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Bounded Negativity, Miyaoka–Sakai Inequality, and Elliptic Curve Configurations

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Similar to the linear Harbourne constant recently introduced in [2], we study the elliptic H-constants of $\mathbb{P}^2$ and of Abelian surfaces. We also study the Harbourne indices of curves on these surfaces. In particular, we show that there are configurations of smooth plane cubic curves whose Harbourne indices are arbitrarily close to -4. Consequently, we obtain that the H-constant of any surface X is less than or equal to -4. Related to these problems, we moreover give a new inequality for the number and multiplicities of singularities of elliptic curves arrangements on Abelian surfaces, inequality which has a close similarity to the one of Hirzebruch for lines arrangements on the plane.

1 Introduction

The Harbourne constant (for short H-constant) of a surface and some related variants of it have been recently introduced in [2], bringing new problems and open questions on curves and their singularities on surfaces. As explained in [2], the H-constant measures the local negativity of curves on surfaces, in analogy with the local positivity measured by Seshadri constants. It has emerged from the context of the bounded negativity conjecture (BNC):

Conjecture. Let $X_{/\mathbb{C}}$ be a smooth projective surface. There exists an integer $b(X)$ such that for every (reduced) curve C on X, one has $C^2 \geq -b(X)$. □

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This ancient and now intensively studied conjecture [2, 3, 5, 8, 15, 18] is trivially true for the plane, but we do not know the behavior of the problem if one takes blow-ups of it. The H-constant was introduced to approach that question. For a blow-up $X' \to X$ of a surface X at a set $\mathcal{P}$ of $s > 0$ distinct points and $C \hookrightarrow X$ a curve, we denote by $\bar{C}$ the strict transform of C in X' and we define the quantity:

$$H(C, \mathcal{P}) = \frac{(\bar{C})^2}{s}.$$

The H-constant of X is defined by:

$$H_X := \inf_{C,\mathcal{P}} H(C, \mathcal{P}),$$

where the infimum is taken over every reduced curves C and finite non-empty sets of points $\mathcal{P}$ on X. If finite, the H-constant has the interesting property that whenever BNC holds for X, then it holds for any of its blow-ups at different points. Let us define the Harbourne index (H-index for short) of a curve C on X by:

$$H(C) := \inf_{\mathcal{P}} H(C, \mathcal{P}) \in \mathbb{R},$$

where $\mathcal{P}$ varies among non-empty finite sets of points on X, so that $H_X = \inf_C H(C)$.

The linear H-constant $H_{L,\mathbb{P}^2} = \inf_C H(C)$ for the plane is defined in [2]; here $X = \mathbb{P}^2$ and the infimum is over every unions C of lines in $\mathbb{P}^2$. Using Hirzebruch bounds on the singularities of lines configurations [1], the authors prove that $H_{L,\mathbb{P}^2} \geq -4$. They moreover give an example of a configuration of lines C with very negative H-index: $H(C) = -225/67$, therefore one knows that $-225/67 \geq H_{L,\mathbb{P}^2} \geq -4$, and $-225/67 \geq H_{L,\mathbb{P}^2} \geq H_{\mathbb{P}^2}$.

In this aritcle, we similarly study the elliptic H-constant $H_{El,X}$ of a surface X, which we define by:

$$H_{El,X} = \inf_{C,\mathcal{P}} H(C, \mathcal{P}) = \inf_{C} H(C)$$

where the infimum is over unions C of elliptic curves on the surface X (throughout this aritcle, "elliptic curve" means a smooth genus 1 curve, without specifying a group structure). Let C be a configuration of elliptic curves on an Abelian surface A. Since an elliptic curve on A is the image of a line in $\mathbb{C}^2$ (the universal cover of A), the curve C has only ordinary singularities, as for the configurations of lines in $\mathbb{P}^2$ studied in [2]. For $k \geq 2$, we denote by t_k the number of k-points on C, that is, the points with multiplicity k. Let $f_0 = \sum t_k$ and $f_1 = \sum k t_k$.

Theorem 1. Let $Sing(C)$ be the set of singularities of C. We have

$$H(C) = H(C, Sing(C)) = -\frac{f_1}{f_0} \geq \frac{t_2 + \frac{1}{4}t_3}{f_0} - 4 \geq -4. \quad (1.1)$$

The elliptic H-constant of A satisfies $H_{El,A} \geq -4$. The H-index of C equals -4 if and only if all the singularities on C are 4-points.

The H-constant and elliptic H-constant are isogeny invariants: if A and B are isogenous, then $H_A = H_B$ and $H_{El,A} = H_{El,B}$. □

Let A be either the surface $(\mathbb{C}/\mathbb{Z}[j])^2$, where $j^2 + j + 1 = 0$, or $(\mathbb{C}/\mathbb{Z}[i])^2$, $i^2 = -1$. By the constructions of Hirzebruch [11] and Holzapfel [13, Example 5.4], there exist configurations C of elliptic curves on A such that their H-indices satisfy $H(C) = -4$, therefore the bound -4 in Theorem 1 is optimal. Actually, using Kobayashi's results [14], one obtains that for any Abelian surface A the equality $H(C) = -4$ is attained by an elliptic curve configuration C if and only if the complement of the strict transform of C in the blow-up of A at the singularity set $Sing(C)$ is isomorphic to an open ball quotient surface. By Theorem 1, this is the case if and only if the singularities of C are 4-points only. Our result thus gives a strong restriction for such ball quotient surfaces (see also [11] on that subject).

