Behaviour of the Stokes operators under domain perturbation *†

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Abstract

Depending of the geometry of the domain, one can define –at least– three different Stokes operators with Dirichlet boundary conditions. We describe how the resolvents of these Stokes operators converge with respect to a converging sequence of domains.

1 Introduction

Let Ω denote an open connected subset of $\mathbb{R}^d$. We do not impose any regularity of the boundary $\partial\Omega$ of the domain Ω and possibly Ω is unbounded. To avoid too many cases, we will however assume that the d-dimensional Hausdorff measure of $\partial\Omega$ is zero.

We denote by $\mathcal{D} = \mathscr{C}_c^\infty(\Omega, \mathbb{R}^d)$ the space of smooth vector fields with compact support in Ω. Let $\mathcal{D}'$ denote its dual, the space of (vector valued) distributions on Ω.

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2 Setting

2.1 The Leray orthogonal decomposition of L^2

We start with a very important and profound result due to de Rham [8, Chapter IV §22, Theorem 17']; see also [11, Chapter I §1.4, Proposition 1.1].

Theorem 2.1 (de Rham). *Let $T \in \mathcal{D}'$ be a distribution. Then the following two properties are equivalent.*

(i) ${}_{\mathcal{D}'}\langle T, \varphi\rangle_{\mathcal{D}} = T(\varphi) = 0$ *for all* $\varphi \in \mathcal{D}$ *with* $\operatorname{div}\varphi = 0$.

(ii) There exists a scalar distribution $S \in \mathscr{C}_c^\infty(\Omega)'$ *such that* $T = \nabla S$ *in* $\mathcal{D}'$.

De Rham's theorem has the following corollary.

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Corollary 2.2. *Let $T \in H^{-1}(\Omega, \mathbb{R}^d)$. Then the following are equivalent.*

(i) ${}_{H^{-1}(\Omega,\mathbb{R}^d)}\langle T, \varphi\rangle_{H_0^1(\Omega,\mathbb{R}^d)} = 0$ *for all* $\varphi \in \mathcal{D}$ *with* $\operatorname{div}\varphi = 0$.

(ii) *There exists a scalar distribution* $\pi \in L^2_{\mathrm{loc}}(\Omega)$ *such that* $T = \nabla\pi$ *in* $\mathcal{D}'$.

Proof. We only have to show (i) $\Longrightarrow$ (ii). By Theorem 2.1 there exists $S \in \mathscr{C}_c^\infty(\Omega)'$ such that $T = \nabla S$. Then $\nabla S \in H^{-1}(\Omega, \mathbb{R}^d)$. Consequently, $S \in L^2_{\mathrm{loc}}(\Omega)$ by [11, Proposition 1.2] (for a direct proof, see also [9, Lemma 2.2.1]). $\square$

Denote by $\mathcal{H} = L^2(\Omega, \mathbb{R}^d)$ the square integrable vector fields on Ω. We endow the vector-valued space $\mathcal{H}$ with the scalar product

$$\langle u, v\rangle_{\mathcal{H}} := \int_\Omega u \cdot v = \sum_{j=1}^d \int_\Omega u_j v_j, \quad u, v \in \mathcal{H}.$$

Then $\mathcal{H}$ is a Hilbert space. We define the subspace $\mathscr{G}$ of $\mathcal{H}$ consisting of gradients by

$$\mathscr{G} := \big\{\nabla\pi; \pi \in L^2_{\mathrm{loc}}(\Omega), \nabla\pi \in \mathcal{H}\big\}. \tag{2.1}$$

As a consequence of Corollary 2.2, $\mathscr{G}$ is a closed subspace of $\mathcal{H}$. We denote by $\mathscr{H}$ the orthogonal subspace of $\mathscr{G}$ in $\mathcal{H}$, that is

$$\mathscr{H} = \big\{u \in \mathcal{H}; \langle u, g\rangle_{\mathcal{H}} = 0 \text{ for all } g \in \mathscr{G}\big\}. \tag{2.2}$$

Obviously, $\mathscr{H}$ is a Hilbert space and one has the orthogonal decomposition

$$\mathcal{H} = \mathscr{H} \overset{\perp}{\oplus} \mathscr{G}. \tag{2.3}$$

The orthogonal projection from $\mathcal{H}$ to $\mathscr{H}$ denoted by $\mathbb{P}$ is called the `Leray projection`. It is the adjoint of the canonical embedding $J : \mathscr{H} \hookrightarrow \mathcal{H}$; it verifies $\mathbb{P}Ju = u$ for all $u \in \mathscr{H}$. Next, define the subspace

$$\mathscr{D} = \big\{u \in \mathcal{D}; \operatorname{div} u = 0 \text{ in } \Omega\big\}. \tag{2.4}$$

Then $\mathscr{D} \subset \mathscr{H}$ and by De Rham's theorem, $\mathscr{D}^\perp = \mathscr{G}$, so that $\mathscr{D}$ is dense in $\mathscr{H}$ with respect to the L^2-norm of $\mathcal{H}$.

