text{ on } \partial\Omega \text{ and } \operatorname{div} L_2 u = 0 \text{ if } \operatorname{div} u = 0. \qquad (45)
\end{aligned}$$

With these potential operators (at this point, only the relations (43) and (45) are needed) and (41), it is easy to prove that (see, e.g., [30])

$$z(z\mathrm{Id} + B_T)^{-1} \text{ is bounded in } H_D^p \text{ and in } G_p \text{ for } p \in \left[\tfrac{6}{5}, 2\right] \text{ uniformly in } z \in \Sigma_{\pi-\theta} \tag{46}$$

where $H_D^p := \left\{u \in L^p(\Omega;\mathbb{R}^3) \text{ s.t. } \operatorname{div} u = 0 \text{ and } \nu \cdot u = 0 \text{ on } \partial\Omega\right\}$ and $G_p := \nabla W^{1,p}(\Omega;\mathbb{R})$ are defined for $p \in (1,\infty)$; if $p = 2$, then $H_D^2 = H_D$ and $G_2 = G$ defined in Sect. 2. With the same reasoning, one can prove that

$$z(z\mathrm{Id} + B_N)^{-1} \text{ is bounded in } H_N^p \text{ and in } G_{p,0} \text{ for } p \in \left[\tfrac{6}{5}, 2\right] \text{ uniformly in } z \in \Sigma_{\pi-\theta} \tag{47}$$

where $H_N^p := \left\{u \in L^p(\Omega;\mathbb{R}^3) \text{ s.t. } \operatorname{div} u = 0\right\}$ and $G_{p,0} := \nabla W_0^{1,p}(\Omega;\mathbb{R})$ are defined for $p \in (1,\infty)$; if $p = 2$, then $H_N^2 = H_N$ and $G_{2,0} = G_0$ defined in Sect. 3.

Proposition 7. *The resolvents* $\{z(z\mathrm{Id}+B_{T,N})^{-1}, z\in\Sigma_{\pi-\theta}\}$ *are uniformly bounded in* $L^p(\Omega;\mathbb{R}^3)$ *for all* $p\in(q_0',q_0)$, *where* $q_0 := \min\{6, 3+\varepsilon\}$ *(*$\varepsilon>0$ *depends on* $\partial\Omega$*).*

Proof. By [19, Theorems 11.1 and 11.2], the projections defined in Sect. 2 and Sect. 3

$$\mathbb{P} \text{ and } \mathbb{P}_N \text{ extend to bounded projections on } L^p(\Omega;\mathbb{R}^3) \text{ for } p\in\big((3+\varepsilon)', 3+\varepsilon\big), \tag{48}$$

where $\varepsilon>0$ depends on $\partial\Omega$ (and $(3+\varepsilon)'=\frac{3+\varepsilon}{2+\varepsilon}<\frac{3}{2}$); if Ω is of class $\mathscr{C}^1$, then $\varepsilon=\infty$. This means in particular that H_D^p coincides with the space $L^p_\sigma(\Omega)$ defined in (6) for all $p\in\big((3+\varepsilon)',3+\varepsilon\big)$. Therefore for all $p\in\big(q_0',2\big]$, the resolvents $\{z(z\mathrm{Id}+B_{T,N})^{-1}, z\in\Sigma_{\pi-\theta}\}$ are uniformly bounded in $L^p(\Omega;\mathbb{R}^3)$. The same result for all $p\in\big[2,q_0\big)$ is obtained by duality. □

Corollary 1. *The semigroups* $(e^{-tB_{T,N}})_{t\geq 0}$ *extend to bounded analytic semigroups on* $L^p(\Omega;\mathbb{R}^3)$ *for* $p\in\big(q_0',q_0\big)$ *and satisfy*

$$\big\|\sqrt{t}\,\mathrm{div}\,(e^{-tB_{T,N}}f)\big\|_p \leq C_p\|f\|_p \quad \big\|\sqrt{t}\,\mathrm{curl}\,(e^{-tB_{T,N}}f)\big\|_p \leq C_p'\|f\|_p \tag{49}$$

$$\big\|t\nabla\mathrm{div}\,(e^{-tB_{T,N}}f)\big\|_p \leq K_p\|f\|_p \quad \big\|t\,\mathrm{curl}\,\mathrm{curl}\,(e^{-tB_{T,N}}f)\big\|_p \leq K_p'\|f\|_p \tag{50}$$

for all $f\in L^p(\Omega;\mathbb{R}^3)$.

Proof. The estimates (49) and (50) in the corollary above come from the fact that for $p\in\big(q_0',q_0\big)$, the negative generators $B^p_{T,N}$ of the semigroups $(e^{-tB_{T,N}})_{t\geq 0}$ satisfy

$$\begin{aligned}\mathsf{D}(B^p_{T,N}) = &\{u\in L^p(\Omega;\mathbb{R}^3); \mathrm{div}\,u\in W^{1,p}(\Omega;\mathbb{R}^3), \mathrm{curl}\,u\in L^p(\Omega;\mathbb{R}^3),\\ &\mathrm{curl}\,\mathrm{curl}\,u\in L^p(\Omega;\mathbb{R}^3), \nu\cdot u=0 \text{ and } \nu\times\mathrm{curl}\,u=0 \text{ on } \partial\Omega\}\end{aligned} \tag{51}$$

$$B^p_{T,N}u = -\Delta u, \quad u\in\mathsf{D}(B^p_{T,N}).$$

This can be proved the same way we proved Proposition 5, (case $p=2$) using the fact that $\mathbb{P}$ and $\mathbb{P}_N$ are bounded in $L^p(\Omega;\mathbb{R})$. □

Remark 8. Let $w\in L^2(\Omega;\mathbb{R}^3)$ such that $\mathrm{curl}\,w\in L^2(\Omega;\mathbb{R}^3)$ and $\nu\times w=0$ on $\partial\Omega$. Then $\nu\cdot\mathrm{curl}\,w=0$ in $H^{-\frac{1}{2}}(\partial\Omega)$.

If the operator B_T is restriced on H_D and the operator B_N on H_N, the following Hodge-Stokes operators A_T and A_N defined by

$$\mathsf{D}(A_T) = \Big\{u\in H_D\cap W_T; \mathrm{curl}\,\mathrm{curl}\,u\in L^2(\Omega;\mathbb{R}^3) \text{ and } \nu\times\mathrm{curl}\,u=0 \text{ on } \partial\Omega\Big\}$$

$$A_T u = \operatorname{curl}\operatorname{curl} u \quad \text{for } u \in \mathsf{D}(A_T)$$

and

$$\mathsf{D}(A_N) = \Big\{u \in H_N \cap W_N; \operatorname{curl}\operatorname{curl} u \in L^2(\Omega;\mathbb{R}^3)\Big\},$$
$$A_N u = \operatorname{curl}\operatorname{curl} u \quad \text{for } u \in \mathsf{D}(A_N)$$

are obtained. The construction of the Hodge-Stokes operators is strongly related to the particular Hodge boundary conditions. As seen in Sect. 2 and 3, this doesn't hold in general. The reason behind this is that the Helmholtz projection $\mathbb{P}$ commutes with B_T and $\mathbb{P}_N$ commutes with B_N. Remark 8 ensures that if $u \in \mathsf{D}(A_T)$ as defined above, $\nu \cdot \operatorname{curl}\operatorname{curl} u = 0$ on $\partial\Omega$, so that $\operatorname{curl}\operatorname{curl} u \in H_D$.

The properties (46) and (47), together with a duality argument and the fact that the projections $\mathbb{P}$ and $\mathbb{P}_N$ are bounded on $L^p(\Omega;\mathbb{R}^3)$ for $p \in \big((3+\varepsilon)', 3+\varepsilon\big)$, prove that $(e^{-tA_T})_{t\geq 0}$ extends to an analytic semigroup on H^p_D (its generator is denoted by $-A_{T,p}$) and $(e^{-tA_N})_{t\geq 0}$ extends to an analytic semigroup on H^p_N (its generator is denoted by $-A_{N,p}$) for all $p \in \big[\frac{6}{5}, q_0\big)$. Moreover, the estimates (49) and (50) are valid if $B_{T,N}$ is replaced by $A_{T,N}$ for all $p \in \big[\frac{6}{5}, q_0\big)$.