The inequality $-\frac{f_1}{f_0} \geq \frac{t_2+\frac{1}{4}t_3}{f_0} - 4$ in Theorem 1 is a corollary of a general result stated in Theorem 9. For any configuration with ordinary singularities $C = \sum_{i=0}^{i=d} C_i$ of smooth curves C_i, Theorem 9 gives an inequality involving the geometric genus and the number and multiplicities of singularities of C. This Theorem 9 is proved by using $(\mathbb{Z}/n\mathbb{Z})^d$ covers of Abelian surfaces. In particular, we obtain the following result:

Theorem 2. For a configuration of elliptic curves on an Abelian surface, one has

$$t_2 + \frac{3}{4}t_3 \geq \sum_{k\geq 5}(2k-9)t_k.$$ □

This is the exact analog of the well-known inequality $t_2 + \frac{3}{4}t_3 \geq d + \sum_{k\geq 5}(2k - 9)t_k$ due to Hirzebruch for a configuration of $d \geq 6$ lines in $\mathbb{P}^2$ such that $t_d = t_{d-1} = t_{d-2} = 0$ (see [12, eq. (9)]). Observe that for lines on the plane, Hirzebruch inequality implies that there are always nodes or 3-points ; on the contrary, that restriction does not apply for elliptic curves on Abelian surfaces, but the condition $t_2 = t_3 = 0$ implies that the configuration is related to ball quotient surfaces. There is a huge literature on arrangements of lines on the plane, our approach for arrangements of elliptic curves on

Abelian surfaces shows a close similarity and opens new questions on the construction of such arrangements.

About the elliptic and H-constants of the plane, we obtain the following results:

Theorem 3. There exist configurations C_n of smooth cubic curves in $\mathbb{P}^2$ such that:

$$\lim_n H(C_n, Sing(C_n)) = -4, \qquad \square$$

(where $Sing(C_n)$ is the set of singular points of C_n) and therefore $-4 \geq H_{El,\mathbb{P}^2} \geq H_{\mathbb{P}^2}$.

From Theorem 3 and functorial properties of the H-constants, we get the following corollary on the Harbourne constant of any surfaces:

Corollary 4. Let X be a smooth surface. Then $-4 \geq H_{\mathbb{P}^2} \geq H_X$. $\square$

The article is organized as follows. In the second section, we prove Theorem 3 and Corollary 4 concerning the elliptic H-constant and the Harbourne constant of the plane. In the third section, we prove Theorem 2, and Theorem 1 is proved in the fourth section. In the last section, we discuss some questions and problems raised by the definitions of the H-index of a curve and H-constant of a surface.

2 Elliptic Curve Configurations on the Plane

The main result of this section is as follows:

Theorem 5. There exist a sequence $\{C_n\}_{n\in 3\mathbb{N}^*}$ of configurations of smooth cubic curves on $\mathbb{P}^2$ such that:

$$\lim_n H(C_n, Sing(C_n)) = -4. \qquad \square$$

Let us recall that a k-point ($k \in \mathbb{N}$, $k \geq 2$) on a curve C is an ordinary singularity of multiplicity k.

Let $p : Z \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the 12 singular points $\mathcal{P}_{\mathrm{Fe}} = \{p_1, \dots, p_{12}\}$ of the Fermat configuration of 9 lines:

$$\{(x^3 - y^3)(x^3 - z^3)(y^3 - z^3) = 0\}.$$

Each line contains four points in $\mathcal{P}_{\mathrm{Fe}}$ and each point in $\mathcal{P}_{\mathrm{Fe}}$ is a 3-point. In [20], end of Section 4, some elliptic curves configurations $\mathcal{H}_n$ (with $n \in 3\mathbb{N}^*$) on Z with the following properties are constructed:

(i) $\mathcal{H}_n$ is the union of $\frac{4}{3}(n^2-3)$ elliptic curves $\mathcal{E}_i$, which are fibers of some elliptic fibrations of Z (in particular $\mathcal{E}_i^2=0$),

(ii) The singularities of $\mathcal{H}_n$ are $\frac{1}{3}(n^2-3)(n^2-9)$ 4-points and $4(n^2-3)$ 3-points.

(iii) Each elliptic curve $\mathcal{E}_i$ of $\mathcal{H}_n$ contains n^2-nine 4-points and nine 3-points, these 3-points are on 9 of the 12 exceptional divisors above the points in $\mathcal{P}_{\mathrm{Fe}}$.

(iv) Each exceptional divisor above the points in $\mathcal{P}_{\mathrm{Fe}}$ contains $\frac{1}{3}(n^2-3)$ 3-points of $\mathcal{H}_n$.

