A special configuration of 12 conics and a related K3 surface

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Joint work with David Kohel and Alessandra Sarti

A smooth projective surface X is called a K3 surface if

$$K_X \simeq \mathcal{O}_X \text{ and } H^1(X, \mathcal{O}_X) = \{0\}.$$

By the adjonction formula, the self-intersection of a smooth rational curve C on X is $C^2 = -2$, sothat such a curve is called a (-2)-curve.

Example Let A be an Abelian surface. The quotient $A/[-1]$ of A by the involution $[-1] : z \to -z$ is a surface with 16 nodal singularities $\mathbf{A}_1$, which are the image of the 16 2-torsion points of A. The minimal desingularization of $A/[-1]$ is denoted $\mathrm{Km}(A)$ and is called a Kummer surface. A Kummer surface is K3, it contains a $16\mathbf{A}_1$-configuration, which means 16 disjoint (-2)-curves.

Theorem (Nikulin) Let X be a K3 surface such that there exists a $16\mathbf{A}_1$-configuration on it. Then there exist an Abelian surface A such that $X \simeq \mathrm{Km}(A)$ and under that isomorphism, the $16\mathbf{A}_1$ on X comes from the resolution of $A/[-1]$.

Shioda raised the following question:
if $\mathrm{Km}(A) \simeq \mathrm{Km}(B)$, is it true that $A \simeq B$?
This is false in general (Gritsenko-Huleck, Hosono-Lian-Oguiso-Yau).
In a previous work with A. Sarti, we constructed the first explicit counter-examples.

Let A be an Abelian surface and $\sigma : A \to A$ be an order 3 symplectic automorphism. σ fixes 9 3-torsion points and A/σ has 9 cusps. The minimal resolution is a K3 surface, denoted by $\mathrm{Km}_3(A)$, and called a generalised Kummer surface. A cusp singularity is resolved by a $\mathbf{A}_2$-configuration i.e. two (-2)-curves C_1, C_2 such that $C_1 C_2 = 1$. The surface $\mathrm{Km}(A)$ contains a $9\mathbf{A}_2$-configuration.

Theorem (Bertin) Let X be a K3 surface such that there exists a $9\mathbf{A}_2$-configuration on it. Then X is a generalised Kummer surface: there exist an Abelian surface A and a symplectic order 3 automorphism σ such that $X \simeq \mathrm{Km}_3(A)$ and under that isomorphism, the $9\mathbf{A}_2$-configuration on X comes from the resolution of A/σ.

Generalised Shioda question: if $\mathrm{Km}_3(A) \simeq \mathrm{Km}_3(B)$, is it true that $A \simeq B$?

Generalised Shioda question: if $\mathrm{Km}_3(A) \simeq \mathrm{Km}_3(B)$, is it true that $A \simeq B$?

Let us fix $X = \mathrm{Km}_3(A)$. It is not difficult to prove that:
Proposition The isomorphism classes of Abelian surfaces B such that $X \simeq \mathrm{Km}_3(B)$ are in one to one correspondance with the $\mathrm{Aut}(X)$-orbits of $9\mathbf{A}_2$-configurations on X.

In particular, the generalized Shioda question has affirmative answer for the generalized Kummer $X = \mathrm{Km}_3(A)$ if and only if the number of $\mathrm{Aut}(X)$-orbits of $9\mathbf{A}_2$-configurations on X is 1.

Our initial aim was to construct a counter-example, as we did for the classical Kummer surfaces. In this talk we explain how to construct 9 more $9\mathbf{A}_2$-configurations on certain generalised Kummer surfaces X. We will then construct automorphisms of the K3 surface X which send the configurations to each others, so that one stays in the same orbit under $\mathrm{Aut}(X)$.
We do not answer to the generalised Shioda question, but thanks to that question, we know much more on the geometry of the K3 surface we study, and on its automorphism group.

We do not answer to the generalised Shioda question, but thanks to that question, we know much more on the rich geometry of the K3 surface we study, and on its automorphism group.