Lemma 5. *If $u \in H^3_D$ and $\operatorname{curl} u \in L^3(\Omega;\mathbb{R}^3)$, then $u \in H^p_D$ for all $p \in \big[3, q_0\big)$.*

Proof. Thanks to the relation (42),

$$u = \mathbb{P}u = \mathbb{P}\big(R_2 \operatorname{curl} u + K_1 u\big)$$

since $\mathbb{P}\nabla R_1 u = 0$. The mapping properties of R_2 and K_1 show that $R_2 \operatorname{curl} u + K_1 u \in L^3(\Omega,\mathbb{R}^3) \cap L^6(\Omega,\mathbb{R}^3)$, which proves the claim of the lemma. This has been done in, e.g., [35, Sections 3 and 4]. □

Remark 9. One can actually prove that the operator $-A_{T,p}$ generates an analytic semigroup in H^p_D for all $p \in (1, 3+\varepsilon)$. The same holds for $-A_{N,p}$ on H^p_N. See [30] for more details.

Remark 10. In [54], M.E. Taylor conjectured that the Dirichlet-Stokes operator generates an analytic semigroup in H^p_D for $p \in \big((3+\varepsilon)', 3+\varepsilon\big)$, which was proved in [51]. The question of optimality of this range is still open; the counterexample provided by P. Deuring in [14] is for $p > 6$. We see here that, for the Hodge-Stokes operator, one can allow all $p \in (1, 3+\varepsilon)$.

4.2 The Nonlinear Hodge-Navier-Stokes Equations

The nonlinear Hodge-Navier-Stokes system ((NS'), (Hbc))

$$\begin{cases} \partial_t u - \Delta u + \nabla \pi - u \times \operatorname{curl} u = 0 & \text{in } (0,T)\times\Omega, \\ \operatorname{div} u = 0 & \text{in } (0,T)\times\Omega, \\ \nu \cdot u = 0, \quad \nu \times \operatorname{curl} u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

is considered for initial data u_0 in the critical space H^3_D in the abstract form:

$$u'(t) + A_{T,p}u(t) - \mathbb{P}\big(u(t)\times \operatorname{curl} u(t)\big) = 0, \quad u_0 \in H^3_D. \tag{52}$$

The idea to solve (52) is to apply the same method as in Sect. 2 and 3.

With the properties of the Hodge-Stokes semigroup listed in the previous subsection (and more particularly Lemma 5), the following existence result for (52) is almost immediate. For $T \in (0,\infty]$, define the space $\mathscr{G}_T$ by

$$\mathscr{G}_T = \Big\{ u \in \mathscr{C}_b([0,T); H^3_D) \cap \mathscr{C}((0,T); H^{3(1+\delta)}_D); \operatorname{curl} u \in \mathscr{C}((0,T); L^3(\Omega,\mathbb{R}^3)) \\ \text{with } \sup_{0<s<T} \big(\| s^{\frac{\delta}{2(1+\delta)}} u(s)\|_{3(1+\delta)} + \|\sqrt{s}\operatorname{curl} u(s)\|_3 \big) < \infty \Big\}$$

endowed with the norm

$$\|u\|_{\mathscr{G}_T} = \sup_{0<s<T} \big(\|u(s)\|_3 + \| s^{\frac{\delta}{2(1+\delta)}} u(s)\|_{3(1+\delta)} + \|\sqrt{s}\operatorname{curl} u(s)\|_3 \big),$$

where $0 < \delta < \frac{\varepsilon}{3}$ ($\varepsilon > 0$ coming from (48)).

Theorem 6. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $u_0 \in H^3_D$. Let γ and Φ be defined by*

$$\gamma(t) = e^{-tA_{T,p}} u_0, \quad t \geq 0,$$

and for $u, v \in \mathscr{G}_T$, and $t \in (0,T)$,

$$\Phi(u,v)(t) = \int_0^t e^{-(t-s)A_{T,3/2}} (\tfrac{1}{2}\mathbb{P})\big((u(s)\times \operatorname{curl} v(s) + v(s)\times \operatorname{curl} u(s)\big)\, \mathrm{d}s.$$

(i) If $\|u_0\|_3$ is small enough, then there exists a unique $u \in \mathscr{G}_\infty$ solution of $u = \gamma + \Phi(u,u)$.
(ii) For all $u_0 \in H^3_D$, there exists $T > 0$ and a unique $u \in \mathscr{G}_T$ solution of $u = \gamma + \Phi(u,u)$.

For a complete proof of this theorem, we refer to [35, Section 5].

5 Robin Boundary Conditions

As studied in [5], the system $\big((\text{NS'}), (\text{Rbc})\big)$ can also be considered. See also [22, 38, 46, 52]. Recently, this has also been investigated in an L^2-setting for smooth domains Ω but with the friction coefficient α replaced by a (time-dependent) matrix $[0,T] \times \partial\Omega \ni (t,x) \mapsto \beta(t,x) \in \mathscr{M}_3(\mathbb{R})$ with $L^\infty_{t,x}$ coefficients, admitting $\nu(x)$ as eigenvector for almost every (t,x); see [43]. It is also worth mentioning that the material here is part of a project with Jürgen Saal [44]. In the following, consider $\alpha \geq 0$ a constant. Note that the proofs in this section go through if $\alpha : \partial\Omega \to [0,\infty)$ is an L^∞-function.

5.1 The Robin-Hodge-Laplacian

Recall the notations at the beginning of Sect. 4.1: $H = L^2(\Omega;\mathbb{R}^3)$ and

$$W_T := \big\{u \in H; \operatorname{curl} u \in H, \operatorname{div} u \in L^2(\Omega;\mathbb{R}) \text{ and } \nu \cdot u = 0 \text{ on } \partial\Omega\big\}.$$

On $W_T \times W_T$, define the form:

$$b_\alpha : W_T \times W_T \to \mathbb{R}, \quad b_\alpha(u,v) = \langle \operatorname{curl} u, \operatorname{curl} v\rangle_\Omega + \langle \operatorname{div} u, \operatorname{div} v\rangle_\Omega + \langle \alpha\, u, v\rangle_{\partial\Omega}.$$

Recall that according to (36), any $u \in W_T$ admits an L^2-trace on $\partial\Omega$, so that $\langle \alpha u, v\rangle_{\partial\Omega}$ makes sense for every $u, v \in W_T$.

Remark 11. The previous property holds also in L^p, $1 < p < \infty$, provided Ω is of class $\mathscr{C}^1$. More precisely, any $u \in L^p(\Omega,\mathbb{R}^3)$ with $\operatorname{curl} u \in L^p(\Omega,\mathbb{R}^3)$, $\operatorname{div} u \in L^p(\Omega,\mathbb{R})$, and $\nu \cdot u = 0$ on $\partial\Omega$ admits an L^p-trace on $\partial\Omega$ which satisfies

$$\|u_{|\partial\Omega}\|_{L^p(\partial\Omega;\mathbb{R}^3)} \leq C\big(\|u\|_p + \|\operatorname{curl} u\|_p + \|\operatorname{div} u\|_p\big).$$

See, e.g., [33, Proposition 6.2]: in the case of a $\mathscr{C}^1$ domain Ω, the exponent q_Ω in that result (related to the solvability of the Poisson problem for Neumann boundary data and the regularity of the Poisson problem for Dirichlet boundary data) is equal to ∞.

The form b_α is continuous, bilinear, symmetric, coercive, and sectorial, so that the associated operator B_α on H is self-adjoint; $-B_\alpha$ generates an analytic semigroup of contractions and $D(B_\alpha^{1/2}) = W_T$. The operator B_α is called the Hodge-Robin-Laplacian. It has the following description:

$$\mathsf{D}(B_\alpha) = \Big\{u \in W_T; \nabla \operatorname{div} u \in H, \operatorname{curl}\operatorname{curl} u \in H \text{ and } \nu \times \operatorname{curl} u = \alpha\, u \text{ on } \partial\Omega\Big\}$$

$$B_\alpha u = -\,\Delta u, \quad u \in \mathsf{D}(B_\alpha). \tag{53}$$

Remark that for $u \in W_T$, $u_{|\partial\Omega} \in L^2(\partial\Omega;\mathbb{R}^3)$ and if moreover $\operatorname{curl}\operatorname{curl} u \in H$, the tangential vector field $\nu \times \operatorname{curl} u$ belongs to $H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)$. Therefore, the identity $\nu \times \operatorname{curl} u = \alpha\, u$ above holds in $H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)$. The proof of (53) follows the lines of the proof of Proposition 5, thanks to the following result (see, e.g., [43, Lemma 2.3], inspired by [33, Proof of Proposition 2.4 (iii)]) of which we also give the proof.