Let us recall briefly how the configurations $\mathcal{H}_n$ are obtained. Let A be the Abelian surface $A=(\mathbb{C}/\mathbb{Z}[j])^2$, where $j^2+j+1=0$. Let us define $T_0=\{y=0\}$, $T_\infty=\{x=0\}$, the horizontal and vertical axes, respectively, $T_1=\{x=y\}$ the diagonal and $T_j=\{x=-jy\}$. The elliptic curves configuration

$$\mathcal{C}_1=T_0+T_I+T_\infty+T_j \tag{2.1}$$

has one singularity only ; it is a 4-point. Let $[m]:A\to A$ the multiplication by $m\in\mathbb{Z}[j]$ map. Let $n\in\mathbb{N}$; the curve $\mathcal{C}_n=[n]^*\mathcal{C}_1$ is a configuration of $4n^2$ elliptic curves, its singularity set is the set of the n^4 n-torsion points, and every singularity of $\mathcal{C}_n$ is a 4-point. These configurations $\mathcal{C}_n$ were discovered by Hirzebruch in [11] ; we use them in Section 4.

The endomorphism $[j]$ is an order 3 automorphism of A, it fixes a set $\mathcal{P}_9$ of nine isolated points. Let $b:\bar{A}\to A$ be the blow-up of A at $\mathcal{P}_9$. The automorphism $[j]$ acts on $\bar{A}$ and fixes the nine exceptional divisors above $\mathcal{P}_9$. The quotient surface $\bar{A}/[j]$ is the surface Z (see e.g., [20]) ; the images by the quotient map $\pi:A\to Z$ of the nine exceptional divisors on $\bar{A}$ are the strict transform on Z of the nine lines of the Fermat configuration. The 12 exceptional divisors of the blow-up $Z\to\mathbb{P}^2$ are the images by π_*b^* of the 12 elliptic curves on A going through three among the nine points in $\mathcal{P}_9$. The irreducible components of $\mathcal{H}_n$ are the smooth genus 1 irreducible components of the support of $\pi_*b^*\mathcal{C}_n$.

Let C_n be the image on $\mathbb{P}^2$ of $\mathcal{H}_n$ by the blow-up map $p:Z\to\mathbb{P}^2$. Then we see that:

(i) C_n is the union of $\frac{4}{3}(n^2-3)$ smooth degree 3 curves E_i.

(ii) Each curve E_i contains $n^2 - 9$ 4-points and goes through 9 of the 12 points in $\mathcal{P}_{\mathrm{Fe}}$.

(iii) The singularities of C_n are $\frac{1}{3}(n^2-3)(n^2-9)$ 4-points and the 12 points in $\mathcal{P}_{\mathrm{Fe}}$ have multiplicity (n^2-3).

These configurations C_n are constructed anew in [4] by a different approach, via Halphen cubics.

Let C be a plane curve of degree d, and let $\mathcal{P}$ be a set of $s>0$ points in $\mathbb{P}^2$. Let us denote by m_p the multiplicity of C at a point $p \in \mathbb{P}^2$ (where $m_p = 0$ means $p \notin C$). By the definition of $H(C,\mathcal{P})$ given in Section 1, one has:

$$H(C,\mathcal{P}) = \frac{d^2 - \sum_{p\in\mathcal{P}} m_p^2}{s}.$$

One now applies that formula to the curve C_n of degree $d = 4(n^2-3)$ and to the set of singular points $Sing(C_n)$ of C_n ; one obtains:

$$H(C_n, Sing(C_n)) = -4\frac{n^4 - 30n^2 + 81}{n^4 - 12n^2 + 63},$$

and it proves Theorem 5. The value $H(C_n, Sing(C_n))$ could have been obtained in an other way, by computing the self-intersection of the strict transform $\bar{C}_n$ of the curve C_n on the blow-up X_n of $\mathbb{P}^2$ at $Sing(C_n)$, using the fact that the surface X_n is also the blow-up of Z at every 4-points of $\mathcal{H}_n$ and the curve $\bar{C}_n$ is the strict transform of $\mathcal{H}_n$.

Remark 6.

(1) For example, one has $H(C_{21}, Sing(C_{21})) = -\frac{20148}{5257} \simeq -3.83$.

(2) The above configurations $\{C_n\}_{n\in 3\mathbb{N}^*}$ are strongly linked to some compactifications $\{X_n\}_{n\in\mathbb{N}}$ of some open ball quotient surfaces constructed by Hirzebruch in [11] and for which $\lim_n \frac{c_1^2}{c_2}(X_n) = 3$, that is, one is close to the upper bound in the Miyaoka–Yau inequality. □

In the following Lemma 7 and Corollary 8, we write $H_X(C,\mathcal{P})$ for $H(C,\mathcal{P})$ when C is a curve on a surface X and $\mathcal{P}$ is a finite non-empty set of points of X.

Let $f : X \to Y$ be a dominant morphism between two smooth surfaces. Let C be a reduced curve on Y. Suppose that C does not contain components of the branch locus B of f and let $\mathcal{P}$ be a set of $s>0$ points in Y, disjoint from B (so that f^*C and $f^*\mathcal{P}$ are reduced of pure dimensions 1 and 0, respectively).