- Hesse Pencil, Hesse and dual Hesse configurations
- An interesting $(9_8, 12_6)$-configuration of points and conics
- Construction of the generalised Kummer surface X and its natural fibration
- Nine new $9\mathbf{A}_2$-configurations on X
- Construction of automorphisms of X
- Another geometric model of X

Let $\mathcal{P}, \mathcal{L}$ be two finite sets, of respective order m, n. Recall that a (m_s, n_t)-configuration is the data of a subset $\mathcal{R} \subset \mathcal{P} \times \mathcal{L}$ such that for any fixed $p \in \mathcal{P}$, one has:

$$|\{(p, \ell) \in \mathcal{R} \mid \ell \in \mathcal{L}\}| = s,$$

and for any fixed $\ell \in \mathcal{L}$, one has:

$$|\{(p, \ell) \in \mathcal{R} \mid p \in \mathcal{P}\}| = t.$$

That implies $ms = nt$. If $m = n$, this is called a m_s-configuration.

In practice $\mathcal{R}$ is rarely made explicit.

Example You may imagine that $\mathcal{P}$ is a set of points, and $\mathcal{L}$ is a set of lines. If on all lines in $\mathcal{L}$ there are t points in $\mathcal{P}$, and through all points of $\mathcal{P}$ there pass s lines, that gives an example of a (m_s, n_t)-configuration.

For $\lambda \in \mathbb{P}^1$, let E_λ be the cubic

$$E_\lambda : \ \ x^3 + y^3 + z^3 - 3\lambda xyz = 0.$$

That cubic is smooth if and only if $\lambda \notin 1, \omega, \omega^2, \infty$, for $\omega^3 = 1$. The pencil

$$\{x^3 + y^3 + z^3 - 3\lambda xyz \mid \lambda \in \mathbb{P}^1\}$$

is called the Hesse pencil.

The base locus $\mathcal{T}_9$ of the Hesse Pencil is also the set of flex points of the elliptic curves E_λ, this is also the set of 3-torsion points of these curves.

Let $\mathcal{L}_{12}$ be the set of lines that contains at least 2 points in $\mathcal{T}_9$. Then

- $|\mathcal{L}_{12}| = 12$.
- Each line $L \in \mathcal{L}_{12}$ contains 3 points in $\mathcal{T}_9$.
- Each point $q \in \mathcal{T}_9$ is contained on 4 lines.

Thus the sets $(\mathcal{T}_9, \mathcal{L}_{12})$ form a $(9_4, 12_3)$-configuration of points and lines. That configuration is called the Hesse configuration.

Each of the 4 singular cubics E_λ, $\lambda \in 1, \omega, \omega^2, \infty$ is a union 3 lines.
In fact the 12 lines in $\mathcal{L}_{12}$ are the union of these 4 triangles. The lines in $\mathcal{L}_{12}$ meet in $12 = 4 \cdot 3$ other points.
Let $W \to \mathbb{P}^2$ be the blow-up at $\mathcal{T}_9$, the 9 base points of the elliptic pencil.
There exists a natural fibration $W \to \mathbb{P}^1$ with fiber E_λ at λ, whose 4 singular fibers are 'triangles' i.e. of type $\tilde{\mathbf{A}}_2$.

Remark Removing a point in $\mathcal{T}_9$ and the 4 lines through it, one gets a 8_3-configuration of points and lines.

The dual Hesse configuration is obtained by taking as a set of points $\mathcal{P}_{12}$ the lines in $\mathcal{L}_{12}$ viewed as points in the dual plane, and as a set $\mathcal{L}_9$ the lines in the dual plane containing at least 2 of the points in $\mathcal{P}_{12}$. The sets $\mathcal{P}_{12}, \mathcal{L}_9$ form a $(12_3, 9_4)$-configuration, called the dual Hesse configuration.

The Hesse and dual Hesse configurations are unique, up to projective automorphisms.