The canonical embedding $J_0 : \mathscr{D} \hookrightarrow \mathcal{D}$ is the restriction of J to $\mathscr{D}$. Its adjoint $J_0' = \mathbb{P}_1 : \mathcal{D}' \to \mathscr{D}'$ is therefore an extension of the Leray projection $\mathbb{P}$. A reformulation of de Rham's theorem (Thm 2.1) is

$$\ker\mathbb{P}_1 = \big\{T \in \mathcal{D}'; \mathbb{P}_1 T = 0\big\} = \big\{\nabla S; S \in \mathscr{C}_c^\infty(\Omega)'\big\}.$$

2.2 Another orthogonal decomposition of L^2

Since we made the assumption that the d-dimensional Hausdorff measure of $\partial\Omega$ is zero, we can identify $\mathcal{H} = L^2(\Omega, \mathbb{R}^d)$ with $\big\{U_{|\Omega}; U \in L^2(\mathbb{R}^d, \mathbb{R}^d), U = 0 \text{ a.e. in } {}^c\overline{\Omega}\big\}$ and define the space $\mathscr{E}$ to be the closure in $L^2(\Omega, \mathbb{R}^d)$ of

$$\mathscr{W} := \big\{U_{|\Omega}; U \in H^1(\mathbb{R}^d, \mathbb{R}^d), U = 0 \text{ a.e. in } {}^c\overline{\Omega} \text{ and } \operatorname{div} U = 0 \text{ in } \mathbb{R}^d\big\}. \tag{2.5}$$

The space $\mathscr{E}$ is closed in $\mathcal{H}$ by definition and contains $\mathscr{D}$, and then $\mathscr{H}$. The following decomposition of $\mathcal{H}$ holds

$$\mathcal{H} = \mathscr{E} \overset{\perp}{\oplus} \mathscr{F}, \tag{2.6}$$

where $\mathscr{F} = \mathscr{E}^{\perp}$. Since $\mathscr{H} \subset \mathscr{E}$, it is obvious that $\mathscr{F} \subset \mathscr{G}$. It is also obvious that $\{\nabla q_{|\Omega}; q \in \dot{H}^1(\mathbb{R}^d)\} \subset \mathscr{F}$: let $u = U_{|\Omega} \in \mathscr{W}$ and $q \in \dot{H}^1(\mathbb{R}^d)$; then

$$\langle u, \nabla q \rangle_{\mathcal{H}} = \langle U, \nabla q \rangle_{L^2(\mathbb{R}^d, \mathbb{R}^d)} = 0.$$

For further use, we will denote by $L : \mathscr{E} \hookrightarrow \mathcal{H}$ the canonical embedding; its adjoint $L' = \mathbb{Q} : \mathcal{H} \to \mathscr{E}$ is the orthogonal projection from $\mathcal{H}$ to $\mathscr{E}$. The operators L and $\mathbb{Q}$ verify $\mathbb{Q}Lu = u$ for all $u \in \mathscr{E}$, as do J and $\mathbb{P}$ in the above setting.

Remark 2.3. When $\Omega \subset \mathbb{R}^d$ is bounded and smooth enough, say with Lipschitz boundary, the spaces $\mathscr{H}$ and $\mathscr{E}$ coincide: they are equal to

$$\mathbb{L}^2_\sigma(\Omega) := \{u \in L^2(\Omega, \mathbb{R}^d); \operatorname{div} u = 0 \text{ in } \Omega \text{ and } \nu \cdot u = 0 \text{ on } \partial\Omega\},$$

where $\operatorname{div} u$ is to be taken in the sense of distributions and $\nu(x)$ denotes the exterior normal unit vector at $x \in \partial\Omega$, defined for almost every x in the case of a Lipschitz boundary $\partial\Omega$. Here, $\nu \cdot u \in H^{-1/2}(\partial\Omega)$ is defined via the integration by parts formula

$$_{H^{-1/2}}\langle \nu \cdot u, \varphi \rangle_{H^{1/2}} = \int_\Omega u \cdot \nabla\Phi + \int_\Omega \operatorname{div} u \cdot \Phi$$

for all $\varphi \in H^{1/2}(\partial\Omega)$ and $\Phi \in H^1(\Omega)$ satisfying $\operatorname{Tr}_{|\partial\Omega}\Phi = \varphi$.

The fact that $\mathscr{H} = \mathbb{L}^2_\sigma(\Omega)$ in the case of a bounded domain with Lipschitz boundary was proved in [11, Thm 1.4]. If $\Omega \subset \mathbb{R}^d$ has a continuous boundary as in [2, Prop. 2.2] (see also [10]), $\mathscr{W} = \{u \in H^1_0(\Omega, \mathbb{R}^d); \operatorname{div} u = 0\}$. According to [11, Thm 1.6], this latter space is the closure of $\mathscr{D}$ in $H^1(\Omega, \mathbb{R}^d)$ if the boundary of Ω is Lipschitz, so that $\mathscr{E} = \mathbb{L}^2_\sigma(\Omega) = \mathscr{H}$.