Lemma 6. *1. Let $g \in L^2(\partial\Omega,\mathbb{R}^3)$. Then there exists $w \in H$ with $\operatorname{curl} w \in H$ such that for all $\phi \in W_T$*

$$\langle g,\phi\rangle_{\partial\Omega} = \langle \operatorname{curl} w,\phi\rangle_\Omega - \langle w,\operatorname{curl}\phi\rangle_\Omega. \tag{54}$$

Moreover, there exists $C > 0$ such that

$$\|w\|_H + \|\operatorname{curl} w\|_H \le C\,\|g\|_{L^2(\partial\Omega,\mathbb{R}^3)}. \tag{55}$$

2. If in addition $g \in L^2_{\tan}(\partial\Omega;\mathbb{R}^3)$ (which means that $g \in L^2(\partial\Omega;\mathbb{R}^3)$ and $\nu\cdot g = 0$ on $\partial\Omega$), then there exists $w \in H$ such that $\operatorname{curl} w \in H$ and (54) holds for all $\phi \in H^1(\Omega)$. And in that case $g = \nu \times w$ in $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$.

Proof. 1. Define the space $X := \{(\phi,\operatorname{curl}\phi);\phi \in W_T\}$. It is a subspace of $H \times H$. As already mentioned, every $\phi \in W_T$ admits an L^2-trace at the boundary $\partial\Omega$ and therefore $\nu \times \phi \in L^2(\partial\Omega;\mathbb{R}^3)$ for all $\phi \in W_T$. Since $g \in L^2(\partial\Omega;\mathbb{R}^3)$, it is immediate that $\nu \times g \in L^2(\partial\Omega;\mathbb{R}^3) = \big(L^2(\partial\Omega;\mathbb{R}^3)\big)'$. Thus, $\nu \times g$ acts as a linear functional on X as follows:

$$(\nu \times g)(\phi,\operatorname{curl}\phi) := \langle \nu \times g,\nu\times\phi\rangle_{\partial\Omega} \quad \text{for all } \phi \in W_T.$$

By the Hahn-Banach theorem, there exist $(v_1,v_2) \in H \times H$ such that

$$(\nu \times g)(\phi,\operatorname{curl}\phi) = \langle v_1,\operatorname{curl}\phi\rangle_\Omega + \langle v_2,\phi\rangle_\Omega \quad \text{for all } \phi \in W_T,$$

where $\big(H \times H\big)'$ has been identified with $H \times H$. Choose $\phi \in H^1_0(\Omega;\mathbb{R}^3) \subset W_T$ and obtain that

$$0 = {}_{H^{-1}}\langle \operatorname{curl} v_1 + v_2,\phi\rangle_{H^1_0}.$$

This gives that $\operatorname{curl} v_1 + v_2 = 0$ in $H^{-1}(\Omega;\mathbb{R}^3)$. Set $w := -v_1 \in H$, so that $\operatorname{curl} w = v_2 \in H$. Moreover,

$$\langle \nu \times g,\nu\times\phi\rangle_{\partial\Omega} = -\langle w,\operatorname{curl}\phi\rangle_\Omega + \langle \operatorname{curl} w,\phi\rangle_\Omega \quad \text{for all } \phi \in W_T. \tag{56}$$

Since $\phi \in W_T$, $\phi_{|\partial\Omega} \in L^2_{\tan}(\partial\Omega,\mathbb{R}^3)$, and it is clear that $\phi = (\nu\times\phi)\times\nu$, so that the left-hand side of (56) coincides with

$$\langle g, \phi\rangle_{\partial\Omega} \qquad \text{for all } \phi \in W_T, \tag{57}$$

which proves (54).

The existence of $C > 0$ such that (55) holds follows from the closed graph theorem since $\{u \in H; \operatorname{curl} u \in H\}$ is complete for the norm $\|u\|_2 + \|\operatorname{curl} u\|_2$.

2. Assume now that $g \in L^2_{\tan}(\partial\Omega; \mathbb{R}^3)$. Let $w \in H$ such that $\operatorname{curl} w \in H$ and (54) holds. Since $\nu \times g \in L^2(\partial\Omega; \mathbb{R}^3)$, we can approach it in $L^2(\partial\Omega; \mathbb{R}^3)$ by a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of vector fields $\varphi_n \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$. In particular,

$$\varphi_n \times \nu \longrightarrow (\nu \times g) \times \nu = g \quad \text{in } L^2(\partial\Omega; \mathbb{R}^3) \text{ as } n \to \infty.$$

By assertion 1, for each $n \in \mathbb{N}$, there exists $w_n \in H$ such that $\operatorname{curl} w_n \in H$ satisfying

$$\langle \varphi_n \times \nu, \phi\rangle_{\partial\Omega} = \langle \operatorname{curl} w_n, \phi\rangle_\Omega - \langle w_n, \operatorname{curl} \phi\rangle_\Omega \quad \text{for all } \phi \in W_T.$$

Thanks to the estimate (55), it is immediate that

$$w_n \xrightarrow[n\to\infty]{} w \quad \text{and} \quad \operatorname{curl} w_n \xrightarrow[n\to\infty]{} \operatorname{curl} w \quad \text{in } H.$$

Let now $\phi \in H^1(\Omega; \mathbb{R}^3)$. For $\varepsilon > 0$, let $\phi_\varepsilon = (1 + \varepsilon B_T)^{-1}\phi$. Then $\phi_\varepsilon \in W_T$, and thanks to Lemma 4,

$$\phi_\varepsilon \xrightarrow[\varepsilon\to 0]{} \phi \quad \text{and} \quad \operatorname{curl} \phi_\varepsilon = (1 + \varepsilon B_N)^{-1} \operatorname{curl} \phi \xrightarrow[\varepsilon\to 0]{} \operatorname{curl} \phi \quad \text{in } H.$$

This implies also that

$$\nu \times \phi_\varepsilon \xrightarrow[\varepsilon\to 0]{} \nu \times \phi \quad \text{in } H^{-1/2}(\partial\Omega; \mathbb{R}^3).$$

Therefore, for all $\varepsilon > 0$ and $n \in \mathbb{N}$

$$\langle \nu \times \phi_\varepsilon, \varphi_n\rangle_{\partial\Omega} = \langle \varphi_n \times \nu, \phi_\varepsilon\rangle_{\partial\Omega} = \langle \operatorname{curl} w_n, \phi_\varepsilon\rangle_\Omega - \langle w_n, \operatorname{curl} \phi_\varepsilon\rangle_\Omega.$$

First take the limit as ε goes to 0 and obtain (recall that $\varphi_n \in H^{1/2}(\partial\Omega; \mathbb{R}^3)$)

$${}_{H^{-1/2}}\langle \nu \times \phi, \varphi_n\rangle_{H^{1/2}} = \langle \operatorname{curl} w_n, \phi\rangle_\Omega - \langle w_n, \operatorname{curl} \phi\rangle_\Omega.$$

Since $\phi \in H^1(\Omega, \mathbb{R}^3)$, the first term of the latter equation is also equal to $\langle \varphi_n \times \nu, \phi\rangle_{\partial\Omega}$. Taking the limit as n goes to ∞ yields

$$\langle g, \phi\rangle_{\partial\Omega} = \langle \operatorname{curl} w, \phi\rangle_\Omega - \langle w, \operatorname{curl} \phi\rangle_\Omega$$

which proves the claim made in 2.

□

Remark 12. If Ω is of class $\mathscr{C}^1$, one can prove that Lemma 6 is also valid in L^p instead of L^2 for all $p \in (1, \infty)$, identifying the dual of L^p with $L^{p'}$ (noting that q_0 defined in Proposition 7 is equal to ∞).

Proof of (53). For the time being, denote by D_α the set on the right-hand side of (53). Let $u \in D_\alpha$: $\Delta u = -\operatorname{curl}\operatorname{curl} u + \nabla \operatorname{div} u \in H$, and for all $v \in W_T \cap H^1(\Omega; \mathbb{R}^3)$,

$$\begin{aligned}
\langle -\Delta u, v\rangle_\Omega &= \langle \operatorname{curl}\operatorname{curl} u, v\rangle_\Omega - \langle \nabla \operatorname{div} u, v\rangle_\Omega \\
&= \langle \operatorname{curl} u, \operatorname{curl} v\rangle_\Omega + \langle \nu \times \operatorname{curl} u, v\rangle_{\partial\Omega} + \langle \operatorname{div} u, \operatorname{div} v\rangle_\Omega \\
&= \langle \operatorname{curl} u, \operatorname{curl} v\rangle_\Omega + \langle \operatorname{div} u, \operatorname{div} v\rangle_\Omega + \alpha\langle u, v\rangle_{\partial\Omega} \\
&= b_\alpha(u, v).
\end{aligned}$$

The second equality comes from the integration by parts formula. In the third equality, the characterization of elements in D_α has been used. Thanks to the density of $W_T \cap H^1(\Omega; \mathbb{R}^3)$ in W_T, this proves the inclusion $D_\alpha \subseteq D(B_\alpha)$ and that $B_\alpha u = -\Delta u$ for $u \in D_\alpha$.