Let f_λ be the form $f_\lambda = x^3 + y^3 + z^3 - 3\lambda xyz$. Start with the cubic $E_\lambda = \{f_\lambda = 0\} \hookrightarrow \mathbb{P}^2$. The dual curve $C_\lambda \hookrightarrow \mathbb{P}^2$ is the image of the cubic E_λ by the dual map $\nabla_\lambda : \mathbb{P}^2 \to \mathbb{P}^2$, $(x : y : z) \to (\frac{\partial f_\lambda}{\partial x} : \frac{\partial f_\lambda}{\partial y} : \frac{\partial f_\lambda}{\partial z})$. Geometrically, it maps a point p to the point $[TE_{\lambda,p}]$ in the dual projective space representing the tangent line to E_λ at p. The curve C_λ is

$$\begin{array}{rl} C_\lambda : & (x^6 + y^6 + z^6) + 2(2\lambda^3 - 1)(x^3y^3 + x^3z^3 + y^3z^3) \\ & -6\lambda^2 xyz(x^3 + y^3 + z^3) - 3\lambda(\lambda^3 - 4)x^2y^2z^2 = 0. \end{array}$$

It is singular at 9 points, and the singularities are $\mathbf{a}_2$ singularities (local equation $x_1^2 = x_2^3$). This set $\mathcal{P}_9 = \mathcal{P}_9(\lambda)$ of 9 cusps is the image of the base locus $\mathcal{T}_9$ by the dual map; it is:

$$\begin{array}{lll} p_1 = (\lambda : 1 : 1), & p_4 = (\lambda : \omega : \omega^2), & p_7 = (\lambda : \omega^2 : \omega) \\ p_2 = (1 : \lambda : 1), & p_5 = (\omega^2 : \lambda : \omega), & p_8 = (\omega : \lambda : \omega^2) \\ p_3 = (1 : 1 : \lambda), & p_6 = (\omega : \omega^2 : \lambda), & p_9 = (\omega^2 : \omega : \lambda). \end{array}$$

The union of the points $p_k = p_k(\lambda)$, $\lambda \in \mathbb{P}^1$ is a line L_k, and the 9 lines $L_1, \dots, L_9$ are the 9 lines of the dual Hesse configuration $(\mathcal{P}_{12}, \mathcal{L}_9)$.

A $(9_8, 12_6)$-configuration of points and conics

There is a unique conic passing through 5 points in general position.

Theorem

Let $\mathcal{C}_{12} = \mathcal{C}_{12}(\lambda)$ be the set of conics that contains at least 6 points in $\mathcal{P}_9 = \mathcal{P}_9(\lambda)$. Then

- $|\mathcal{C}_{12}| = 12$, *each conic is irreducible*
- *The sets $(\mathcal{P}_9, \mathcal{C}_{12})$ form a $(9_8, 12_6)$-configuration.*
- *The union $\cup C$, $C \in \mathcal{C}_{12}$ has singular set $\mathcal{P}_9(\lambda) \cup \mathcal{P}_{12}$, where we recall that $\mathcal{P}_{12}$ is the set of intersection points of the lines in $\mathcal{L}_9$, the dual Hesse configuration.*
- *Let $C \in \mathcal{C}_{12}$ be a conic. The intersections with the other conics C' are transverse. One has $C \cap C' \subset \mathcal{P}_9$, unless for a unique conic C', for which $C \cap C'$ contains a unique point in $\mathcal{P}_{12}$.*
- *For $q \in \mathcal{P}_9$ define $\mathcal{P}_q = \mathcal{P}_9 \setminus \{q\}$ and let $\mathcal{C}_q$ be the 8 conics that contains q. The set $(\mathcal{P}_q, \mathcal{C}_q)$ form a 8_5-configuration $\tilde{A}_q$.*

I. Dolgachev, A. Laface, U. Persson, G. Urzua have also independently discovered that configuration of points and conics.
P. Pokora and T. Szemberg studied the freeness of that configuration.

Warning : the set $\mathcal{P}_9 = \mathcal{P}_9(\lambda)$ of cusps of the sextic C_λ depends on the parameter λ, and so is $\mathcal{C}_{12} = \mathcal{C}_{12}(\lambda)$. We take the parameter λ generic.

Proposition The linear system δ of sextic curves passing through $\mathcal{P}_9$ with multiplicity at least 2 is a pencil. It contains the sextic curve C_λ and a unique non-reduced curve, $2\mathrm{Ca}(\lambda)$, where

$$\mathrm{Ca}(\lambda) : \; x^3 + y^3 + z^3 - \tfrac{(\lambda^3+2)}{\lambda} xyz = 0, \tag{1}$$

is the so-called Cayleyan curve of C_λ.