3 Spaces of divergence-free vector fields

In this section, we introduce several spaces which yield different suitable definitions of the Stokes operator with Dirichlet boundary conditions.

We start with

$$\mathcal{V} = H^1_0(\Omega, \mathbb{R}^d).$$

Then $\mathcal{V}$ is the closure of $\mathcal{D}$ in $H^1(\Omega, \mathbb{R}^d)$. We provide $\mathcal{V}$ with the norm induced from $H^1(\Omega, \mathbb{R}^d)$. Next, we define the space

$$\mathcal{W} := \{U_{|\Omega} : U \in H^1(\mathbb{R}^d, \mathbb{R}^d) \text{ and } U = 0 \text{ a.e. in } \overline{\Omega}^c\}.$$

Then $\mathcal{W}$ is a closed subspace of $H^1(\Omega, \mathbb{R}^d)$ and we provide $\mathcal{W}$ with the norm induced from $H^1(\Omega, \mathbb{R}^d)$.

It is clear that $\mathcal{D} \subset \mathcal{V} \subseteq \mathcal{W} \subset \mathcal{H}$. If Ω has a continuous boundary, then $\mathcal{V} = \mathcal{W}$ (see [10, pages 24-26]), but in general $\mathcal{V} \neq \mathcal{W}$, as shown in [2, Section 7]). Identifying $\mathcal{H}$ with its dual, we obtain the Gelfand triples $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$ and $\mathcal{W} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{W}'$.

Let $\mathscr{V}$ be the closure of $\mathscr{D}$ in $\mathcal{V} = H^1_0(\Omega, \mathbb{R}^d)$ and let $\mathscr{X} := \mathcal{V} \cap \mathscr{H}$. It is straightforward that $\mathscr{V} \subseteq \mathscr{X} \subseteq \mathscr{W}$. If Ω is bounded with Lipschitz boundary, then $\mathscr{V} = \mathscr{X} = \mathscr{W}$ (see [4, Section 3] and [6, Theorem 2.2]), but not in general. The famous example for which $\mathscr{V}$ is different from $\mathscr{X}$ is the unbounded smooth aperture domain (see [4, Theorem 17]). The three spaces $\mathscr{V}$, $\mathscr{X}$ and $\mathscr{W}$ all contain $\mathscr{D}$, $\mathscr{V}$ and $\mathscr{X}$ are dense subspaces of $\mathscr{H}$, $\mathscr{W}$ is a dense subspace of $\mathscr{E}$ by definition. Moreover, $\mathscr{V}$ and $\mathscr{X}$ are closed in $\mathcal{V}$ and $\mathscr{W}$ is closed in $\mathcal{W}$.

3.1 Weak- and pseudo-Dirichlet Laplacians

We now briefly describe how to define the Laplacian with homogeneous Dirichlet boundary conditions in a weak sense: depending on how the boundary conditions are modelled, different operators appear. Recall that since we do not impose any regularity on the boundary of our domain Ω, it does not make sense to talk about traces. We start by defining the bilinear form $\mathfrak{a} : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ by

$$\mathfrak{a}(u,v) := \langle \nabla u, \nabla v \rangle_{\mathcal{H}} = \sum_{j=1}^{d} \langle \partial_j u, \partial_j v \rangle_{\mathcal{H}}. \tag{3.1}$$

The forms $\mathfrak{a}$ and $\mathfrak{a}|_{\mathcal{V}\times\mathcal{V}}$ are associated with analytic semigroups of contractions on $\mathcal{H}$ (see, e.g., [5, §VI.2]). Let $-\Delta_D^{\Omega}$ be the operator associated with the form $\mathfrak{a}|_{\mathcal{V}\times\mathcal{V}}$ and let $-\Delta_D^{\overline{\Omega}}$ be the operator associated with the form $\mathfrak{a}$. Following [2], we call Δ_D^{Ω} the `(weak-)Dirichlet Laplacian` and $\Delta_D^{\overline{\Omega}}$ the `pseudo-Dirichlet Laplacian`. They are self-adjoint (unbounded) operators in $\mathcal{H}$. The interest of considering the weak-Dirichlet and the pseudo-Dirichlet Laplacians lies in particular in domain perturbation problems.

Let $\Omega, \Omega_1, \Omega_2, \ldots$ be bounded open subsets of $\mathbb{R}^d$. We say that $\Omega_n \uparrow \Omega$ as $n \to \infty$ if $\Omega_n \subset \Omega_{n+1}$ for all $n \in \mathbb{N}$ and for each compact subset $K \subset \Omega$ there exists an $n \in \mathbb{N}$ with $K \subset \Omega_n$. We say that $\Omega_n \downarrow \Omega$ as $n \to \infty$ if $\Omega_n \supset \Omega_{n+1} \supset \overline{\Omega}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} |(\Omega_n \cap B) \setminus \overline{\Omega}| = 0$ for every ball B, where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^d$.