Lemma 7. Under the above assumptions on C, one has $H_X(f^*C, f^*\mathcal{P}) = H_Y(C, \mathcal{P})$. □

Proof. Let d be the degree of f and let p be a point of $\mathcal{P}$. Since f is étale over $Y \setminus B$, the d points above p have the same multiplicity m_p inside $C' = f^*C$ than p inside C and:

$$H_X(f^*C, f^*\mathcal{P}) = \frac{\bar{C'}^2}{ds} = \frac{dC^2 - d\sum m_p^2}{ds} = \frac{\bar{C}^2}{s} = H_Y(C, \mathcal{P}).$$ ■

Lemma 7 and Theorem 5 imply:

Corollary 8. Let X be a smooth surface. Then $H_X \leq H_{\mathbb{P}^2} \leq -4$. □

Proof. Let $f : X \to \mathbb{P}^2$ be a generic projection of X on to the plane. Let C be a curve in $\mathbb{P}^2$ and let $\mathcal{P}$ be a finite set of points. Let $g \in PGL_3(\mathbb{C})$ be an automorphism of the plane such that the curve $C' = g^*C \simeq C$ do not contain any components of the branch divisor B of f and $\mathcal{P}' = g^*\mathcal{P}$ is disjoint from B. We then apply Lemma 7 to f, C' and $\mathcal{P}'$, to obtain

$$H_X(f^*C', f^*\mathcal{P}') = H_{\mathbb{P}^2}(C', \mathcal{P}').$$

We can see that $H_{\mathbb{P}^2}(C', \mathcal{P}') = H_{\mathbb{P}^2}(C, \mathcal{P})$, and by taking the infimum over every curves and finite sets in X (respectively, $\mathbb{P}^2$), we obtain the inequality $H_X \leq H_{\mathbb{P}^2}$. Moreover, by Theorem 5, one has $H_{\mathbb{P}^2} \leq -4$. ■

3 Arrangements of Curves on Abelian Surfaces and $(\mathbb{Z}/n\mathbb{Z})^d$ Covers

Let A be an Abelian surface and let $C = \sum_{i=1}^d C_i$ be a reduced divisor with only ordinary singularities (i.e., singularities resolved after one blow-up), a union of $d \geq 2$ smooth divisors C_i (e.g., C_i may be the union of genus 1 fibers of a fibration of A on to an elliptic curve). As in [17, point G, p. 408], let g be the geometric genus of C, that is,

$$g - 1 = \sum g_j - 1,$$

where g_j is the genus of the irreducible component C_i. Let us denote by t_k the number of k-points on C, that is, the number of singularities with multiplicity k.

Theorem 9. We have

$$10g - 10 + t_2 + \frac{3}{4}t_3 \geq \sum_{k \geq 5}(2k - 9)t_k,$$

and

$$H(C, Sing(C)) = \frac{2g-2-f_1}{f_0} \geq \frac{2t_2 + \frac{9}{8}t_3 + \frac{1}{2}t_4 + 8 - 8g}{f_0} - \frac{9}{2}. \tag{3.1}$$

Suppose that C is a configuration of elliptic curves. Then

$$H(C) = -\frac{f_1}{f_0} \geq \frac{t_2 + \frac{1}{4}t_3}{f_0} - 4 \geq -4, \tag{3.2}$$

where $H(C)$ is the H-index: $H(C) = \min_{\mathcal{P}} H(C, \mathcal{P})$. Moreover $H(C) = -4$ if and only if all the singularities on C are 4-points. □

The remaining of this section is the proof of Theorem 9. Let us recall a Theorem of Namba on branched covers. Let M be a manifold, $D_1, \ldots, D_s$ be irreducible reduced divisors on M, and $n_1, \ldots, n_s$ be positive integers. We denote by D the divisor $D = \sum n_i D_i$. Let $Div(M, D)$ be the sub-group of the $\mathbb{Q}$-divisors generated by the entire divisors and:

$$\frac{1}{n_1}D_1, \ldots, \frac{1}{n_s}D_s.$$

Let $\sim$ be the linear equivalence in $Div(M, D)$, where $G \sim G'$ if and only if $G - G'$ is an entire principal divisor. Let $Div(M, D)/\sim$ be the quotient and let $Div^0(M, D)/\sim$ be the kernel of the Chern class map

$$\begin{array}{ccc} Div(M, D)/\sim & \to & H^{1,1}(M, \mathbb{R}) \\ G & \to & c_1(G) \end{array}.$$

Theorem 10 (Namba, [19, Theorem 2.3.20]). There exists a finite Abelian cover which branches at D with index n_i over D_i for all $i = 1, \ldots, s$ if and only if for every $j = 1, \ldots, s$ there exists an element of finite order $v_j = \sum \frac{a_{ij}}{n_i} D_i + E_j$ of $Div^0(M, D)/\sim$ (where E_j an entire divisor and $a_{ij} \in \mathbb{Z}$) such that a_{jj} is coprime to n_j.