Let $f : Z \to \mathbb{P}^2$ be the blow-up at the 9 points in $\mathcal{P}_9$. Let $E_1, \dots, E_9$ be the 9 exceptional curves.
The moving part of the pull-back to Z of the pencil of sexics is now base point free and define a morphism $\varphi' : Z \to \mathbb{P}^1$ for which the curves E_k are 2-sections, which means $E_k \cdot (Fiber) = 2$.

The strict transform $C'_\lambda \simeq E_\lambda$ on Z of C_λ is a curve that intersect the exceptional divisors tangentially at one point.

Proposition The exists a double cover $X_\lambda \to Z$ of Z branched over C'_λ. The surface $X = X_\lambda$ is a K3 surface.The pull-back to X of an exceptional curves E_k is the union $A_k + A'_k$ of two (-2)-curves.
(We recall that a (-2)-curve on a K3 surface is a smooth rational curve).
The elliptic fibration $\varphi' : Z \to \mathbb{P}^1$ pull-backs to an elliptic fibration

$$\varphi : X \to \mathbb{P}^1.$$

The curves $A_k, A'_k, \kappa = 1, \dots, 9$ are sections of the elliptic fibration.

Another view point: If instead we consider the double cover $Y \to \mathbb{P}^2$ branched over the sextic curve C_λ, we see that X_λ is the minimal resolution of the 9 cusps singularities over the points in $\mathcal{P}_9$.

Since the K3 surface X_λ contains 9 disjoint $\mathbf{A}_2$-configurations, it is a generalised Kummer surface. In our case Birkenhake and Lange proved that $X = \mathrm{Km}_3(A)$ with $A = E_\lambda \times E_\lambda$.

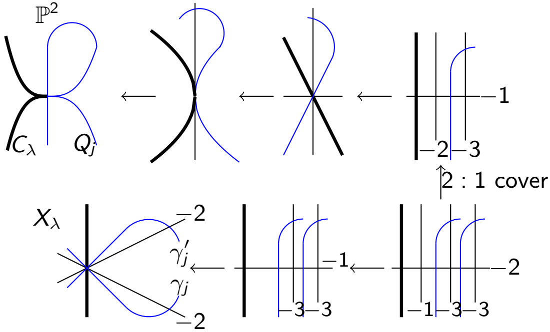
Recall that we say that two (-2)-curves C, C' on a K3 surfaces such that $CC' = 1$ form a $\mathbf{A}_2$-configuration. The K3 surface $X = X_\lambda$ contains 9 disjoint $\mathbf{A}_2$-configurations, thus it is a generalized Kummer Surface.
We want to know more $9\mathbf{A}_2$-configurations on X_λ. For that aim, we use the pull-back to X_λ of the special set $\mathcal{C}_{12}$ of 12 conics. Let us denote by C_{ijklmn} the conic that pass through points p_s, $s \in \{i, j, k, l, m, n\}$ (when it exists). The following 4 sextic curves are element of the pencil δ of sextics with multiplicity ≥ 2 at the p_k's:

$$C_{123456} + C_{123789} + C_{456789},\ C_{124578} + C_{134679} + C_{235689},$$
$$C_{124689} + C_{135678} + C_{234579},\ C_{125679} + C_{134589} + C_{234678}.$$

The strict transform of these curves by the blow-up $Z \to \mathbb{P}^2$ at points p_k are 4 singular fibers of type $\tilde{\mathbf{A}}_2$. The pull-back to X of these 4 fibers via the double cover $X_\lambda \to Z$ are 8 fibers of type $\tilde{\mathbf{A}}_2$. These are all the singular fibers of $\varphi : X_\lambda \to \mathbb{P}^1$.
That way, we obtain $3^8 = 6561$ $8\mathbf{A}_2$-configurations. For each $j \in \{1, ..., 9\}$, the pull-back $\mathcal{A}'_j$ to X_λ of the set $\mathcal{C}_{p_j}$ of 8 conics containing a fixed point $p_j \in \mathcal{P}_9$ is among these $8\mathbf{A}_2$-configurations.

Theorem

For $j \in \{1,,\dots,9\}$, there exists a unique rational quartic curve Q_j containing $\mathcal{P}_9$ with a unique singularity at the point p_j. That curve singularity has type $\mathbf{d}_5$. The strict transform on X_λ of the curve Q_j by $\eta : X \to \mathbb{P}^2$ is the union of two (-2)-curves γ_j, γ_j' which form an $\mathbf{A}_2$-configuration.
The curves γ_j, γ_j' and the 16 curves in $\mathcal{A}_j'$ form a $9\mathbf{A}_2$-configuration $\mathcal{A}_j$.



