On the geometry of K3 surfaces with finite automorphism group

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Notations

A part of this talk is on a joint work with M. Artebani and C. Correa.

- X smooth complex projective K3 surface, K3 means:

$$K_X \simeq \mathcal{O}_X \textit{ and } H^1(X, \mathcal{O}_X) = 0.$$

- Examples: double cover branched over a sextic curve in $\mathbb{P}^2$, quartics in $\mathbb{P}^3$, degree 6 complete intersection surface in $\mathbb{P}^4$...
- Let $NS(X)$ be the Néron-Severi group of X.
  The Picard number of X is $\rho_X = \mathit{rank}\, NS(X)$.
- One has $H^2(X, \mathbb{Z}) \simeq \mathbb{Z}^{22}$, thus $h^{1,1} = 20$ and

$$1 \leq \rho_X \leq 20.$$

- A smooth rational curve on the K3 surface X is called a (-2)-curve.
  ! We will often (by abuse) confuse a (-2)-curve and its class in $NS(X)$. !
- $\equiv$ denotes linear equivalence between divisors

History: the Nikulin and Vinberg classification

- It is a classical result that the automorphism group of a smooth curve is finite if and only if its genus is > 1.
- For surfaces there is also a classification according to the Kodaira dimension, initiated by the Italian Algebraic Geometry school. (Think to Cremona group for $\mathbb{P}^2$ vs surfaces of general type which have finite number of birational maps).
- In the 70's and early 80's the K3 surfaces X with finite automorphism group have been classified by
  - Piatetski-Shapiro, Shafarevich (cases $\rho_X = 1, 2$)
  - Nikulin (cases $\rho_X \in \{3, 5, ..., 19, 20\}$)
  - Vinberg (case $\rho_X = 4$)

  This talk is on K3's with finite automorphism group and $\rho_X > 2$

The 118 families of K3 with finite Aut. group, Notations

The output of Nikulin-Vinberg classification is a list of 118 families of such K3 surfaces.
Each family $\mathcal{M}_L$ is characterised by the lattice L such that there is a primitive embedding $L \hookrightarrow NS(X)$ for each K3 surface X in the family (with equality for the general K3).

Remark: It is a general result that the moduli $\mathcal{M}_L$ of L-polarized K3 surfaces has dimension $20 - Rank(L)$.

More notations
We denote by $\mathbf{A}_n, \mathbf{D}_n, \mathbf{E}_n$ the negative definite lattice associated to the root system with same symbol.
For a symmetric integral matrix M let $[M]$ be the lattice with Gram matrix M.
By example the hyperbolic lattice is $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
$L(m)$ means the lattice L with quadratic form multiplied by m.

A few examples of lattices L among the 118

The K3 surfaces X with $NS(X)$ isomorphic to the lattices below have a finite automorphism group:

$$S_{6,1,1} = \begin{bmatrix} -14 & 2 & 2 \\ 2 & -2 & 4 \\ 2 & 4 & -8 \end{bmatrix},\ L_{24} = \begin{bmatrix} 12 & 2 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix},$$

$$L_{12} = \begin{bmatrix} 0 & 3 \\ 3 & -2 \end{bmatrix} \oplus \mathbf{A}_2$$

$$U(4) \oplus \mathbf{A}_1^{\oplus 3},\ \ U(2) \oplus \mathbf{A}_1^{\oplus 7},\ \ U \oplus \mathbf{D}_4 \oplus \mathbf{A}_1^{\oplus 5},\ \ U \oplus \mathbf{E}_8 \oplus \mathbf{E}_8 \oplus \mathbf{A}_1 \ldots$$

Other viewpoints: Mori dream surfaces, rational curves

- Theorem (Nikulin) A K3 surface X with $\rho_X > 2$ has a finite automorphism group if and only if X contains a finite non-zero number of (-2)-curves.
- Recall that the Cox ring of a variety X is

$$Cox(X) = \bigoplus_{[D] \in Cl(X)} H^0(X, \mathcal{O}_X(D))$$

  Theorem (Artebani-Hausen-Laface and indep. McKernan). For a K3 surface with $\rho_X > 2$, $Cox(X)$ is finitely generated (X is a "Mori dream surface") if and only if the effective cone is polyhedral i.e. there are a finite ($\neq 0$) number of (-2)-curves on X.
- Theorem Any K3 surface contains infinitely many rational curves. (Works of Mori, Mukai, Chen, Lewis, Bogomolov, Hasset, Tschinkel, Tayou, Charles, Li, Chen, Gounelas, Liedke...).