Then the subgroup in $Div^0(M, D)/\sim$ generated by the v_j is isomorphic to the Galois group of such an Abelian cover. □

We find the inequalities among the t_k's in Theorem 9 using $(\mathbb{Z}/n\mathbb{Z})^d$ covers of A ramified above curves related to the curves C_i. These inequalities involve quantities that are "linear" under isogenies, by which we mean that if $\phi : B \to A$ is an isogeny of degree m, then the number of k-points on ϕ^*C (a reduced curve), the intersections between the ϕ^*C_i's and ϕ^*C, the geometric genus minus 1 of $\phi^*C \ldots$ are the ones of C, $C_i \ldots$ multiplied by m. By that property, inequalities involving linear terms in the t_k's and C_i^2 proved on abelian surface B are then inequalities for A.

Let $\phi = [m] : A \to A$ be the multiplication by $m \in \mathbb{N}$ map. Recall that $\phi^* D \sim \frac{m(m+1)}{2} D + \frac{m(m-1)}{2} [-1]^* D$ for any divisor D (see [6, Proposition 2.3.5]). By taking $m = 2n$, one can therefore suppose that the divisors C_i are n-divisible, that is, there exists divisors L_i such that $C_i \sim nL_i$. The divisor $v_i = \frac{1}{n} C_i - L_i$ is in $Div^0(A, nC)/\sim$, has order n, and the multiplicity of an irreducible component C_i' in $\frac{1}{n} C_i$ is $\frac{1}{n}$. The group generated by divisors $\frac{1}{n} C_i - L_i$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^d$ and there exists a $(\mathbb{Z}/n\mathbb{Z})^d$ cover of A branched with index n over C. For the computation of the Chern numbers of the resolution X_n of that cover, we refer to the local analysis of the $(\mathbb{Z}/n\mathbb{Z})^d$-branched covers of the plane constructed by Hirzebruch in [11] (see also the geometric approach of [9]).

The following quantities f_0, f_1, f_2 are linear under isogenies:

$$f_0 = \sum_{k \geq 2} t_k,\ f_1 = \sum_{k \geq 2} k t_k,\ f_2 = \sum_{k \geq 2} k^2 t_k.$$

Let $\pi : Z \to A$ be the blow-up at the $f_0 - t_2 = \sum_{k \geq 3} t_k$ singularities of C of multiplicities $k \geq 3$ and let $\bar{C} = \sum \bar{C}_i$ be the strict transform of C in Z. For a singularity p of C of multiplicity $k_p \geq 3$, we denote by $E_p \hookrightarrow Z$ the exceptional curve over p. There exists a degree n^d map $f : X_n \to Z$ branched with index n above the curve $\bar{C}$. Above E_p lies n^{d-r} $(r = k_p)$ copies in X_n of a smooth curve F_p, which is a $(\mathbb{Z}/n\mathbb{Z})^{r-1}$ cover of E_p ramified with index n at r points, thus

$$e(F_p) = n^{r-1}(2 - r) + r n^{r-2} = n^{r-2}(2n + r(1 - n)).$$

Since the Galois group permutes these curves, we have $(F_p)^2 = -n^{r-2}$. If a singularity p of C is a node, then X_n is smooth over p and the fiber of f at p has only n^{d-2} points.

We have

$$e(C) = 2 - 2g + f_0 - f_1,\ e(C \setminus Sing(C)) = 2 - 2g - f_1,\ e(A \setminus C) = -e(C) = 2g - 2 + f_1 - f_0,$$

and $C^2 = \sum C_i^2 + f_2 - f_1 = 2g - 2 + f_2 - f_1$. Therefore we obtain

$$e(X_n \setminus f^{-1} E_p) = n^d e(A \setminus C) + n^{d-1} e(C \setminus Sing(C)) + n^{d-2} t_2$$

and

$$\frac{1}{n^{d-2}} e(X_n \setminus f^{-1} E_p) = n^2(2g - 2 + f_1 - f_0) + n(2 - 2g - f_1) + t_2.$$

Above each exceptional divisor E_p in Z, there are n^{d-k} curves with Euler number $e(F_p)$, thus

$$e(X_n) = e(X_n \setminus f^{-1}E_p) + \sum_{k\geq 3} n^{d-2} t_k(2n + k(1-n))$$

and

$$\frac{1}{n^{d-2}} e(X_n) = (2g - 2 + f_1 - f_0)n^2 + 2(1 - g + f_0 - f_1)n + f_1 - t_2.$$

Let us now compute the canonical divisor: K_{X_n} is numerically equivalent to the pullback of

$$K = \sum E_p + \frac{n-1}{n}\left(\sum E_p + \pi^* C - \sum k_p E_p\right) = \sum_p \frac{2n - 1 + k_p(1-n)}{n} E_p + \frac{n-1}{n}\pi^* C.$$

We get

$$K^2 = \sum_{k\geq 3} -\frac{(2n-1+k(1-n))^2}{n^2} t_k + \left(\frac{n-1}{n}\right)^2 C^2,$$

and we obtain

$$\frac{1}{n^{d-2}} K_{X_n}^2 = (2g - 2 + 3f_1 - 4f_0)n^2 + 4(f_0 - f_1 - g + 1)n - f_0 + f_1 + t_2 + 2g - 2.$$