Theorem

A model of X_λ in $\mathbb{P}^1 \times \mathbb{P}^2$ with coordinates $t \in \mathbb{P}^1$, $(x:y:z) \in \mathbb{P}^2$ is

$$X_\lambda : \quad x^3 + y^3 + z^3 + \frac{\lambda^3(t^2+3) - 4t^2}{\lambda^2(t^2-1)} xyz = 0. \tag{2}$$

The Mordell-Weil group of sections of the natural fibration $\varphi : X_\lambda \to \mathbb{P}^1$ is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2$; it is generated by the 18 curves A_k, A_k''s.
The torsion part acts transitively on the 9 $9A_2$-configurations we found.

The equation of X_λ is obtained using Magma software (non trivial).
Shimada computed all possible Mordell-weil group of elliptic K3 surfaces. For the assertion on the generators, we use Shioda's theory of Mordell-Weil lattice, for that aim we compute a base of the Néron-Severi group $\mathrm{NS}(X_\lambda)$ of X_λ, and its discriminant.
For the action of the torsion part on the 9 $9A_2$-configurations, we use the explicit addition law on the model (2), (one knows explicit equations of the (-2)-curves in the fibers).

Recall that we denoted by $A_1, A_1', \dots, A_9, A_9'$ the 18 (-2)-curves of the natural $9\mathbf{A}_2$-configuration $\mathcal{C}$ (these are sections of φ).
Let $\mathcal{C}' = B_1, B_1', \dots, B_9, B_9'$ be one of the $9\mathbf{A}_2$-configurations we constructed.

Theorem

There exists an automorphism ϕ of X_λ such that $\phi(\mathcal{C}) = \mathcal{C}'$.

Suppose that such ϕ exists.The orthogonal complement of $\mathcal{C}$ is generated by D_2, the pull-back of a line in $\mathbb{P}^2$. One computes that the orthogonal complements of $\mathcal{C}'$ is also generated by divisor D_2' of square 2. Since ϕ^* must preserve the Néron-Severi lattice, we have $\phi^*(D_2) = D_2'$.
There are $2^9 9! = 185.794.560$ bijective maps

$$\mu : \{A_1, A_1', \dots, A_9, A_9'\} \to \{B_1, B_1', \dots, B_9, B_9'\}$$

which preserves the incidence relations in $\mathcal{C}$ and $\mathcal{C}'$. Each map extends uniquely to an isometry $\tilde{\phi}_\mu$ of $\mathrm{NS}(X_\lambda) \otimes \mathbb{Q}$. We compute that among these maps, only 864 are isometries of $\mathrm{NS}(X_\lambda)$ (we know a base of it).
Among the 864 maps, only 36 are effective Hodge isometries, and thus by the Torelli Theorem these 36 maps lift to automorphisms ϕ of the K3 X_λ such that $\phi(\mathcal{C}) = \mathcal{C}'$.

Theorem A model of X_λ in $\mathbb{P}^1 \times \mathbb{P}^2$ is

$$X_\lambda : \quad x^3 + y^3 + z^3 + \frac{\lambda^3(t^2+3) - 4t^2}{\lambda^2(t^2-1)} xyz = 0.$$

Let $\pi_2 : X_\lambda \to \mathbb{P}^2$ be the projection on the second factor, the map π_2 contracts the curves $A_1, \dots, A_9$. By computing the branch locus of π_2, one gets

Theorem

The surface X_λ is the minimal desingularization of the double cover of the plane branched over the union of $E_\lambda : \quad x^3 + y^3 + z^3 - 3\lambda xyz = 0$ and the Hessian of E_λ:

$$\mathrm{He}(\lambda) : \quad x^3 + y^3 + z^3 + \tfrac{(\lambda^3 - 4)}{\lambda^2} xyz = 0.$$

The images in $\mathbb{P}^2$ by π_2 of the 24 irreducible components of the 8 singular fibers of the fibration $\varphi : X_\lambda \to \mathbb{P}^1$ are the 12 lines of the Hesse configuration.

Thank you !