If $f \in \mathcal{H}$, then we denote by $\tilde{f} \in L^2(\mathbb{R}^d, \mathbb{R}^d)$ the extension by 0 of f to $\mathbb{R}^d$.

The following results have been established in [2, §3]. See also [1, §6] and [3, §6 and §7].

Proposition 3.1. *Let $\Omega, \Omega_1, \Omega_2, \ldots$ be bounded open subsets of $\mathbb{R}^d$.*

a. Suppose that $\Omega_n \uparrow \Omega$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \widetilde{\left[\left(\mathrm{I} + (-\Delta_D^{\Omega_n})\right)^{-1}(f_{|\Omega_n})\right]}_{|\Omega} = \left(\mathrm{I} + (-\Delta_D^{\Omega})\right)^{-1} f \quad \textit{and}$$
$$\lim_{n\to\infty} \widetilde{\left[\left(\mathrm{I} + (-\Delta_D^{\overline{\Omega}_n})\right)^{-1}(f_{|\Omega_n})\right]}_{|\Omega} = \left(\mathrm{I} + (-\Delta_D^{\Omega})\right)^{-1} f$$

in $\mathcal{H}$ for all $f \in \mathcal{H}$.

b. Suppose that $\Omega_n \downarrow \Omega$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \left[\left(\mathrm{I} + (-\Delta_D^{\overline{\Omega}_n})\right)^{-1}(\tilde{f}_{|\Omega_n})\right]_{|\Omega} = \left(\mathrm{I} + (-\Delta_D^{\overline{\Omega}})\right)^{-1} f \quad \textit{and}$$
$$\lim_{n\to\infty} \left[\left(\mathrm{I} + (-\Delta_D^{\Omega_n})\right)^{-1}(\tilde{f}_{|\Omega_n})\right]_{|\Omega} = \left(\mathrm{I} + (-\Delta_D^{\overline{\Omega}})\right)^{-1} f$$

in $\mathcal{H}$ for all $f \in \mathcal{H}$.

Remark 3.2. Strictly speaking, Proposition 3.1 has been proved in [2] for scalar valued functions $f \in L^2(\Omega, \mathbb{R})$, and only the first part of a ([2, Proposition 3.2]) and the second part of b ([2, Proposition 3.5]) can be found in that reference. Using [2, Proposition 2.3] establishing monotonicity properties of the resolvents of the weak-Dirichlet Laplacian and the pseudo-Dirichlet Laplacian with respect to the inclusion of domains, the other two limits are immediate.

3.2 The weak-Dirichlet Stokes operator

Since the spaces $\mathscr{V}$ and $\mathscr{X}$ are dense subspaces of the Hilbert space $\mathscr{H}$, one can define two Dirichlet types of Stokes operators in $\mathscr{H}$. Recall the form $\mathfrak{a} : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ from (3.1)

$$\mathfrak{a}(u, v) := \langle \nabla u, \nabla v \rangle_{\mathcal{H}} = \sum_{j=1}^{d} \langle \partial_j u, \partial_j v \rangle_{\mathcal{H}}.$$

Then $\mathfrak{a}|_{\mathscr{X} \times \mathscr{X}}$ is a positive symmetric densely defined closed form in $\mathscr{H}$. Let $\mathcal{B}$ be the operator associated with $\mathfrak{a}|_{\mathscr{X} \times \mathscr{X}}$. Then $\mathcal{B}$ is self-adjoint and $\mathcal{B}$ is the Stokes operator considered in [7].

Since $\mathscr{V} \subset \mathscr{X}$ we can also define $\mathcal{B}_0$ to be the self-adjoint operator in $\mathscr{H}$ associated with the form $\mathfrak{a}|_{\mathscr{V} \times \mathscr{V}}$. We call $\mathcal{B}_0$ the `weak-Dirichlet Stokes operator`. This Stokes operator is the one which was considered by H. Sohr in [9, Chapter 3, §2.1].

The operators $\mathcal{B}$ and $\mathcal{B}_0$ are both negative generators of analytic semigroups in $\mathscr{H}$.

Each of the cases above models differently spaces of divergence free vector fields with zero boundary conditions. As already mentioned before, they coincide in the case of bounded Lipschitz domains and consequently then also the two operators $\mathcal{B}$ and $\mathcal{B}_0$ coincide.