  The last indomitable cases that remained were K3 surfaces with finite automorphism group, $\rho_X = 4$ and no elliptic fibrations.

Aims: from the abstract lattices to equations

- Starting from the knowledge of the lattice L from Nikulin-Vinberg classification, describe the (finite!) set of (-2)-curves on K3 surfaces X such that $L = NS(X)$, by which we mean find their classes in $NS(X)$ and give their intersection matrix or the dual graph.

- Obtain a geometric model of these surfaces, either as a double cover of a known surface, or as a (maybe singular) surface in some $\mathbb{P}^n$ and then, when possible, describe their equations, and the realizations of the (-2)-curves as lines, conics etc...

- Study the unirationality of their moduli spaces $\mathcal{M}_L$.

Plan of the remaining of the talk

No plan, rather a journey through the world of K3 surfaces with finite automorphism groups.

- Explain some of the used tools
- A first series of examples
- A remark on Reid "famous 95" families of K3 surfaces with a link to K3 surfaces with finite automorphisms; unirationality
- Nikulin Star-Shaped lattices and K3 surfaces
- A table with some results
- More involved examples

Tools, Sieve of Eratosthenes-Vinberg for K3's

Let us describe Vinberg's algorithm to find the (-2)-curves on K3's.
Let $L \simeq NS(X)$ be the lattice of a K3 surface X.

We choose an ample class of X as any element $P \in L$ such that $P^2 > 0$ and the negative definite lattice $P^\perp$ does not contain roots ie. elements c such that $c^2 = -2$.

For $d \in \mathbb{N}^*$, define the (finite & computable) set of degree $\leq d$ roots:

$$\mathcal{R}_d^+ = \{c \in L = NS(X) \,|\, 0 < P.c \leq d, \text{ and } c^2 = -2\}.$$

By Riemann-Roch Theorem, a root $c \in \mathcal{R}_d^+$ is the class of an effective divisor on X.
Let $\mathcal{N}_d \subset \mathcal{R}_d^+$ be the set of 'primes' i.e. classes of (-2)-curves up to degree d. One can recognise these irreducible divisors as follows:

Let d_o be the first $d \in \mathbb{N}$ such that $\mathcal{R}_d^+ \neq \emptyset$. If $c \in \mathcal{R}_{d_o}^+$ is not irreducible, one may suppose that $c = u + v$ with u, v effective, $u^2 = -2$.
But then u is a root of degree $0 < P.u < P.c = d_o$, a contradiction, thus $c \in \mathcal{R}_{d_o}^+$ is irreducible, and $\mathcal{N}_{d_o} = \mathcal{R}_{d_o}^+$.

Tools, Sieve of Eratosthenes-Vinberg for K3's

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Suppose that one knows the set $\mathcal{N}_d \subset \mathcal{R}_d^+$ of (-2)-curves up to degree d.

Let $c \in \mathcal{R}_{d+1}^+$ of degree $d+1$. Suppose that $\forall c' \in \mathcal{N}_d, c.c' \geq 0$ and c is not irreducible, then $c = \sum a_i c_i$ with $a_i > 0$ and c_i with $c_i.P < d+1$ and either $c_i^2 \geq 0$ or $c_i \in \mathcal{N}_d$. Then:
$c^2 = (\sum a_i c_i)c = \sum a_i(c_i.c) \geq 0$, which is absurd since $c^2 = -2$.

Thus the set of (-2)-curves of degree $d+1$ is

$$\mathcal{N}_{d+1} = \mathcal{N}_d \cup \{c \in \mathcal{R}_{d+1}^+ \mid P.c = d+1,\ \forall c' \in \mathcal{N}_d,\ c.c' \geq 0\}.$$

The algorithm for finding the (-2)-curves among the (-2)-classes is thus like the Erathostene sieve: if one knows the 'primes' c of degree $\leq d$, every root c' of degree $d+1$ such that $cc' < 0$ must be discard, the remaining roots of degree $d+1$ are 'primes' ie. (-2)-curves.