Since the surface X_n covers an Abelian surface, its Kodaira dimension is non-negative. Then we get by using the Miyaoka–Yau inequality:

$$\frac{1}{n^{d-2}}(3c_2 - K_{X_n}^2) = (f_0 + 4g - 4)n^2 + 2(f_0 - f_1 - g + 1)n + 2f_1 + f_0 - 4t_2 - 2g + 2 \geq 0. \tag{3.3}$$

As in [12], we will use a refinement of the Miyaoka–Yau inequality for the surfaces X_n that contain smooth rational curves and elliptic curves, that is, for $n = 2$ or 3. Let Y be a surface of non-negative Kodaira dimension. Suppose that there exists on Y some smooth disjoint elliptic curves D_j and m disjoint (-2) curves, disjoint also from the curves D_j, then:

Theorem 11. (Miyaoka [16, Corollary 1.3]). We have

$$3c_2(Y) - K_Y^2 \geq \frac{9}{2}m - \sum (D_j)^2.$$

□

For $n = 3$, we get from equation 3.3:

$$\frac{1}{3^{d-2}}(3c_2 - K_{X_3}^2) = 4(7g - 7 + 4f_0 - f_1 - t_2) \geq 0.$$

Taking into account the fact that over the 3-points the surface contains $3^{d-3}t_3$ elliptic curves of self-intersection -3, we can refine that inequality and obtain

$$-\frac{f_1}{f_0} \geq \frac{t_2 + \frac{1}{4}t_3 - 7g + 7}{f_0} - 4.$$

For $g = 1$, that is, C is a configuration of elliptic curves, let $Sing(C)$ be the singularity set of C, then we get:

$$H(C, Sing(C)) = -\frac{f_1}{f_0} \geq \frac{t_2 + \frac{1}{4}t_3}{f_0} - 4.$$

For $\mathcal{P}$ a (non-empty) set of points in A, one can use the same demonstration as in [2, Theorem 3.3] for the linear H-constant of $\mathbb{P}^2$ to conclude that $H(C, \mathcal{P}) \geq H(C, Sing(C))$, therefore $H(C) = H(C, Sing(C))$. Suppose that the bound -4 is attained, then $4f_0 = f_1$, $t_2 = t_3 = 0$ and from equality $\sum_{k\geq 5}(k-4)t_k = 2t_2 + t_3$, we see that $t_k = 0\ \forall k \geq 5$. The only possibility is $t_4 \neq 0$, which indeed exists (see below).

For $n = 2$, the surface X_2 contains $2^{d-3}t_3$ disjoint (-2)-curves and it contains $t_4 2^{d-4}$ elliptic curves of self-intersection -4, therefore

$$\frac{1}{2^{d-2}}(3c_2 - K_{X_2}^2) \geq \frac{9}{4}t_3 + t_4$$

and

$$10g - 10 + 9f_0 \geq 2f_1 + 4t_2 + \frac{9}{4}t_3 + t_4,$$

which implies $10g - 10 + t_2 + \frac{3}{4}t_3 \geq \sum_{k\geq 5}(2k-9)t_k$. Using $H(C, Sing(C)) = \frac{C^2 - f_2}{f_0} = \frac{\sum C_i^2 - f_1}{f_0}$ and $2g - 2 = \sum C_i^2$, we get

$$H(C, Sing(C)) \geq \frac{2t_2 + \frac{9}{8}t_3 + \frac{1}{2}t_4 + 8 - 8g}{f_0} - \frac{9}{2}.$$

4 The Elliptic H-Constants of Abelian Surfaces

Let X be a smooth projective surface and let C be a configuration of smooth disjoint elliptic curves on X. Let us recall the following result:

Theorem 12 ([14, Theorem 2, (1.3)], Kobayashi)**.** The universal cover of $X \setminus C$ is the unit ball if and only if $(K_X + C)^2 = 3e(X \setminus C)$, where e denotes the Euler number. □

Let C be a configuration of elliptic curves on an Abelian surface A. Let $X \to A$ be the blow-up at the singular points of C and let $\bar{C}$ be the strict transform of C in X. We obtain:

Corollary 13. The elliptic curve configuration C on A has H-index $H(C) = -4$ if and only if $X \setminus \bar{C}$ is a ball quotient surface, that is, its universal cover is the ball.

In that case, the curve C has only 4-points singularities and there exists a covering $X_3 \to A$ branched with order 3 over $[3]^*C$ such that X_3 is a smooth compact ball quotient surface: $c_1^2(X_3) = 3c_2(X_3)$. □

Remark 14. Using Abelian covers of $A = (\mathbb{C}/\mathbb{Z}[j])^2$, Hirzebruch also obtained a compact ball quotient surface (see [11]). □

Proof. One computes that $(K_X + \bar{C})^2 = f_1 - f_0$ and $e(X \setminus \bar{C}) = f_0$. Therefore, one has equality $(K_X + \bar{C})^2 = 3e(X \setminus \bar{C})$ if and only if $4f_0 = f_1$, which is equivalent by Theorem 9 to $H(C) = -4$ and to the condition that C has only 4-points.