The relation between the weak-Dirichlet Laplacian and the weak-Dirichlet Stokes operator is described in the following commutative diagram:

$$\begin{tikzcd}
\mathscr{V} \arrow[r, hook, "J_0"] \arrow[d, hook, "d"'] \arrow[dd, bend right=60, "\mathcal{B}_0"'] & \mathcal{V} \arrow[d, hook, "d"] \arrow[dd, bend left=60, "(-\Delta_D^{\Omega})"] \\
\mathscr{H} \arrow[r, hook, shift left, "J"] \arrow[d, hook, "d"'] & \mathcal{H} \arrow[l, shift left, "\mathbb{P}=J'"] \arrow[d, hook, "d"] \\
\mathscr{V}' & \mathcal{V}' \arrow[l, "\mathbb{P}_1=J_0'"]
\end{tikzcd}$$

where J_0 is the restriction of J to $\mathscr{V}$ and $\mathbb{P}_1$, its adjoint operator, is the extension of the Leray projection $\mathbb{P}$ to $\mathcal{V}'$. What this says in particular is that $\mathcal{B}_0 = \mathbb{P}_1(-\Delta_D^{\Omega})J_0$.

3.3 The pseudo-Dirichlet Stokes operator

If we now restrict the form $\mathfrak{a}$ to $\mathscr{W} \times \mathscr{W}$ we obtain a positive symmetric densely defined closed form in $\mathscr{E}$. We then define $\mathcal{A}$ to be the self-adjoint operator in $\mathscr{E}$ associated with $\mathfrak{a}|_{\mathscr{W} \times \mathscr{W}}$. We call $\mathcal{A}$ the `pseudo-Dirichlet Stokes operator`. It is the negative generator of an analytic semigroup in $\mathscr{E}$.

As said before, in the case of a bounded domain Ω with Lipschitz boundary, the spaces $\mathscr{X}$, $\mathscr{V}$ and $\mathscr{W}$ coincide as well as the spaces $\mathscr{H}$ and $\mathscr{E}$, then so do the operators $\mathcal{B}$, $\mathcal{B}_0$ and $\mathcal{A}$.

The relation between the pseudo-Dirichlet Laplacian and the pseudo-Dirichlet Stokes operator is described in the following commutative diagram:

$$\begin{tikzcd}
\mathscr{W} \arrow[r, hook, "L_0"] \arrow[d, hook, "d"'] \arrow[dd, bend right=60, "\mathcal{A}"'] & \mathcal{W} \arrow[d, hook, "d"] \arrow[dd, bend left=60, "(-\Delta_D^{\overline{\Omega}})"] \\
\mathscr{E} \arrow[r, hook, shift left, "L"] \arrow[d, hook, "d"'] & \mathcal{H} \arrow[l, shift left, "\mathbb{Q}=L'"] \arrow[d, hook, "d"] \\
\mathscr{W}' & \mathcal{W}' \arrow[l, "\mathbb{Q}_1=L_0'"]
\end{tikzcd}$$

where L_0 is the restriction of L to $\mathscr{W}$ and $\mathbb{Q}_1$, its adjoint operator, is the extension of the projection $\mathbb{Q}$ from §2.2 to $\mathcal{W}'$. What this says in particular is that $\mathcal{A} = \mathbb{Q}_1(-\Delta_D^{\overline{\Omega}})L_0$.

4 Domain perturbation

Similar results as those stated in Proposition 3.1 hold for the different Dirichlet Stokes operators described above. Roughly speaking, resolvents of the different Stokes operators converge to the resolvent of the weak-Dirichlet Stokes operator in $\mathscr{H}$ in the case of an increasing sequence of open sets and to the resolvent of the pseudo-Dirichlet Stokes operator in $\mathscr{E}$ in the case of a decreasing sequence of open sets.

4.1 Increasing sequence of domains

Theorem 4.1. *Let $\Omega, \Omega_1, \Omega_2, \ldots$ be connected open subsets of $\mathbb{R}^d$. Suppose that $\Omega_n \uparrow \Omega$ as $n \to \infty$. For all $n \in \mathbb{N}$ denote by $\mathbb{P}_n$ the Leray projection from $L^2(\Omega_n, \mathbb{R}^d)$ onto $\mathscr{H}_n$ (the corresponding space of divergence-free vector fields as in* (2.3)*), I_n the identity operator on $\mathscr{H}_n$, $\mathscr{V}_n$ the corresponding form domain and $\mathcal{B}_0^{(n)}$ the corresponding weak-Dirichlet Stokes operator. Then*

$$\lim_{n\to\infty} \widetilde{\Big(\big(I_n + \mathcal{B}_0^{(n)}\big)^{-1} \mathbb{P}_n(f_{|\Omega_n}) \Big)}_{|\Omega} = (I + \mathcal{B}_0)^{-1} f$$

in $H^1(\Omega, \mathbb{R}^d)$ for all $f \in \mathscr{H}$.