Conversely, let $u \in D(B_\alpha)$. Let $\eta = -B_\alpha u \in H$, $g = \alpha u$. Since $u_{|\partial\Omega} \in L^2_{\tan}(\partial\Omega; \mathbb{R}^3)$, Lemma 6 shows the existence of $w \in H$ with $\operatorname{curl} w \in H$ such that $\alpha u = \nu \times w$ on $\partial\Omega$. Therefore, the boundary value $g = \alpha u$ satisfies the conditions of [33, Theorem 1.2] with $p = 2$. Then there exists a unique $\tilde u$ satisfying

$$\begin{cases}
\tilde u \in W_T, \operatorname{curl}\operatorname{curl} \tilde u \in H, \operatorname{div} \tilde u \in H^1(\Omega), \\
\Delta \tilde u = \eta \in H, \\
\nu \times \operatorname{curl} \tilde u = g \in H^{-1/2}(\partial\Omega; \mathbb{R}^3),
\end{cases} \tag{58}$$

For all $v \in W_T$, integrating by parts,

$$\begin{aligned}
\langle \operatorname{curl} \tilde u, \operatorname{curl} v\rangle_\Omega + \langle \operatorname{div} \tilde u, \operatorname{div} v\rangle_\Omega &= \langle -\Delta \tilde u, v\rangle_\Omega - \langle \nu \times \operatorname{curl} \tilde u, v\rangle_{\partial\Omega} \\
&= \langle -\eta, v\rangle_\Omega - \langle g, v\rangle_{\partial\Omega} \\
&= \langle B_\alpha u, v\rangle_\Omega - \langle \alpha u, v\rangle_{\partial\Omega} \\
&= b_\alpha(u, v) - \alpha\langle u, v\rangle_{\partial\Omega} \\
&= \langle \operatorname{curl} u, \operatorname{curl} v\rangle_\Omega + \langle \operatorname{div} u, \operatorname{div} v\rangle_\Omega .
\end{aligned}$$

The second equality comes from the fact that $\tilde u$ is the solution of (58). The third equality is a simple reformulation of the previous line using the notations introduced before. The fourth equality uses the fact that B_α is the operator associated with the form b_α. Finally, the last equality comes directly from the definition of b_α. Therefore, we proved that $v = u - \tilde u \in W_T$ and satisfies $\operatorname{curl} v = 0$ and $\operatorname{div} v = 0$. Since Ω is simply connected, this proves that $v = 0$, or equivalently $u = \tilde u$, and then that $u \in D_\alpha$ from which follows the inclusion $D(B_\alpha) \subseteq D_\alpha$.

Ultimately, it has been proved that $D(B_\alpha) = D_\alpha$. □

As in the case of Proposition 6, Gaffney-type estimates hold.

Proposition 8. *There exist two constants $C, c > 0$ such that for any open sets $E, F \subset \mathbb{R}^3$ such that* $\operatorname{dist}(E, F) > 0$ *and for all $t > 0$, $f \in H$ and*

$$u = (\mathrm{Id} + t^2 B_\alpha)^{-1}(1\!\!1_F f),$$

it holds

$$\|1\!\!1_E u\|_2 + t\|1\!\!1_E \operatorname{div} u\|_2 + t\|1\!\!1_E \operatorname{curl} u\|_2 + t\sqrt{\alpha}\,\|1\!\!1_E u\|_{L^2(\partial\Omega;\mathbb{R}^3)} \leq C e^{-c\frac{\operatorname{dist}(E,F)}{t}}\|1\!\!1_F f\|_2. \tag{59}$$

Proof. The proof goes as in the case $\alpha = 0$ (Proposition 6 for B_T). Choose a smooth cutoff function $\xi : \mathbb{R}^3 \to \mathbb{R}$ satisfying $\xi = 1$ on E, $\xi = 0$ on F, and $\|\nabla\xi\|_\infty \leq \frac{k}{\operatorname{dist}(E,F)}$. Then define $\eta = e^{\delta\xi}$ where $\delta > 0$ is to be chosen later. Next, take the scalar product of the equation:

$$u - t^2\Delta u = 1\!\!1_F f, \quad u \in \mathsf{D}(B_\alpha)$$

with the function $v = \eta^2 u$. Since $\eta = 1$ on F and $\|u\|_2 \leq \|1\!\!1_F f\|_2$, it is easy to check then that

$$\|\eta u\|_2^2 + t^2\|\eta \operatorname{div} u\|_2^2 + t^2\|\eta \operatorname{curl} u\|_2^2 + t^2\alpha\|\eta u\|^2_{L^2(\partial\Omega;\mathbb{R}^3)}$$

$$\leq \|1\!\!1_F f\|_2^2 + 2\alpha\|\nabla\xi\|_\infty t^2\|\eta u\|_2\big(\|\eta \operatorname{div} u\|_2 + \|\eta \operatorname{curl} u\|_2\big)$$

and therefore, using the estimate on $\|\nabla\xi\|_\infty$ and choosing $\delta = \frac{\operatorname{dist}(E,F)}{4kt}$,

$$\|\eta u\|_2^2 + t^2\|\eta \operatorname{div} u\|_2^2 + t^2\|\eta \operatorname{curl} u\|_2^2 + t^2\alpha\|\eta u\|^2_{L^2(\partial\Omega;\mathbb{R}^3)} \leq 2\|1\!\!1_F f\|_2^2.$$

Using now the fact that $\eta = e^\delta$ on E,

$$\|1\!\!1_E u\|_2 + t\|1\!\!1_E \operatorname{div} u\|_2 + t\|1\!\!1_E \operatorname{curl} u\|_2 + t\sqrt{\alpha}\,\|1\!\!1_E u\|_{L^2(\partial\Omega;\mathbb{R}^3)} \leq \sqrt{2}e^{-\frac{\operatorname{dist}(E,F)}{4kt}}\|1\!\!1_F f\|_2,$$

which gives (59) with $C = \sqrt{2}$ and $c = \frac{1}{4k}$. □

As before, with a slight modification of the proof, it can be shown that for all $\theta \in (0, \pi)$ there exist two constants $C, c > 0$ such that for any open sets $E, F \subset \mathbb{R}^3$ such that $\operatorname{dist}(E, F) > 0$ and for all $z \in \Sigma_{\pi-\theta} = \big\{\omega \in \mathbb{C} \setminus \{0\}; |\arg z| < \pi - \theta\big\}$, $f \in H$ and

$$u = (z\mathrm{Id} + B_\alpha)^{-1}(1\!\!1_F f),$$

it holds

$$|z|\|\mathbb{1}_E u\|_2 + |z|^{\frac12}\|\mathbb{1}_E \operatorname{div} u\|_2 + |z|^{\frac12}\|\mathbb{1}_E \operatorname{curl} u\|_2 \\ + |z|^{\frac12}\sqrt{\alpha}\,\|\mathbb{1}_E u\|_{L^2(\partial\Omega;\mathbb{R}^3)} \le C\, e^{-c\,\operatorname{dist}(E,F)|z|^{\frac12}}\|\mathbb{1}_F f\|_2. \tag{60}$$

With the same arguments as for the Hodge-Laplacian, the analogue of Proposition 7 and Corollary 1 can be obtained, as well as (51) for B_α: for all $p \in \big(q_0', q_0\big)$:

$$\big\{z(z\mathrm{Id} + B_\alpha)^{-1}, z \in \Sigma_{\pi-\theta}\big\} \text{ is uniformly bounded in } L^p(\Omega;\mathbb{R}^3); \tag{61}$$

$$(e^{-tB_\alpha})_{t\ge 0} \text{ extends to a bounded analytic semigroup on } L^p(\Omega;\mathbb{R}^3); \tag{62}$$

$$\big\|\sqrt{t}\operatorname{div}(e^{-tB_\alpha} f)\big\|_p \le C_p\|f\|_p, \quad \big\|\sqrt{t}\operatorname{curl}(e^{-tB_\alpha} f)\big\|_p \le C_p'\|f\|_p; \tag{63}$$

$$\big\|t\nabla\operatorname{div}(e^{-tB_\alpha} f)\big\|_p \le K_p\|f\|_p, \quad \big\|t\operatorname{curl}\operatorname{curl}(e^{-tB_\alpha} f)\big\|_p \le K_p'\|f\|_p. \tag{64}$$

Moreover, if Ω is of class $\mathscr{C}^1$, the following description of $B_{\alpha,p}$, the negative generator of $(e^{-tB_\alpha})_{t\ge 0}$ in $L^p(\Omega;\mathbb{R}^3)$ holds:

$$\mathsf{D}(B_{\alpha,p}) = \big\{u \in L^p(\Omega;\mathbb{R}^3); \operatorname{div} u \in W^{1,p}(\Omega;\mathbb{R}^3), \operatorname{curl} u \in L^p(\Omega;\mathbb{R}^3), \\ \operatorname{curl}\operatorname{curl} u \in L^p(\Omega;\mathbb{R}^3), \nu\cdot u = 0 \text{ and } \nu\times\operatorname{curl} u = \alpha\, u \text{ on } \partial\Omega\big\} \tag{65}$$

$$B_{\alpha,p}u = -\Delta u, \quad u \in \mathsf{D}(B_{\alpha,p}),$$

To prove that, the result in Remark 6 has been used, as well as the solvability of (58) in L^p for p in the interval $\big((3+\varepsilon)', 3+\varepsilon\big) = (1,\infty)$ in that case ([33, Theorem 1.2] is also valid in this range of p).