Let $X_n \to A$ be the covering associated to the elliptic curve configuration $[2n]^*C$ in the proof of Theorem 9. By equation 3.3, the value of $\frac{1}{n^{d-2}}(3c_2(X_n) - K_{X_n}^2)$ is $f_0(n-3)^2$. Thus for $n = 3$ we obtain a compact ball quotient surface. ■

Let $j = \frac{-1+i\sqrt{3}}{2}$ with $i^2 = -1$, we have:

Proposition 15. The elliptic H-constants of $(\mathbb{C}/\mathbb{Z}[j])^2$ and $(\mathbb{C}/\mathbb{Z}[i])^2$ are equal to -4. □

Proof. Hirzebruch [11] and Holzapfel [13] found elliptic curves arrangements $\mathcal{C}$ on $(\mathbb{C}/\mathbb{Z}[j])^2$ and $(\mathbb{C}/\mathbb{Z}[i])^2$, respectively, with 4-points singularities only. Therefore by Theorem 9, one has $H(\mathcal{C}) = -4$ and the elliptic H-constant of these two surfaces is -4.

We describe the example on $(\mathbb{C}/\mathbb{Z}[j])^2$ in 2.1. It is the union of four elliptic curves with only one singularity: $t_4 = 1$, $t_k = 0$ for $k \neq 4$.

On the surface $(\mathbb{C}/\mathbb{Z}[i])^2$, the configuration $\mathcal{C}$ has six irreducible components. For $u \in \mathbb{C}$, let E_u be the image of the line $\{y = ux\} \subset \mathbb{C}^2$ by the quotient map $\mathbb{C}^2 \to (\mathbb{C}/\mathbb{Z}[i])^2$ and let E_∞ be the image of the line $\{x = 0\}$. The configuration is

$$\mathcal{C} = E_0 + E_\infty + E_{i-1} + E_{\frac{1}{2}(i-1)} + E'_{-1} + E'_i,$$

where $E'_{-1} = E_{-1} + (\tau, 0)$, $E'_i = E_i + (\tau, 0)$, for $\tau = \frac{1+i}{2}$. The points $0, (\tau, 0), (0, \tau)$ are the only singularities on $\mathcal{C}$ and are 4-points. ■

Remark 16. Hirzebruch's and Holzapfel's examples are elliptic curves configurations on Abelian surfaces with only 4-points singularities. That situation must be compared with the plane where there do not exist configurations of $d > 6$ lines with $t_2 = t_3 = 0$ and $t_d = t_{d-1} = t_{d-2} = 0$ (by inequality 5.3 below). □

Let E, E' be two elliptic curves and let A be an Abelian surface.

Proposition 17. The H-constant and elliptic H-constant of A are invariants of the isogeny class of A. Suppose A is isogenous to $E \times E'$. If E and E' are not isogenous, then $H_{El,A} = -2$. If E and E' are isogenous, then $H_{El,A} \leq -3$. □

Proof. Let $\phi : A \to B$ be an isogeny between two Abelian surfaces ; it is an étale map. Let C be a (reduced) curve on B, and let $\mathcal{P}$ be a set of points on B. By Lemma 7, $H_A(\phi^*\mathcal{P}, \phi^*C) = H_B(C, \mathcal{P})$, thus

$$\inf_{C,\mathcal{P}} H_A(C, \mathcal{P}) \leq \inf_{C,\mathcal{P}} H_B(C, \mathcal{P}).$$

Since there exists an isogeny $\psi : B \to A$ too, we have the reverse inequality. That holds also for the elliptic H-constant, since the pull-back by an isogeny of a genus one curve is a union of genus one curves.

Let A be isogenous to $E \times E'$. Suppose that E and E' are not isogenous. Then a configuration C of elliptic curves on $E \times E'$ is as follows:

$$C = \sum_{k=1}^{m} F_k + \sum_{k=1}^{n} F'_k,$$

where the F_k (respectively, F'_k) are fibers of the fibration of $E \times E'$ on to E (respectively, E'). Then by Theorem 9, $H(C) = -2$, and therefore $H_{A,El} = -2$.

Suppose that E and E' are isogenous. Since the elliptic H-constant is an isogeny invariant, we can suppose that $E = E'$. Let Δ be the diagonal in $E \times E$ and let be $F = \{y = 0\}$, $F' = \{x = 0\}$, where x, y are the coordinates. Then $C = \Delta + F + F'$ has one 3-point in 0, and no other singularities. Thus by Theorem 9, $H(C) = -3 \geq H_{A,El}$. ■

5 Remarks on the Harbourne Indices of Curves

5.1 Irreducible curves with low H-index

Let us recall that the Harbourne index of a curve C is defined by

$$H(C) = \inf_{\mathcal{P}} H(C, \mathcal{P}).$$

In view of the BNC, 1 it would be interesting to know irreducible curves C with low H-index. Then by taking the appropriate blow-up one could get very negative curves and maybe obtain counter examples to the Conjecture. By example, the H-indices of the rational curves C_d of degree d in $\mathbb{P}^2$ with the maximal number of nodes go to -2 when d grows to ∞ (on that subject, see also the introduction of [2]). If we want to go down, we would need to impose singularities with higher multiplicities on the curve C, and it may force C to have "negative genus", that is, to be a union of (at least) two curves. That explains why one rather considers unions of curves than irreducible curves. Moreover considering reducible curves gives more functorialities to the H-constants, for example, when one wishes to compare these constants through a dominant map $f : X \to Y$.