Proof. For all $n \in \mathbb{N}$ define $u_n = \big(\mathrm{I}_n + \mathcal{B}_0^{(n)}\big)^{-1} \mathbb{P}_n(f_{|\Omega_n})$. Then $u_n \in \mathscr{V}_n$ and $\tilde{u}_{n|\Omega} \in \mathscr{V}$ and

$$\int_{\Omega_n} \nabla u_n \cdot \nabla v + \int_{\Omega_n} u_n \cdot v = \int_{\Omega_n} (\mathbb{P}_n(f_{|\Omega_n})) \cdot v = \int_{\Omega_n} f \cdot v \tag{4.1}$$

for all $v \in \mathscr{V}_n$. Choosing $v = u_n$ gives

$$\int_{\Omega_n} |\nabla u_n|^2 + \int_{\Omega_n} |u_n|^2 = \int_{\Omega_n} \mathbb{P}_n(f_{|\Omega_n}) \cdot u_n \leq \|f\|_2 \Big(\int_{\Omega_n} |u_n|^2 \Big)^{1/2}.$$

This implies that $(\tilde{u}_{n|\Omega})_{n\in\mathbb{N}}$ is a bounded sequence in $\mathscr{V}$. Passing to a subsequence if necessary, there exists a $u \in \mathscr{V}$ such that $\lim_{n\to\infty} \tilde{u}_{n|\Omega} = u$ weakly in $\mathscr{V}$. Let $v \in \mathscr{D}$. There exists an $N \in \mathbb{N}$ such that $\tilde{v}_{|\Omega_n} \in \mathscr{D}_n$ for all $n \geq N$. By definition of u_n we then have for all $n \geq N$ that

$$\mathfrak{a}(\tilde{u}_{n|\Omega}, v) + \langle \tilde{u}_{n|\Omega}, v \rangle_{L^2(\Omega,\mathbb{R}^d)} = \langle f, v \rangle_{L^2(\Omega,\mathbb{R}^d)}.$$

Taking the limit as $n \to \infty$ we obtain that

$$\mathfrak{a}(u, v) + \langle u, v \rangle_{L^2(\Omega,\mathbb{R}^d)} = \langle f, v \rangle_{L^2(\Omega,\mathbb{R}^d)}. \tag{4.2}$$

This is true for all $v \in \mathscr{D}$. Then by continuity and density, (4.2) is valid for all $v \in \mathscr{V}$. This shows that $u \in \mathsf{D}(\mathcal{B}_0)$ and $u = (I + \mathcal{B}_0)^{-1} f$.

It remains to show that $\lim_{n\to\infty} \tilde{u}_{n|\Omega} = u$ strongly in $L^2(\Omega, \mathbb{R}^d)$. Since $\lim_{n\to\infty} \tilde{u}_{n|\Omega} = u$ weakly in $L^2(\Omega, \mathbb{R}^d)$,

$$\liminf_{n\to\infty} \|\tilde{u}_{n|\Omega}\|_{L^2(\Omega,\mathbb{R}^d)} \geq \|u\|_{L^2(\Omega,\mathbb{R}^d)}.$$

Comparing $\limsup_{n\to\infty} \|u_n\|_2$ and $\liminf_{n\to\infty} \|u_n\|_2$, it suffices to show that

$$\limsup_{n\to\infty} \|\tilde{u}_{n|\Omega}\|_{L^2(\Omega,\mathbb{R}^d)} \leq \|u\|_{L^2(\Omega,\mathbb{R}^d)}.$$

Let $n \in \mathbb{N}$. Choose $v = u_n$ in (4.1). Then the following equality holds

$$\int_\Omega |\tilde{u}_{n|\Omega}|^2 = \int_\Omega f \cdot \tilde{u}_{n|\Omega} - \int_\Omega \left|\nabla \tilde{u}_{n|\Omega}\right|^2. \tag{4.3}$$

Since $\lim_{n\to\infty} \tilde{u}_{n|\Omega} = u$ weakly in $\mathscr{V}$, and hence in $H^1_0(\Omega, \mathbb{R}^d)$, one deduces that

$$\lim_{n\to\infty} \int_\Omega f \cdot \tilde{u}_{n|\Omega} = \int_\Omega f \cdot u \quad \text{and} \quad \lim_{n\to\infty} \partial_k \tilde{u}_{n|\Omega} = \partial_k u \text{ weakly in } L^2(\Omega, \mathbb{R}^d) \text{ for all } k \in \{1, \ldots, d\}.$$