5.2 The Robin-Hodge-Stokes Operator

From now on, assume that Ω is of class $\mathscr{C}^1$. Let $p \in (1,\infty)$. Let $g \in L^p(\Omega;\mathbb{R}^3)$, with $\operatorname{div} g = 0$. By Remark 1 (also valid for $p \in (1,\infty)$ with the obvious changes), it holds $\nu\cdot g \in B^{-1/p}_{p,p}(\partial\Omega)$ and also $\nu\cdot g$ satisfies the condition ${}_{B^{-1/p}_{p,p}(\partial\Omega)}\langle \nu\cdot g, \mathbb{1}\rangle_{B^{1/p}_{p',p'}(\partial\Omega)} = 0$. By [19, Corollary 9.3], the problem

$$q \in W^{1,p}(\Omega), \quad \Delta q = 0 \text{ in } \Omega, \quad \partial_\nu q = \nu\cdot g \text{ on } \partial\Omega \tag{66}$$

has a unique (modulo constants) solution satisfying moreover

$$\|\nabla q\|_p \lesssim \|\nu\cdot g\|_{B^{-1/p}_{p,p}(\partial\Omega)}. \tag{67}$$

Consider the operator

$$\Gamma_p : \mathsf{D}(B_{\alpha,p}) \longrightarrow W^{1,p}(\Omega), \quad u \longmapsto q$$

where q is the solution of (66) with $g = -\mathrm{curl}\,\mathrm{curl}\,u$.

Lemma 7. *For $p \in (1,\infty)$, $u \in \mathsf{D}(B_{\alpha,p})$, the following estimate holds*

$$\|\nabla \Gamma_p u\|_p \lesssim \alpha\big(\|\mathrm{curl}\,u\|_p + \|\mathrm{div}\,u\|_p\big). \tag{68}$$

Proof. Let $p \in (1,\infty)$ and $u \in \mathsf{D}(B_{\alpha,p})$. Let $\varphi \in B^{1/p}_{p',p'}(\partial\Omega)$. Let $\Phi \in W^{1,p'}(\Omega)$, so that $\Phi_{|\partial\Omega} = \varphi$ (recall that $\frac{1}{p} = 1 - \frac{1}{p'}$). Thanks to the description of $\mathsf{D}(B_{\alpha,p})$ given by (65) and the formula (34) (also valid in L^p), there holds

$$\begin{aligned} {}_{B^{-1/p}_{p,p}(\partial\Omega)}\langle \nu \cdot \mathrm{curl}\,\mathrm{curl}\,u, \varphi\rangle_{B^{1/p}_{p',p'}(\partial\Omega)} &= \langle \mathrm{curl}\,\mathrm{curl}\,u, \nabla\Phi\rangle_\Omega = \langle \nu \times \mathrm{curl}\,u, \nabla\Phi\rangle_{\partial\Omega} \\ &= \alpha\,\langle u, \nabla\Phi\rangle_{\partial\Omega} = \alpha\,\langle \mathrm{curl}\,w, \nabla\Phi\rangle_\Omega, \end{aligned}$$

where $w \in L^p(\Omega;\mathbb{R}^3)$ with $\mathrm{curl}\,w \in L^p(\Omega;\mathbb{R}^3)$ is determined by Lemma 6, 2 (for $g = u$; see Remark 6). Therefore by Remark 11,

$$\begin{aligned} \|\nu \cdot \mathrm{curl}\,\mathrm{curl}\,u\|_{B^{-1/p}_{p,p}(\partial\Omega)} &\le C\,\|\mathrm{curl}\,w\|_p \le C\,\|u\|_{L^p(\partial\Omega;\mathbb{R}^3)} \\ &\le C\big(\|u\|_p + \|\mathrm{curl}\,u\|_p + \|\mathrm{div}\,u\|_p\big). \end{aligned}$$

Since Ω is bounded, $\|u\|_p$ can be estimated in terms of $\|\mathrm{curl}\,u\|_p$ and $\|\mathrm{div}\,u\|_p$, which gives (68). □

Next result links the operator Γ_p and $B_{\alpha,p}$ with the Robin-Hodge-Stokes resolvent problem for $z \in \Sigma_{\pi-\theta}$:

$$\begin{cases} zu - \Delta u + \nabla q = f & \text{in } \Omega, \\ \mathrm{div}\,u = 0 & \text{in } \Omega, \\ \nu \cdot u = 0, \ \nu \times \mathrm{curl}\,u = \alpha\,u & \text{on } \partial\Omega. \end{cases} \tag{69}$$

Proposition 9. *Let $p \in (1,\infty)$. Let $z \in \Sigma_{\pi-\theta}$ and $f \in H^p_D$. Then $(u,q) \in \mathsf{D}(B_{\alpha,p}) \times W^{1,p}(\Omega)$ is a solution of (69) if, and only if, $u \in \mathsf{D}(B_{\alpha,p}) \cap H^p_D$ satisfies $zu - \Delta u + \nabla\Gamma_p u = f$ and in that case $q = \Gamma_p u$.*

Proof. $\Rightarrow$: Assume that $(u,q) \in \mathsf{D}(B_{\alpha,p}) \times W^{1,p}(\Omega)$ is a solution of (69). Applying the divergence to the first equation of (69) and using the fact that $\mathrm{div}\,u = 0$, there holds $\Delta\pi = 0$. Moreover, taking the normal component at the boundary of the same equation, $\partial_\nu q = \nu \cdot \Delta u = -\nu \cdot \mathrm{curl}\,\mathrm{curl}\,u$ (recall that, since $f \in H^p_D$, $\nu \cdot f = 0$ on $\partial\Omega$) and therefore q satisfies (66) with $g = -\mathrm{curl}\,\mathrm{curl}\,u$, which implies by definition of Γ_p that $q = \Gamma_p u$. This shows that $u \in \mathsf{D}(B_{\alpha,p}) \cap H^p_D$ and satisfies $zu - \Delta u + \nabla\Gamma_p u = f$.

$\Leftarrow$: Conversely, let $u \in \mathsf{D}(B_{\alpha,p}) \cap H^p_D$ satisfying $zu - \Delta u + \nabla\Gamma_p u = f$ and define $v := \operatorname{div} u \in W^{1,p}(\Omega)$. Then v satisfies $zv - \Delta v = 0$ in Ω: apply the divergence to $zu - \Delta u + \nabla\Gamma_p u = f$ and remark that $\operatorname{div} f = 0$ and $\operatorname{div} \nabla\Gamma_p u = \Delta\Gamma_p u = 0$. Moreover, taking the normal component of $zu - \Delta u + \nabla\Gamma_p u = f$ at the boundary, $-\partial_\nu v + \nu \cdot \operatorname{curl}\operatorname{curl} u + \partial_\nu \Gamma_p u = 0$ on $\partial\Omega$ (we wrote $-\Delta u = -\nabla v + \operatorname{curl}\operatorname{curl} u$), and therefore $\partial_\nu v = 0$ on $\partial\Omega$. Uniqueness of the Neumann problem for the Laplacian,

$$\big(zv - \Delta v = 0 \text{ in } \Omega \quad \text{and} \quad \partial_\nu v = 0 \text{ on } \partial\Omega\big) \Longrightarrow \big(v = 0\big),$$

shows that $v = \operatorname{div} u = 0$. Therefore, $(u, \Gamma_p u) \in \mathsf{D}(B_{\alpha,p}) \times W^{1,p}(\Omega)$ is a solution of (69).