About the problem of constructing infinitely many irreducible curves with low H-index, the example of totally geodesic curves on a Shimura surface or a ball quotient surface X is particularly interesting. If smooth, such curves C of genus g satisfy $C^2 = -2(g-1)$ or $C^2 = -(g-1)$, respectively. These curves were considered as possible counterexamples for the BNC. But one of the main results of [3] in the Shimura surface case, and in [15], [18] for the ball quotient case, is the fact that on X there are at most a finite number of such curves C with $C^2 < 0$. However, when one takes some blow-up of X, one does not know the behavior of $\bar{C}^2$ for $\bar{C}$ the strict transform of such a curve C.

It is classical that the singularities of a totally geodesic curve C on a Shimura or a ball quotient surface are ordinary (as for elliptic curves on Abelian surfaces, that can be proved by using the universal cover of X; e.g., for a ball quotient $X = \mathbb{B}_2/\Gamma$, C is the image of $L \cap \mathbb{B}_2$ where L is a line). If X contains a totally geodesic curve, then there exists an infinite number of such curves (see e.g., [7]) on X and by the above quoted result, all but a finite number of such curves are singular. Let C be an irreducible singular totally geodesic curve on X. Let $\delta \in \mathbb{N}^*$ be defined by $K_X C + C^2 = 2g - 2 + 2\delta$, for g the geometric genus of C. We have $2\delta = f_2 - f_1$ (where $f_i = \sum_{k\geq 2} k^i t_k$, for t_k the number of k-points on C), thus

$$H(C, Sing(C)) = \frac{C^2 - f_2}{f_0} = \frac{-K_X C + 2g - 2 - f_1}{f_0}.$$

Since C is a totally geodesic curve, we know moreover that $K_X C = 4g - 4 > 0$ in the Shimura case and $K_X C = 3g - 3 > 0$ in the ball quotient case. Therefore we see that

$$H(C, Sing(C)) \leq -\frac{f_1}{f_0}.$$

Since always $f_1 \geq 2f_0$, we obtain two more examples of families of irreducible curves C with

$$\liminf_{C} H(C) \leq -2.$$

5.2 The H-indices of arrangements of lines

In the case of an arrangement of d lines in $\mathbb{P}^2$ with $t_d = t_{d-1} = t_{d-2} = t_{d-3} = 0$ and $d \geq 6$, Hirzebruch [10, p. 140] proved the following inequality

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{k\geq 5}(k-4)t_k. \tag{5.1}$$

Using that result, the authors of [2] obtained the following inequality for the H-index of such a line arrangement C:

$$H(C) \geq B_1 := -4 + \frac{1}{\sum t_k}\left(2d + t_2 + \frac{1}{4}t_3\right). \tag{5.2}$$

A better inequality for the singularities of lines arrangements is given in [12, eq. (9)], which is

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{k\geq 5}(2k-9)t_k. \tag{5.3}$$

Proposition 18. Using inequality 5.3, one obtains:

$$H(C) \geq B_2 := \frac{1}{\sum t_k}\left(\frac{3}{2}d + 2t_2 + \frac{9}{8}t_3 + \frac{1}{2}t_4\right) - \frac{9}{2}. \tag{5.4}$$

One has $B_2 \geq B_1$ and therefore inequality 5.4 is sharper than 5.2. □

As a referee pointed out, inequality 5.2 has the advantage that the bound $H(C) \geq -4$ is immediately clear, whereas at first glance, inequality 5.4 gives only $H(C) \geq 4.5$ (this probably explains the choice of inequality 5.2 in [2]). For the Klein configuration of lines (see [2, Section 4.1]) and the Fermat configurations of 9 and 12 lines, inequalities

5.2 and 5.4 are equalities. For the Fermat configuration of 18 lines, inequality 5.4 is an equality but this is not the case for inequality 5.2.

Proof. Using the combinatoric equality $\begin{pmatrix} d \\ 2 \end{pmatrix} = \sum_{k\geq 2}^{d} t_k \begin{pmatrix} k \\ 2 \end{pmatrix}$, we have

$$H(C, Sing(C)) = \frac{d^2 - \sum_{k\geq 2} k^2 t_k}{\sum t_k} = \frac{d - \sum_{k\geq 2} k t_k}{\sum t_k}.$$

By 5.3, we have $t_2 + \frac{3}{4}t_3 \geq d + \sum_{k\geq 4}(2k-9)t_k$, thus

$$-\sum_{k\geq 2} k t_k \geq \frac{1}{2}\left(d + 4t_2 + \frac{9}{4}t_3 + t_4 - 9\sum t_k\right) \tag{5.5}$$

and we obtain inequality 5.4. The fact that this new inequality is sharper comes from the fact that inequality 5.3 is typically better than 5.1. ∎

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