This implies $\|\partial_k u\|_{L^2(\Omega,\mathbb{R}^d)} \leq \liminf_{n\to\infty} \|\partial_k \tilde{u}_{n|\Omega}\|_{L^2(\Omega,\mathbb{R}^d)}$. Consequently,

$$\begin{aligned}\limsup_{n\to\infty} \|\tilde{u}_{n|\Omega}\|^2_{L^2(\Omega,\mathbb{R}^d)} &= \lim_{n\to\infty} \Big(\int_\Omega f \cdot \tilde{u}_{n|\Omega}\Big) - \liminf_{n\to\infty} \|\nabla \tilde{u}_{n|\Omega}\|^2_{L^2(\Omega,\mathbb{R}^{d\times d})} \\ &\leq \langle f, u\rangle_{L^2(\Omega,\mathbb{R}^d)} - \|\nabla u\|^2_{L^2(\Omega,\mathbb{R}^{d\times d})} = \|u\|^2_{L^2(\Omega,\mathbb{R}^d)},\end{aligned}$$

where the last equality follows from (4.2). Then $\lim_{n\to\infty} \tilde{u}_{n|\Omega} = u$ strongly in $L^2(\Omega, \mathbb{R}^d)$. One concludes by the fact that every sequence for which every subsequence posseses a convergent subsequence to a unique limit is convergent. To prove the convergence in $H^1(\Omega, \mathbb{R}^d)$, rewrite (4.3) as

$$\int_\Omega \left|\nabla \tilde{u}_{n|\Omega}\right|^2 = \int_\Omega f \cdot \tilde{u}_{n|\Omega} - \int_\Omega |\tilde{u}_{n|\Omega}|^2$$

and take the $\limsup_{n\to\infty}$: the right hand-side converges to

$$\langle f, u\rangle_{L^2(\Omega,\mathbb{R}^d)} - \|u\|^2_{L^2(\Omega,\mathbb{R}^d)} = \|\nabla u\|^2_{L^2(\Omega,\mathbb{R}^{d\times d})}$$

and therefore $\big(\nabla \tilde{u}_{n|\Omega}\big)_{n\geq 1}$ converges strongly in $L^2(\Omega, \mathbb{R}^{d\times d})$ as claimed. □

4.2 Decreasing sequence of domains

If $f \in L^2(\Omega, \mathbb{R}^d)$, then we denote by $\tilde{f} \in L^2(\mathbb{R}^d, \mathbb{R}^d)$ the extension by 0 of f to $\mathbb{R}^d$.

Theorem 4.2. *Let $\Omega, \Omega_1, \Omega_2, \ldots$ be connected open subsets of $\mathbb{R}^d$. Suppose that the d-dimensional Hausdorff measure of $\partial\Omega$ and $\partial\Omega_n$ for all $n \in \mathbb{N}$ is zero. Suppose that $\Omega_n \downarrow \Omega$ as $n \to \infty$. For all $n \in \mathbb{N}$ denote by $\mathbb{Q}_n$ the projection from $L^2(\Omega_n, \mathbb{R}^d)$ onto $\mathscr{E}_n$ (the corresponding space of divergence-free vector fields as in (2.6)), $\mathcal{I}_n$ the identity operator on $\mathscr{E}_n$, $\mathscr{W}_n$ the corresponding form domain and $\mathcal{A}_n$ the corresponding pseudo-Dirichlet Stokes operator. Then*

$$\lim_{n\to\infty} \Big(\big(\mathcal{I}_n + \mathcal{A}_n\big)^{-1} \tilde{f}_{|\Omega_n}\Big)_{|\Omega} = (\mathcal{I} + \mathcal{A})^{-1} f$$

in $H^1(\Omega, \mathbb{R}^d)$ for all $f \in \mathscr{E}$.

Proof. First note that $\tilde{f}_{|\Omega_n} \in \mathscr{E}_n$ for all $f \in \mathscr{E}$ and $\tilde{u}_{|\Omega_n} \in \mathscr{W}_n$ for all $u \in \mathscr{W}$ and all $n \in \mathbb{N}$. Fix $f \in \mathscr{E}$. For all $n \in \mathbb{N}$ define $u_n := \big(\mathcal{I}_n + \mathcal{A}_n\big)^{-1}(\tilde{f}_{|\Omega_n})$. Then $u_n \in \mathscr{W}_n$ and

$$\int_{\mathbb{R}^d} \nabla\tilde{u}_n \cdot \nabla\tilde{v} + \int_{\mathbb{R}^d} \tilde{u}_n \cdot \tilde{v} = \int_{\Omega_n} \tilde{f}_{|\Omega_n} \cdot \tilde{v} = \int_{\mathbb{R}^d} \tilde{f} \cdot \tilde{v}, \tag{4.4}$$

for all $v \in \mathscr{W}_n$. Choosing $v = u_n$ gives

$$\|\nabla\tilde{u}_n\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^d)} + \|\tilde{u}_n\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^d)} \le \|\tilde{f}\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)}\|\tilde{u}_n\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)}.$$

Hence $(\tilde{u}_n)_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\mathbb{R}^d,\mathbb{R}^d)$. Passing to a subsequence if necessary, there exists a $U \in H^1(\mathbb{R}^d,\mathbb{R}^d)$ such that $\lim_{n\to\infty} \tilde{u}_n = U$ weakly in $H^1(\mathbb{R}^d,\mathbb{R}^d)$.