Ernest Borisovich Vinberg (1937–2020)

(Picture from EMS Newsletter December 2016)

Tools

On a K3 surface X with finite automorphism group, the effective cone is polyhedral, and so is the dual cone, which is the Nef cone.

In order to check that one has the complete list of (-2)-curves, we use the characterization (by Vinberg, but not formulated in that language, and M. Artebani, C. Correa Diesler and A. Laface):
Proposition The convex polyhedral cone generated by (-2)-curves $A_1, \dots, A_m$, $(m \geq \rho_X)$ is the effective cone if and only if its facets are demi-definite negative.

Knowing the Nef cone, one can then easily compute interesting nef or ample classes, which classes we use in order to obtain projective models of the surface X.

Classical tool: double covers and (-2)-curves

Let X be a K3 surface. A base point free linear system $|D_2|$ where $D_2^2 = 2$ defines a double cover $f : X \to \mathbb{P}^2$ of the plane branched over a sextic curve C_6.
f contracts the (-2)-curves A such that $D_2 A = 0$ to ADE singularities of the curve C_6.

Proposition i) Suppose $D_2 \equiv A_1 + A_2$ with A_1, A_2 two (-2)-curves and $D_2 A_1 = D_2 A_2 = 1$.
Then there exists a line $L \hookrightarrow \mathbb{P}^2$ such that $D_2 \equiv f^* L$. The line is tritangent to the branch locus C_6.
ii) Suppose $2D_2 \equiv A_1 + A_2$ with A_1, A_2 two (-2)-curves and $D_2 A_1 = D_2 A_2 = 2$.
Then there exists a conic $C \hookrightarrow \mathbb{P}^2$ such that $2D_2 \equiv f^* C$. The conic C is 6-tangent to the branch locus C_6.

Example 1. The case $S_{6,1,1}$

Let X be a K3 surface with Néron-Severi group

$$NS(X) \simeq S_{6,1,1} = \begin{bmatrix} -14 & 2 & 2 \\ 2 & -2 & 4 \\ 2 & 4 & -8 \end{bmatrix}.$$

There are 4 (-2)-curves $A_1, \dots, A_4$ on X, with intersection matrix

$$\begin{pmatrix} -2 & 4 & 2 & 2 \\ 4 & -2 & 2 & 2 \\ 2 & 2 & -2 & 10 \\ 2 & 2 & 10 & -2 \end{pmatrix}$$

The curves A_1, A_2, A_3 generates $NS(X)$. The divisor $D_4 = A_1 + A_2$ has square 4 and $2D_4 = A_3 + A_4$.

Proposition i) The surface X is a quartic in $\mathbb{P}^3$ with a hyperplane section which is the union of two conics, and a quadric section which is the union of two degree 4 smooth rational curves.

ii) The moduli $\mathcal{M}_{S_{6,1,1}}$ of K3 surfaces X with $NS(X) \simeq S_{6,1,1}$ is unirational.

Example 1. The case $S_{6,1,1}$

Proposition i) The surface X is a quartic in $\mathbb{P}^3$ with an hyperplane section D_4 such that $D_4 = A_1 + A_2$ and $2D_4 \equiv A_3 + A_4$.
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Proof. i) One has $D_4A_1 = D_4A_2 = 2$, $D_4A_3 = D_4A_4 = 4$, thus D_4 is ample. The very-ampleness is verified using Saint-Donat's criterias.
ii) Let $Q_4 \hookrightarrow \mathbb{P}^3$ be a generic quartic that contains a conic C_1 and a degree 4 rational curve C_3 in $\mathbb{P}^3$ such that $Degree(C_1 \cap C_3) = 2$.
The plane containing the conic C_1 cuts Q_4 into a residual conic C_2.
One can prove that there is a unique quadric $Q_2 \subset \mathbb{P}^3$ containing C_3.
Thus the intersection $Q_4 \cap Q_2$ splits as $H = C_3 + C_4$ where C_4 has degree 4. We have $8 = C_3.2H = C_3.(C_3 + C_4) = -2 + C_3C_4$ thus $C_3C_4 = 10$ and computing $(C_3 + C_4)^2$, one get $C_4^2 = -2$. $(C_1C_3 = C_2C_3 = C_1C_4 = 2)$.
Therefore the surface Q_4 is such that $S_{6,1,1} \subset NS(Q_4)$, and one has equality by genericity assumption.
Thus to construct such a surface is the same as constructing the curves C_1, C_3 with $C_1C_3 = 2$ and taking a quartic in the linear system containing them, that construction is made using rational parameters. $\square$