□

Proposition 9 allows to define the part of $B_{\alpha,p}$ in H^p_D. In this case, the pressure term $\nabla\Gamma_p u$ appears as a small perturbation of the main term given by $-\Delta u$. This happened already in the case of Hodge boundary conditions ($\alpha = 0$): $\nabla\Gamma_p u = 0$; see (68) and the comment Sect. 4 after the definitions of A_T and A_N.

Definition 3. Let $p \in (1, \infty)$. The Robin-Hodge-Stokes operator denoted by $A_{\alpha,p}$ is an unbounded operator in H^p_D defined by

$$\mathsf{D}(A_{\alpha,p}) = \mathsf{D}(B_{\alpha,p}) \cap H^p_D, \qquad A_{\alpha,p} u = -\Delta u + \nabla\Gamma_p u, \quad u \in \mathsf{D}(A_{\alpha,p}). \tag{70}$$

Remark 13. If $p = 2$, it is easy to see that $A_{\alpha,2}$ is the operator associated with the continuous, bilinear, symmetric, coercive form a_α defined as follows:

$$a_\alpha : (W_T \cap H_D) \times (W_T \cap H_D) \to \mathbb{R}, \quad a_\alpha(u, v) := \langle \operatorname{curl} u, \operatorname{curl} v\rangle_\Omega + \langle \alpha\, u, v\rangle_{\partial\Omega}.$$

Therefore, $A_{\alpha,2}$ is self-adjoint, and $-A_{\alpha,2}$ is the generator of an analytic semigroup of contractions in H_D.

Lemma 8. *Let $p \in [2, \infty)$ and $u \in \mathsf{D}(A_{\alpha,p})$. Then $u \in L^{\frac{9p}{4}}(\Omega; \mathbb{R}^3)$.*

Proof. By definition, if $u \in \mathsf{D}(A_{\alpha,p})$, then $u, \operatorname{curl} u \in L^p(\Omega; \mathbb{R}^3)$, $\operatorname{div} u = 0 \in L^p(\Omega)$, and $\nu \cdot u = 0$ on $\partial\Omega$. By [32, Theorem 11.2] (note that $B^{1/p}_{p,p} \hookrightarrow L^{\frac{3p}{2}}$ in dimension 3), there holds $u \in L^{\frac{3p}{2}}(\Omega; \mathbb{R}^3)$. Apply the same reasoning to $\operatorname{curl} u$: $\operatorname{curl} u, \operatorname{curl}\operatorname{curl} u \in L^p(\Omega; \mathbb{R}^3)$, $\operatorname{div}\operatorname{curl} u = 0 \in L^p(\Omega)$, and $\nu \times \operatorname{curl} u = \alpha\, u \in L^p(\partial\Omega; \mathbb{R}^3)$, so that $\operatorname{curl} u \in L^{\frac{3p}{2}}(\Omega; \mathbb{R}^3)$. Using again that $\nu \cdot u = 0$ on $\partial\Omega$, there holds $u \in L^{\frac{9p}{4}}(\Omega; \mathbb{R}^3)$. □

Theorem 7. *For all $p \in (1, \infty)$, the operator $-A_{\alpha,p}$ generates an analytic semigroup in H^p_D satisfying the estimates*

$$\left\| \sqrt{t}\, \mathrm{curl}\, (e^{-tA_{\alpha,p}} f) \right\|_p \le C_p \| f \|_p \quad \textit{and} \quad \left\| t\, \mathrm{curl}\, \mathrm{curl}\, (e^{-tA_{\alpha,p}} f) \right\|_p \le K_p \| f \|_p, \tag{71}$$

for all $f \in H_D^p$ *and all* $t > 0$ *if* $p \ge 2$.

Proof. Let $z \in \Sigma_{\pi-\theta}$. By Proposition 9,

$$(z\mathrm{Id} + A_{\alpha,p}) = \big(\mathrm{Id} - \nabla\Gamma_p (z\mathrm{Id} + B_{\alpha,p})^{-1}\big)(z\mathrm{Id} + B_{\alpha,p}).$$

Lemma 7 and (63) imply that for all $f \in L^p(\Omega;\mathbb{R}^3)$,

$$\begin{aligned} \|\nabla\Gamma_p (z\mathrm{Id} + B_{\alpha,p})^{-1} f\|_p &\lesssim \alpha\big(\|\mathrm{curl}\,(z\mathrm{Id} + B_{\alpha,p})^{-1} f\|_p \\ &\quad + \|\mathrm{div}\,(z\mathrm{Id} + B_{\alpha,p})^{-1} f\|_p\big) \le C\, \tfrac{\alpha}{\sqrt{|z|}}\, \|f\|_p. \end{aligned}$$

This proves that, for $|z|$ large enough ($|z| \ge 4C^2\alpha^2$), $z\mathrm{Id} + A_{\alpha,p} : \mathsf{D}(A_{\alpha,p}) \to H_D^p$ is invertible with

$$(z\mathrm{Id} + A_{\alpha,p})^{-1} = (z\mathrm{Id} + B_{\alpha,p})^{-1}\big(\mathrm{Id} - \nabla\Gamma_p (z\mathrm{Id} + B_{\alpha,p})^{-1}\big)^{-1}$$

and

$$\left\| z(z\mathrm{Id} + A_{\alpha,p})^{-1} \right\|_{\mathscr{L}(H_D^p)} \le 2 \left\| z(z\mathrm{Id} + B_{\alpha,p})^{-1} \right\|_{\mathscr{L}(L^p(\Omega;\mathbb{R}^3))} \lesssim 1.$$

Moreover, the same reasoning gives

$$\begin{aligned} &\left\| \sqrt{|z|}\, \mathrm{curl}\,(z\mathrm{Id} + A_{\alpha,p})^{-1} \right\|_{\mathscr{L}(H_D^p; L^p(\Omega;\mathbb{R}^3))} \\ &\quad \le 2 \left\| \sqrt{|z|}\, \mathrm{curl}\,(z\mathrm{Id} + B_{\alpha,p})^{-1} \right\|_{\mathscr{L}(L^p(\Omega;\mathbb{R}^3))} \lesssim 1 \end{aligned} \tag{72}$$

and

$$\left\| \mathrm{curl}\,\mathrm{curl}\,(z\mathrm{Id} + A_{\alpha,p})^{-1} \right\|_{\mathscr{L}(H_D^p; L^p(\Omega;\mathbb{R}^3))} \le 2 \left\| \mathrm{curl}\,\mathrm{curl}\,(z\mathrm{Id} + B_{\alpha,p})^{-1} \right\|_{\mathscr{L}(L^p(\Omega;\mathbb{R}^3))} \lesssim 1 \tag{73}$$

To prove that $z\mathrm{Id} + A_{\alpha,p} : \mathsf{D}(A_{\alpha,p}) \to H_D^p$ is invertible if $z \in \Sigma_{\pi-\theta}$ with $|z| \le 4C^2\alpha^2$, proceed by induction. The assertion is proved for $p \ge 2$ (the range is obtained $1 < p \le 2$ by duality since $A_{\alpha,2}$ is self-adjoint in H_D). Assume first that $p \in \big[2, \frac{9}{2}\big]$, so that $\mathsf{D}(A_{\alpha,2}) \hookrightarrow H_D^p$ by Lemma 8. Let $z \in \Sigma_{\pi-\theta}$ with $|z| \le 4C^2\alpha^2$ and let $\omega = z + 8C^2\alpha^2$. There holds $\omega \in \Sigma_{\pi-\theta}$ and $|\omega| \ge 8C^2\alpha^2 - |z| \ge 4C^2\alpha^2$. Therefore, for $f \in H_D^p \hookrightarrow H_D$,

$$(z\mathrm{Id} + A_{\alpha,2})^{-1} f = (\omega\mathrm{Id} + A_{\alpha,p})^{-1} f + 8C^2\alpha^2 (\omega\mathrm{Id} + A_{\alpha,p})^{-1}(z\mathrm{Id} + A_{\alpha,2})^{-1} f,$$

which gives

$$\left\|(z\mathrm{Id} + A_{\alpha,2})^{-1} f\right\|_p \le C_\alpha \|f\|_p,$$

and this proves that $z\mathrm{Id} + A_{\alpha,p} : \mathsf{D}(A_{\alpha,p}) \to H^p_D$ is invertible with the norm of its inverse controlled by a constant depending on α. For any $p \ge 2$, the previous procedure can be iterated using again Lemma 8 valid for all $p \ge 2$. Estimates of the form (72) and (73) are straightforward. Eventually, the result claimed in Theorem 7 is obtained for $p \ge 2$. As mentioned earlier, the case $1 < p \le 2$ is obtained by duality. □

5.3 The Nonlinear Robin-Hodge-Navier-Stokes Equations

The nonlinear Robin-Hodge-Navier-Stokes system $\big($(NS'), (Rbc)$\big)$

$$\begin{cases} \partial_t u - \Delta u + \nabla \pi - u \times \operatorname{curl} u = 0 & \text{in } (0,T) \times \Omega, \\ \operatorname{div} u = 0 & \text{in } (0,T) \times \Omega, \\ \nu \cdot u = 0, \quad \nu \times \operatorname{curl} u = \alpha\, u & \text{on } (0,T) \times \partial\Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

for initial data u_0 is considered in the critical space H^3_D in the abstract form:

$$u'(t) + A_{\alpha,p} u(t) - \mathbb{P}\big(u(t) \times \operatorname{curl} u(t)\big) = 0, \quad u_0 \in H^3_D. \tag{74}$$

Recall that $\mathscr{C}^1$ domains Ω are considered here. The idea to solve (74) is to apply the same method as in previous sections.