We next show that $U = 0$ a.e. on $\overline{\Omega}^{\mathrm{c}}$. Let $\Phi \in C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)$ and suppose that $\operatorname{supp}\Phi \subset \overline{\Omega}^{\mathrm{c}}$. Let $R > 0$ such that $\operatorname{supp}\Phi$ is contained in the ball $B = B(0,R)$. If $n \in \mathbb{N}$, then

$$\Big|\int_{\mathbb{R}^d} \tilde{u}_n \cdot \Phi\Big| \le \|\tilde{u}_n\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)}\|\Phi\|_{L^2(\Omega_n,\mathbb{R}^d)} \le \|\tilde{f}\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)}\|\Phi\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^d)}|(\Omega_n \setminus \overline{\Omega}) \cap B|^{1/2}.$$

Since $\lim_{n\to\infty} |(\Omega_n \setminus \overline{\Omega}) \cap B| = 0$ it follows that

$$\int_{\mathbb{R}^d} U \cdot \Phi = \lim_{n\to\infty} \int_{\mathbb{R}^d} \tilde{u}_n \cdot \Phi = 0.$$

So $U = 0$ a.e. on $\overline{\Omega}^{\mathrm{c}}$. Set $u = U|_\Omega$. Then $u \in \mathcal{W}$.

To prove that $u \in \mathscr{W}$, it remains to prove that $\operatorname{div} U = 0$ in $\mathbb{R}^d$. This is straightforward since for all $\nabla p \in L^2(\mathbb{R}^d,\mathbb{R}^d)$ and for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^d} U \cdot \nabla p \xleftarrow[\infty\leftarrow n]{} \int_{\mathbb{R}^d} \tilde{u}_n \cdot \nabla p = 0.$$

Now, taking the limit as n goes to ∞ in (4.4) for $v \in \mathscr{W}$, we obtain that

$$\int_{\mathbb{R}^d} \nabla U \cdot \nabla\tilde{v} + \int_{\mathbb{R}^d} U \cdot \tilde{v} = \mathfrak{a}(u,v) + \langle u,v\rangle = \langle f,v\rangle = \int_{\mathbb{R}^d} \tilde{f} \cdot \tilde{v}. \tag{4.5}$$

Therefore, $u \in \mathsf{D}(\mathcal{A})$. It remains to prove that $u_{n|\Omega} \xrightarrow[n\to\infty]{} u$ strongly in $L^2(\Omega,\mathbb{R}^d)$. The proof is similar to the proof of Theorem 4.1, comparing $\liminf_{n\to\infty}\|u_{n|\Omega}\|$ and $\limsup_{n\to\infty}\|u_{n|\Omega}\|$. By weak convergence of $\big(u_{n|\Omega}\big)_{n\in\mathbb{N}}$ to u in $L^2(\Omega,\mathbb{R}^d)$, the inequality $\liminf_{n\to\infty}\|u_{n|\Omega}\|_2 \ge \|u\|_2$ holds. The proof of $\limsup_{n\to\infty}\|u_{n|\Omega}\|_2 \le \|u\|_2$, uses (4.4) with $v = u_n$ and the fact that $\big(\tilde{u}_n\big)_{n\in\mathbb{N}}$ converges weakly to U in $L^2(\mathbb{R}^d,\mathbb{R}^d)$, so that

$$\begin{aligned}\limsup_{n\to\infty} \|u_{n|\Omega}\|_2^2 \le \limsup_{n\to\infty} \|\tilde{u}_n\|_2^2 &= \lim_{n\to\infty} \int_{\mathbb{R}^d} \tilde{u}_n \cdot \tilde{f} - \liminf_{n\to\infty}\|\nabla\tilde{u}_n\|_2^2 \\ &\le \int_{\mathbb{R}^d} U \cdot \tilde{f} - \|\nabla U\|_2^2 = \langle u,f\rangle - \|\nabla u\|_2^2 = \|u\|_2^2\end{aligned}$$

by (4.5) with $v = u$. The strong convergence of $\big(\nabla u_{n|\Omega}\big)_{n\ge1}$ to ∇u in $L^2(\Omega,\mathbb{R}^{d\times d})$ is proved using the same arguments as in the proof of Theorem 4.1. $\square$

4.3 Comments

The reader may want to compare Theorem 4.2 and Theorem 4.1 with Proposition 3.1 and ask whether one can approximate the pseudo-Dirichlet Stokes operator in Ω with weak-Dirichlet Stokes operators in Ω_n where $\Omega_n \downarrow \Omega$ and the weak-Dirichlet Stokes operator in Ω with pseudo-Dirichlet Stokes operators in Ω_n where $\Omega_n \uparrow \Omega$. This is obviously true if the approximation domains Ω_n are smooth and bounded since in this case, the weak-Dirichlet Stokes operator and the pseudo-Dirichlet Stokes operator coincide. In the case of increasing or decreasing sequences of arbitrary domains, the strategy followed by [2] (comparison of resolvents with respect to the inclusion of domains as in Remark 3.2) doesn't work: the Stokes problem is purely vector-valued and the spaces involved are not Banach lattices.

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