Ex.2, one construction, 2 lattices: twin cases L_{12} and L_{24}

Let X, X' be two K3 surfaces with respective Néron-Severi group:

$$L_{12} = \begin{bmatrix} 0 & 3 \\ 3 & -2 \end{bmatrix} \oplus \mathbf{A}_2, \quad L_{24} = \begin{bmatrix} 12 & 2 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

Proposition There are exactly 6 (-2)-curves on X and X'. One can order these 6 curves so that their intersection matrices are respectively:

$$\left(\begin{array}{cccccc} -2 & 3 & 0 & 1 & 0 & 1 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 1 & 0 & 3 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 & 3 \\ 1 & 0 & 0 & 1 & 3 & -2 \end{array}\right), \left(\begin{array}{cccccc} -2 & 3 & 0 & 1 & 0 & 1 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 3 & 0 & 1 \\ 1 & 0 & 3 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 3 \\ 1 & 0 & 1 & 0 & 3 & -2 \end{array}\right).$$

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Ex.2, one construction, 2 lattices: twin cases L_{12} and L_{24}

Proposition 1. Let $A_1, \dots, A_6$ be the 6 (-2)-curves on X (resp. X').
i) The divisor $D_2 = A_1 + A_2$ is ample, of square 2, and

$$D_2 \equiv A_3 + A_4 \equiv A_5 + A_6.$$

ii) The linear system $|D_2|$ defines a double cover of $\mathbb{P}^2$ branched over a smooth sextic curve C_6.
The 3 curves $A_1 + A_2$, $A_3 + A_4$, $A_5 + A_6$ are mapped onto 3 lines in $\mathbb{P}^2$, which lines are tritangent to C_6.
iii) Conversely, let Y be the double cover of $\mathbb{P}^2$ branched over a smooth sextic curve which has three tritangent lines. Suppose that $\rho_Y = 4$. Then either $NS(Y) \simeq L_{12}$ or $NS(Y) \simeq L_{24}$.
iv) (Recall that X is such that $NS(X) \simeq L_{12}$). The branch locus C_6 of the double cover $X \to \mathbb{P}^2$ has the form

$$\ell_1\ell_2\ell_3 g - f^2 = 0,$$

where the ℓ_j are linear forms, and f, g are cubic forms.
v) The moduli space $\mathcal{M}_{L_{12}}$ of K3's X with $NS(X) \simeq L_{12}$ is unirational.

Ex.2, one construction, two lattices: twin cases L_{24} and L_{12}

Here is an example of a K3 surface Y with $NS(Y) \simeq L_{24}$:

Example 1. Let ℓ_1, ℓ_2, q_4, f be the forms

$$\begin{gathered}\ell_1 = x + y + 2z, \quad \ell_2 = -3x + 2y + z, \\ q_4 := 8x^4 + x^3y + x^2y^2 + 3xy^3 - 2y^4 - 20x^3z - 2x^2yz - xy^2z + 3y^3z \\ -12x^2z^2 + xyz^2 + 4yz^3, \\ f = 5x^3 - 3x^2y + xy^2 + 4y^3 + 2x^2z - 3xyz - 3y^2z + 5xz^2 + 4yz^2.\end{gathered}$$

The smooth sextic curve $C_6 = \{\ell_1\ell_2q_4 - f^2 = 0\}$ has three tritangent lines which are $\ell_1 = 0$, $\ell_2 = 0$ and the line $\{y = 0\}$. Let $Y \to \mathbb{P}^2$ be the double cover branched over C_6. One has $L(24) \hookrightarrow NS(Y)$. Using its reduction modulo 13, point counting and the Artin-Tate formula, one obtains that the Picard number of Y is 4, thus $NS(Y) \simeq L_{24}$.

Remark: 1. A K3 surface X' such that $NS(X') \simeq L_{24}$ has no elliptic fibrations, contrary to X.
2. $Aut(X') \simeq \mathbb{Z}/2\mathbb{Z}$ for X' (Kondo).