With the properties of the Robin-Hodge-Stokes semigroup listed in particular in Theorem 7, the following existence result for (74) is almost immediate. For $T \in (0,\infty]$, define the space $\mathscr{H}_T$ by

$$\mathscr{H}_T = \Big\{ u \in \mathscr{C}_b([0,T); H^3_D); \operatorname{curl} u \in \mathscr{C}((0,T); L^3(\Omega,\mathbb{R}^3)) \\ \text{with } \sup_{0<s<T} \|\sqrt{s} \operatorname{curl} u(s)\|_3 < \infty \Big\}$$

endowed with the norm

$$\|u\|_{\mathscr{H}_T} = \sup_{0<s<T} \big(\|u(s)\|_3 + \|\sqrt{s} \operatorname{curl} u(s)\|_3 \big).$$

Theorem 8. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $u_0 \in H^3_D$. Let γ and Φ be defined by*

$$\gamma(t) = e^{-tA_{\alpha,3}} u_0, \quad t \ge 0,$$

and for $u, v \in \mathscr{H}_T$, and $t \in (0, T)$,

$$\Phi(u, v)(t) = \int_0^t e^{-(t-s)A_{\alpha,2}}(\tfrac{1}{2}\mathbb{P})\big((u(s) \times \operatorname{curl} v(s) + v(s) \times \operatorname{curl} u(s)\big)\, \mathrm{d}s.$$

(i) If $\|u_0\|_3$ is small enough, then there exists a unique $u \in \mathscr{H}_\infty$ solution of $u = \gamma + \Phi(u, u)$.
(ii) For all $u_0 \in H^3_D$, there exists $T > 0$ and a unique $u \in \mathscr{H}_T$ solution of $u = \gamma + \Phi(u, u)$.

Elements of the proof. Remark that, as in Lemma 5, for $u \in \mathscr{H}_T$, (thanks to (42)) there holds $u = \mathbb{P}(R_2 \operatorname{curl} u + K_1 u) \in \mathscr{C}((0, T); H^6_D)$ with $\sup_{0<s<T} \sqrt{s}\|u(s)\|_6 \le \|u\|_{\mathscr{H}_T}$. The proof goes as in the previous sections. □

6 Conclusion

In the case of a smooth bounded domain in $\mathbb{R}^n$, it was proved by Y. Giga and T. Miyakawa in [23] that the Dirichlet-Navier-Stokes system admits a local mild solution for initial values in L^n (critical space for the system in dimension n). Their method relies on the fact that the Dirichlet-Stokes operator, as defined in Sect. 2, extends to all L^p spaces and is the negative generator of an analytic semigroup there, which was proved in [21]. The situation in Lipschitz domains is different. For instance, P. Deuring provided in [14] an example of a domain with one conical singularity such that the Dirichlet-Stokes semigroup does not extend to an analytic semigroup in L^p for p large, away from 2 (in this example, $p > 6$).

As already mentioned, E. Fabes, O. Mendez, and M. Mitrea proved in [19] that the orthogonal projection $\mathbb{P}$ defined in Sect. 2 on $L^2(\Omega; \mathbb{R}^3)$ extends to a bounded projection on $L^p(\Omega; \mathbb{R}^3)$ for p in an open interval containing $\left[\frac{3}{2}, 3\right]$ (if Ω is $\mathscr{C}^1$, then this interval is $(1, \infty)$). This led M. Taylor in [54] to formulate the conjecture that the Dirichlet-Stokes semigroup defined originally on H_D extends to an analytic semigroup on L^p for p in the same interval as in [19]. This is actually true as shown in Sect. 2.1.2. It is not known whether this range is optimal, i.e., for any $p > 3$ (or any $p < \frac{3}{2}$), is there a bounded Lipschitz domain such that the Dirichlet-Stokes semigroup $(e^{-tA_D})_{t \ge 0}$ does not extend to a bounded analytic semigroup in H^p_D? When considering Hodge boundary conditions (Hbc), the range where $(e^{-tA_T})_{t \ge 0}$ extends to a bounded analytic semigroup in H^p_D is however larger (see Remark 9, based on results in [30]).

To apply the Fujita-Kato scheme as in Sect. 2.2, proving that the Dirichlet-Stokes semigroup $(e^{-tA_D})_{t \ge 0}$ extends to an analytic semigroup in H^3_D seems to be the first step to obtain mild solutions of the Navier-Stokes system with Dirichlet boundary conditions. The next step is to be able to estimate ∇e^{-tA_D} in the L^3 norm, which is not as straightforward as in the L^2 case where $\|\nabla e^{-tA_D} f\|_2 = \|A_D^{1/2} e^{-tA_D} f\|_2$.

Finally, it would be very satisfactory to obtain a theory for Robin boundary conditions (Rbc) in Lipschitz domains as studied in Sect. 5 for $\mathscr{C}^1$ domains.

Acknowledgements The first author was partially supported by: Harmonic Analysis at its Boundaries, ANR-12-BS01-0013-01.

Cross-References

- ▶ Critical Function Spaces for Well-Posedness of the Navier-Stokes Initial Value Problems
- ▶ Derivation of Model Problems
- ▶ Equations for Incompressible Viscous Fluids in Geophysics
- ▶ Stokes Equation in the Lp-Setting Well-Posedness and Regularity Properties
- ▶ Stokes Semigroup, Strong, Weak and Very Weak Solutions for General Domains

References

1. T. Akiyama, H. Kasai, Y. Shibata, M. Tsutsumi, On a resolvent estimate of a system of Laplace operators with perfect wall condition. Funkcial. Ekvac. **47**(3), 361–394 (2004)
2. H. Amann, On the strong solvability of the Navier-Stokes equations. J. Math. Fluid Mech. **2**(1), 16–98 (2000)
3. C. Amrouche, C. Bernardi, M. Dauge, V. Girault, Vector potentials in three-dimensional non-smooth domains. Math. Methods Appl. Sci. **21**(9), 823–864 (1998)
4. C. Bègue, C. Conca, F. Murat, O. Pironneau, Les équations de Stokes et de Navier-Stokes avec des conditions aux limites sur la pression. in *Nonlinear Partial Differential Equations and Their Applications. Collège de France Seminar, Vol. IX (Paris, 1985–1986)*. Pitman Research Notes in Mathematics Series, vol. 181 (Longman Scientific & Technical, Harlow, 1988), pp 179–264
5. H. Beirão da Veiga, Remarks on the Navier-Stokes evolution equations under slip type boundary conditions with linear friction. Port. Math. (N.S.) **64**(4), 377–387 (2007)
6. R.M. Brown, The method of layer potentials for the heat equation in Lipschitz cylinders. Am. J. Math. **111**, 339–379 (1989)
7. R.M. Brown, Z. Shen, Estimates for the Stokes operator in Lipschitz domains. Indiana Univ. Math. J. **44**(4), 1183–1206 (1995)
8. L. Caffarelli, I. Peral, On $W^{1,p}$ estimates for elliptic equations in divergence form. Commun. Pure Appl. Math. **51**, 1–21 (1998)
9. M. Costabel, A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains. Math. Methods Appl. Sci. **12**(4), 365–368 (1990)
10. M. Costabel, A. McIntosh, On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains. Math. Z. **265**(2), 297–320 (2010)
11. B. Dahlberg, C. Kenig, G. Verchota, Boundary value problems for the system of elastostatics in Lipschitz domains. Duke Math. J. **57**, 795–818 (1988)
12. G. de Rham, *Differentiable Manifolds*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 266 (Springer, Berlin, 1984), Forms, currents, harmonic forms, Translated from the French by F.R. Smith, With an introduction by S.S. Chern
13. L. de Simon, Un'applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine. Rend. Sem. Mat. Univ. Padova **34**, 205–223 (1964)