Ex.3 $U(2) \oplus \mathbf{A}_1^{\oplus 7}$, 120 conics, degree 1 Del Pezzo surfaces

Using lattice theoretic considerations, Kondo proved that the generic K3 surface X with $NS(X) \simeq U(2) \oplus \mathbf{A}_1^{\oplus 7}$ has automorphism group $(\mathbb{Z}/2\mathbb{Z})^2$. Let $\sigma_1, \sigma_2 \in Aut(X)$ be the non-symplectic involutions.

One construction of such surface X is natural, but there is a second unexpected construction:

Proposition i) The quotient surface X/σ_1 is a del Pezzo surface Z of degree 1 and the quotient map $f_1 : X \to Z$ is branched over a genus 2 curve C. There exist a blow-down map $Z \to \mathbb{P}^2$ such that the image of C is a sextic curve with 8 nodes, each at the 8 contracted (-1)-curves.
ii) Conversely, starting with a sextic plane curves having 8 nodes in general position, the minimal desingularisation of the double cover is a K3 surface with Néron-Severi group $U(2) \oplus \mathbf{A}_1^{\oplus 7}$.
The K3 surface X contains 240 (-2)-curves, these are the pull back of the 240 (-1)-curves of the del Pezzo surface.
iii) The pull back on X of the pencil $|-K_Z|$ gives another double cover $f_2 : X \to \mathbb{P}^2 = X/\sigma_2$. Its branch locus is a smooth sextic curve C_6 to which 120 conics are tangent at every intersection points with C_6. These 120 conics are the images of the 240 (-2)-curves on X.

Some remarks on the "famous 95"

Reid and independently Yonemura classified K3 surfaces which are anticanonical divisors with at most Gorenstein singularities in weighted projective threefolds $\mathbb{WP}^3 = \mathbb{WP}^3(\tilde{a})$.

There are 95 families of such surfaces, called the "famous 95", classified according to the 95 possible weights $\tilde{a}$ of $\mathbb{WP}^3(\tilde{a})$.
Belcastro studied the Néron-Severi group of these surfaces and the Néron-Severi group of their mirrors. We remark that

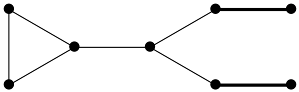
Remark i) The "95 famous" moduli spaces are unirational (check that the linear system $|-K_{\mathbb{PW}^3}|$ contains all the K3 surfaces of the family).
Among the 95 famous families, 25 are such that the K3 surfaces have finite automorphism group.
ii) Among the mirror surfaces, 7 families are K3 with finite automorphism group and are not listed among the 95.

We obtain in that way that 25 modulus among the 118 modulus of K3's with finite automorphism group are unirational.

The star shaped lattices of Nikulin: list of (-2)-curves

Proposition (Nikulin) Let X be a K3 surface such that $NS(X) \simeq U \oplus K$.
i) There is an elliptic pencil $\pi : X \to \mathbb{P}^1$ with a section; a fiber and the section generate the lattice isomorphic to the hyperbolic lattice U.
ii) Suppose $K = \bigoplus \mathbf{G}_i$ with the lattices $\mathbf{G}_i$ are among the lattices $\mathbf{A}_l$, $\mathbf{D}_m$, $\mathbf{E}_n$. Then the non-irreducible fibers of the fibration π are of Kodaira Néron type $\tilde{\mathbf{G}}_i$, where $\tilde{\mathbf{A}}_l, \tilde{\mathbf{B}}_m, \tilde{\mathbf{E}}_n$ are the extended Dynkin diagram.

The dual graph formed by the(-2)-curves of the singular fibers and by the section of the fibration is called the star of the lattice $U \oplus \bigoplus \mathbf{G}_i$.
By example the star of $NS(X) = U \oplus \mathbf{A}_2 \oplus \mathbf{A}_1^{\oplus 2}$ is



The star shaped lattices of Nikulin: list of (-2)-curves

Theorem (Nikulin) Let X be a K3 surface with Néron-Severi group isomorphic to $U \oplus K$, where K is among the following 29 lattices:

$$\mathbf{A}_2;\ \mathbf{A}_1 \oplus \mathbf{A}_2, \mathbf{A}_3;\ \mathbf{A}_1^{\oplus 2} \oplus \mathbf{A}_2, \mathbf{A}_2^{\oplus 2}, \mathbf{A}_1 \oplus \mathbf{A}_3, \mathbf{A}_4;$$
$$\mathbf{A}_1 \oplus \mathbf{A}_2^{\oplus 2}, \mathbf{A}_1^{\oplus 2} \oplus \mathbf{A}_3, \mathbf{A}_2 \oplus \mathbf{A}_3, \mathbf{A}_1 \oplus \mathbf{A}_4, \mathbf{A}_5, \mathbf{D}_5;$$
$$\mathbf{A}_2^{\oplus 3}, \mathbf{A}_3^{\oplus 2}, \mathbf{A}_2 \oplus \mathbf{A}_4, \mathbf{A}_1 \oplus \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_2 \oplus \mathbf{D}_4, \mathbf{A}_1 \oplus \mathbf{D}_5, \mathbf{E}_6; \mathbf{A}_7,$$
$$\mathbf{A}_3 \oplus \mathbf{D}_4, \mathbf{A}_2 \oplus \mathbf{D}_5, \mathbf{D}_7, \mathbf{A}_1 \oplus \mathbf{E}_6; \mathbf{A}_2 \oplus \mathbf{E}_6; \mathbf{A}_2 \oplus \mathbf{E}_8; \mathbf{A}_3 \oplus \mathbf{E}_8.$$

Then the K3 surface has finite automorphism group and the star of $U \oplus K$ is the dual graph of all the (-2)-curves on X.

That result gives a way of constructing these K3 surfaces, and knowing their (-2)-curves, by searching for a Weierstrass model of the natural fibration.

Nikulin also gives the number and configuration of (-2)-curves for some other cases, in particular when the Picard number is $\rho_X = 3$.

The classification table

In order to understand the tables in the next page, we explain the notations here:
These tables give the number of (-2)-curves: this is a lower bound if there is the sign $\dagger$ in the next column, it is the exact number otherwise,

An aleph $\aleph$ means that the lattice is among the 95 famous families,

an angle $\angle$ means that the lattice is a mirror of one of the 95 famous, but not one of these 95,

A star $\star$ means that their (-2)-curves configuration is predicted by Nikulin as star shaped lattices,

A $\mathbf{u}$ means unirational moduli space ; at least 60 modulis among the 118 are unirational. Unless for rank $\rho_X = 3, 4$, we have not systematically searched if the moduli spaces were unirational. It is very likely that, except perhaps a few cases, all are unirational.

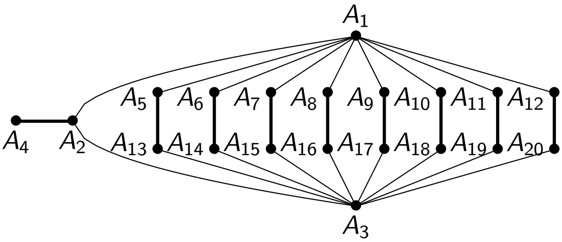
$\mathcal{M}_L$ unirational means in practice that one knows how to construct all the surfaces of the family.

The classification table

See last pages of the Atlas paper:
https://arxiv.org/abs/2003.08985

Example 4. The lattice $U \oplus \mathbf{D}_4 \oplus \mathbf{A}_1^{\oplus 5}$

A K3 surface X with $NS(X) \simeq U \oplus \mathbf{D}_4 \oplus \mathbf{A}_1^{\oplus 5}$ contains 90 (-2)-curves. The 20 first curves $A_1, \dots, A_{20}$ have the following dual graph



where a thin edge between two curves mean intersection number 1, a bold edge means intersection 2. For $k \in \{5, \dots, 12\}$, we have

$$F = A_2 + A_4 \equiv A_k + A_{k+8} = F_k,$$

these divisors $F, F_k,\ k = 5, ..., 12$ are 9 singular fibers of a fibration with sections A_1, A_3.