14. P. Deuring, The Stokes resolvent in 3D domains with conical boundary points: nonregularity in L^p-spaces. Adv. Differential Equations **6**(2), 175–228 (2001)
15. P. Deuring, W. von Wahl, Strong solutions of the Navier-Stokes system in Lipschitz bounded domains. Math. Nachr. **171**, 111–148 (1995)
16. M. Dindoš, M. Mitrea, The stationary Navier-Stokes system in nonsmooth manifolds: the Poisson problem in Lipschitz and C^1 domains. Arch. Ration. Mech. Anal. **174**(1), 1–47 (2004)
17. L. R. Duduchava, D. Mitrea, M. Mitrea, Differential operators and boundary value problems on hypersurfaces, Math. Nachr. **279**(9–10), 996–1023 (2006)
18. E. Fabes, C. Kenig, G. Verchota, The Dirichlet problem for the Stokes system on Lipschitz domains. Duke Math. J. **57**, 769–793 (1988)
19. E. Fabes, O. Mendez, M. Mitrea, Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains. J. Funct. Anal. **159**(2), 323–368 (1998)
20. H. Fujita, T. Kato, On the Navier-Stokes initial value problem. I. Arch. Ration. Mech. Anal. **16**, 269–315 (1964)
21. Y. Giga, Analyticity of the semigroup generated by the Stokes operator in L_r spaces, Math. Z. **178**(3), 297–329 (1981)
22. Y. Giga, The nonstationary Navier-Stokes system with some first order boundary condition. Proc. Jpn. Acad. Ser. A Math. Sci. **58**(3), 101–104 (1982)
23. Y. Giga, T. Miyakawa, Solutions in L_r of the Navier-Stokes initial value problem. Arch. Ration. Mech. Anal. **89**(3), 267–281 (1985)
24. V. Girault, Curl-conforming finite element methods for Navier-Stokes equations with nonstandard boundary conditions in $\mathbf{R}^3$, in *The Navier-Stokes Equations* (Oberwolfach, 1988), Lecture Notes in Mathematics, vol. 1431 (Springer, Berlin, 1990), pp. 201–218
25. T. Iwaniec, C.A. Nolder, Hardy-Littlewood inequality for quasiregular mappings in certain domains in $\mathbf{R}^n$. Ann. Acad. Sci. Fenn. Ser. A I Math. **10** 267–282 (1985)
26. D. Jerison, C.E. Kenig, The inhomogeneous Dirichlet problem in Lipschitz domains. J. Funct. Anal. **130**(1), 161–219 (1995)
27. C.E. Kenig, *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems*. CBMS Regional Conference Series in Mathematics, vol. 83 (Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, 1994)
28. P.G. Lemarié-Rieusset, *Recent Developments in the Navier-Stokes Problem*. Chapman & Hall/CRC Research Notes in Mathematics, vol. 431 (Chapman & Hall/CRC, Boca Raton, 2002)
29. J.-L. Lions, Espaces d'interpolation et domaines de puissances fractionnaires d'opérateurs. J. Math. Soc. Jpn. **14**, 233–241 (1962)
30. A. M^{c}Intosh, S. Monniaux, Hodge-Dirac, Hodge-Laplacian, and Hodge-Stokes operators in L^p spaces on Lipschitz domains (2016, submitted), 36 pp. Available on arXiv:1608.01797
31. D. Mitrea, M. Mitrea, S. Monniaux, The Poisson problem for the exterior derivative operator with Dirichlet boundary condition in nonsmooth domains. Commun. Pure Appl. Anal. **7**(6), 1295–1333 (2008)
32. D. Mitrea, M. Mitrea, M. Taylor, Layer potentials, the Hodge Laplacian, and global boundary problems in nonsmooth Riemannian manifolds. Mem. Am. Math. Soc. **150**(713), x+120 (2001)
33. M. Mitrea, Sharp Hodge decompositions, Maxwell's equations, and vector Poisson problems on nonsmooth, three-dimensional Riemannian manifolds. Duke Math. J. **125**(3), 467–547 (2004)
34. M. Mitrea, S. Monniaux, The regularity of the Stokes operator and the Fujita-Kato approach to the Navier-Stokes initial value problem in Lipschitz domains. J. Funct. Anal. **254**(6), 1522–1574 (2008)
35. M. Mitrea, S. Monniaux, The nonlinear Hodge-Navier-Stokes equations in Lipschitz domains. Differ. Integral Equ. **22**(3–4), 339–356 (2009)
36. M. Mitrea, S. Monniaux, On the analyticity of the semigroup generated by the Stokes operator with Neumann-type boundary conditions on Lipschitz subdomains of Riemannian manifolds. Trans. Am. Math. Soc. **361**(6), 3125–3157 (2009)

37. M. Mitrea, S. Monniaux, M. Wright, The Stokes operator with Neumann boundary conditions in Lipschitz domains. J. Math. Sci. (N.Y.) **176**(3), 409–457 (2011)
38. T. Miyakawa, The L^p approach to the Navier-Stokes equations with the Neumann boundary condition. Hiroshima Math. J. **10**(3), 517–537 (1980)
39. S. Monniaux, Uniqueness of mild solutions of the Navier-Stokes equation and maximal L^p-regularity. C. R. Acad. Sci. Paris Sér. I Math. **328**(8), 663–668 (1999)
40. S. Monniaux, On uniqueness for the Navier-Stokes system in 3D-bounded Lipschitz domains. J. Funct. Anal. **195**(1), 1–11 (2002)
41. S. Monniaux, Navier-Stokes equations in arbitrary domains: the Fujita-Kato scheme. Math. Res. Lett. **13**(2–3), 455–461 (2006)
42. S. Monniaux, Maximal regularity and applications to PDEs, *Analytical and Numerical Aspects of Partial Differential Equations* (Walter de Gruyter, Berlin, 2009), pp. 247–287
43. S. Monniaux, E.M. Ouhabaz, The Incompressible Navier–Stokes System with Time-Dependent Robin-Type Boundary Conditions. J. Math. Fluid Mech. **17**(4), 707–722 (2015)
44. S. Monniaux, J. Saal, Robin-Hodge boundary conditions for the Navier-Stokes equations in 3D-bounded Lipschitz domains (2016, in preparation)
45. B. O'Neill, *Elementary Differential Geometry*, 2nd edn. (Elsevier/Academic Press, Amsterdam, 2006)
46. J. Saal, Stokes and Navier-Stokes equations with Robin boundary conditions in a half-space. J. Math. Fluid Mech. **8**(2), 211–241 (2006)
47. Z. Shen, Boundary value problems for parabolic Lamé systems and a nonstationary linearized system of Navier-Stokes equations in Lipschitz cylinders. Am. J. Math. **113**, 293–373 (1991)
48. Z. Shen, Resolvent estimates in L^p for elliptic systems in Lipschitz domains. J. Funct. Anal. **133**, 224–251 (1995)
49. Z. Shen, Bounds of Riesz transforms on L^p spaces for second order elliptic operators. Ann. Inst. Fourier (Grenoble) **55**, 173–197 (2005)
50. Z. Shen, Necessary and sufficient conditions for the solvability of the L^p Dirichlet problems on Lipschitz domains. Math. Ann. **336**, 697–724 (2006)
51. Z. Shen, Resolvent estimates in L^p for the Stokes operator in Lipschitz domains. Arch. Ration. Mech. Anal. **205**(2), 395–424 (2012)
52. Y. Shibata, R. Shimada, On a generalized resolvent estimate for the Stokes system with Robin boundary condition. J. Math. Soc. Jpn. **59**(2), 469–519 (2007)
53. M.E. Taylor, *Partial Differential Equations. I.* Applied Mathematical Sciences, vol. 115 (Springer, New York, 1996). Basic theory
54. M.E. Taylor, Incompressible fluid flows on rough domains, in *Semigroups of Operators: Theory and Applications* (Newport Beach, 1998). Progress in Nonlinear Differential Equations and Their Applications, vol. 42 (Birkhäuser, Basel, 2000), pp. 320–334
55. R. Temam, *Navier-Stokes Equations*, revised ed., Studies in Mathematics and its Applications, vol. 2 (North-Holland Publishing Co., Amsterdam/New York, 1979), Theory and Numerical Analysis, With an appendix by F. Thomasset
56. G. Verchota, Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. J. Funct. Anal. **59**, 572–611 (1984)