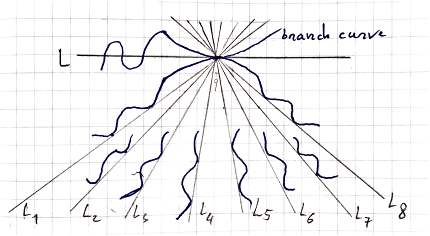
The divisor

$$D_2 = A_1 + 2A_2 + A_3 + A_4$$

is nef of square 2, with $D_2A_1 = D_2A_2 = D_2A_3 = 0$, $D_2A_j = 1$ for $j \in \{5, \dots, 20\}$ and $D_2A_j = 2$ for $j = 4$ or $j > 20$.
The linear system $|D_2|$ defines is a double cover $X \to \mathbb{P}^2$ branched over a sextic curve C_6 which has a $\mathbf{a}_3$ singularity q. Using the fibration, we get:

$$D_2 \equiv A_1 + A_2 + A_3 + A_k + A_{k+8}, \ \forall k \in \{5, \dots, 12\}.$$

The curves A_1, A_2, A_3 are contracted to q, the curve A_4 is mapped onto the 'tangent' L of C_6 at singularity q, the curves A_k, A_{k+8} with $k \in \{5, \dots, 12\}$ are mapped to 8 lines L_k going through q and which are tangent to the sextic at any other intersection points.



Example 4. The lattice $U \oplus \mathbf{D}_4 \oplus \mathbf{A}_1^{\oplus 5}$

For any subset $J = \{i, j, k, l\}$ of $\{5, \dots, 11\}$ of order 4 (35 such choices), let us define:

$$A_J = 2A_1 - A_4 + \sum_{t \in J} A_t$$
$$B_J = 4A_2 + 2A_3 + 3A_4 - \sum_{t \in J} A_t.$$

The classes A_J and B_J are the classes of the remaining 70 (-2)-curves $A_{21}, \dots, A_{90}$. Moreover we see that

$$2D_2 \equiv A_J + B_J,\ \forall J = \{i, j, k, l\} \subset \{5, \dots 12\},\ \#\{i, j, k, l\} = 4$$

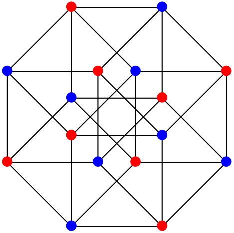
and therefore there exists 35 conics that are 6-tangent to C_6.
Let J, J' be two subsets of order 4 of $\{5, \dots, 11\}$. The configuration of curves $A_J, A_{J'}, B_J, B_{J'}$ is as follows:

$$A_J A_{J'} = B_J B_{J'} = 6 - 2\#(J \cap J'),$$
$$A_J B_{J'} = -2 + 2\#(J \cap J').$$

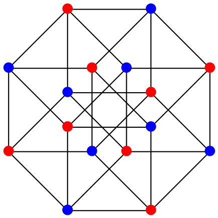
Example 5. The lattice $U(4) \oplus \mathbf{A}_1^{\oplus 3}$

A K3 surface X with $NS(X) \simeq U(4) \oplus \mathbf{A}_1^{\oplus 3}$ contains 24 (-2)-curves. The surface X is a double cover $X \to \mathbb{P}^2$ branched over a smooth sextic curve C_6. There exists 12 conics which are tangent to C_6 at all intersection points. The 24 (-2)-curves on X are irreducible components of the pull-back of these 12 conics.

There exists a partition of the 24 (-2)-curves into 3 sets S_1, S_2, S_3 of 8 curves each, such that for curves B, B' in two different sets S, S', one has $BB' = 0$ or 4. In the following graph, red vertices are curves in S, blue vertices are curves in S' and an edge links a red curve to a blue curve if and only if their intersection is 4:



Example 5. The lattice $U(4) \oplus \mathbf{A}_1^{\oplus 3}$

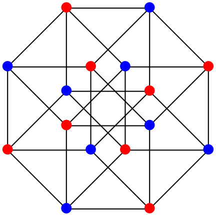


Proposition i) Up to a projective transformation, there exist a projective model of X as a quartic surface with a node at point $(1:1:1:1)$ and with equation

$$xyzt - q_2 q_2' = 0,$$

where q_2, q_2' are quadrics.
ii) The moduli space $\mathcal{M}_{U(4)\oplus \mathbf{A}_1^{\oplus 3}}$ is rational.

Each hyperplane section $x = 0, y = 0, z = 0, t = 0$ is union of two conics. It seems difficult to find the other (-2)-curves.



Thank you !