CANONICAL SURFACES WITH BIG COTANGENT BUNDLE

XAVIER ROULLEAU and ERWAN ROUSSEAU

Abstract

Surfaces of general type with positive second Segre number are known to have big cotangent bundle. We give a new criterion, ensuring that a surface of general type with canonical singularities has a minimal resolution with big cotangent bundle. This provides many examples of surfaces with negative second Segre number and big cotangent bundle.

1. Introduction

Projective varieties with positive cotangent bundle have attracted a lot of attention because of the strong geometric properties they have. In particular, they are Kobayashi-hyperbolic and algebraically hyperbolic (see [7] for an introduction).

Surfaces of general type with ample cotangent bundle are known to have positive second Segre class $s_2 := c_1^2 - c_2 > 0$ (see [8]). In fact, Bogomolov [1] proved that if X is a surface of general type with positive second Segre class, then the family of curves on X of fixed geometric genus is bounded. The numerical positivity ensures that these surfaces have many symmetric tensors; in other words, their cotangent bundle is big. More recent generalizations of these results have been obtained, including the algebraic degeneracy of entire curves in surfaces of general type with positive second Segre class by McQuillan [15] and effective results on the canonical degree of irreducible curves of genus g in such surfaces by Miyaoka [18]. Many examples of surfaces with positive second Segre class are known: complete intersection surfaces in sufficiently big projective spaces, surfaces with ample cotangent bundle, and so on (see, e.g., [16]).

On the other hand, among surfaces with negative second Segre class, we know rather few examples that have big cotangent bundle. Smooth surfaces in $\mathbb{P}^3$ are well known to have no symmetric tensors [21], but Bogomolov and De Oliveira [3] made the interesting observation that minimal resolutions of singular surfaces may provide

DUKE MATHEMATICAL JOURNAL
Vol. 163, No. 7, © 2014 DOI 10.1215/00127094-2681496
Received 19 March 2013. Revision received 16 September 2013.
2010 *Mathematics Subject Classification*. Primary 14J60, 14J70; Secondary 14J25, 32Q45.
Roulleau's work partially supported by Agence Nationale de la Recherche project POSITIVE, grant ANR-2010-BLAN-0119-01.

such examples. In [3], nodal surfaces are considered and a numerical condition on the number of nodes is given to ensure that the resolution will have big cotangent bundle. Unfortunately, this statement turns out not to be completely correct (see Section 4.1 below for details).

In this article, we would like to give a new general criterion, ensuring that a surface of general type has big cotangent bundle. Let us describe our result.

Recall that if we take a minimal model Y of a surface of general type (i.e., smooth with K_Y nef and big), then the curves E with $K_Y.E = 0$ form bunches of (-2)-curves and can be contracted to *canonical* singularities (also known as ADE or Du Val singularities in the case of surfaces; see, e.g., [19] or [13]).

Let X be a *canonical* surface, that is, a projective surface with positive canonical divisor K_X and at worst canonical singularities. In dimension 2, canonical singularities are known to be quotients of $\mathbb{C}^2$ by finite subgroups of $\mathrm{SL}(2,\mathbb{C})$. Therefore we can attach two objects to the surface X. On the one hand, we consider $Y \to X$ to be its minimal resolution. On the other hand, we let $\mathcal{X} \to X$ be the orbifold (or stack) attached to X.

We denote by $s_2(Y) = c_1^2(Y) - c_2(Y)$ and $s_2(\mathcal{X}) = c_1^2(\mathcal{X}) - c_2(\mathcal{X})$ the second Segre numbers of Y and $\mathcal{X}$, respectively.

THEOREM 1
Let X be a canonical surface, let $Y \to X$ be its minimal resolution, and let $\mathcal{X} \to X$ be the orbifold associated to X. If

$$s_2(Y) + s_2(\mathcal{X}) > 0,$$

then the cotangent bundle of Y is big. In particular, Y has only finitely many rational or elliptic curves.

Thanks to the work of McQuillan [15], this result also provides surfaces of general type satisfying the Green–Griffiths–Lang conjecture, since entire curves in such surfaces will be contained in a proper algebraic subvariety.

As applications, we obtain many examples of surfaces with big cotangent bundle and negative second Segre number. Among them, we have the following results.

THEOREM 2
Let $X \subset \mathbb{P}^3$ be a hypersurface of degree d with ℓ singularities A_k, and let $Y \to X$ be its minimal resolution. If

$$\ell > \frac{4(k+1)}{k(k+2)}(2d^2 - 5d),$$

then Y has big cotangent bundle.

As a consequence, we obtain the existence of surfaces with big cotangent sheaf for all degrees $d \geq 13$. In the following application of Theorem 2, the singularities of the surface considered are A_{d-1}.

THEOREM 3
Let $X \subset \mathbb{P}^3$ be the ramified cover of $\mathbb{P}^2$ of degree $d = \sum d_j$, branched along the normal crossing divisor $D = \bigcup_{j=1}^{j=k} D_j \subset \mathbb{P}^2$, where D_j is a curve of degree d_j. Suppose that $d_i \geq c$ for $i = 1, \dots, k$ and

$$k(k-1) > \frac{8d^2(2d-5)}{c^2(d^2-1)};$$

then the minimal resolution $Y \to X$ has big cotangent bundle.

This result provides examples of hypersurfaces with big cotangent bundle for all degrees $d \geq 15$.

The main technical difference with the approach developed in [3] using reflexive sheaves is that here we follow a "stacky" approach. Our work is also inspired by the orbifold techniques of Campana [5], though his notion of orbifolds is more general than the one we adopt here.

The paper is structured as follows. In Section 2 we present the orbifold setting and the results we will use. In Section 3 we prove our main theorem (Theorem 1). In Section 4 we give applications proving Theorem 2 and Theorem 3. Section 5 is devoted to the geography of surfaces of general type with big cotangent bundle.

2. Orbifold basics

For the reader's convenience, we recall in this section the basic facts on orbifolds (referring to [20] for details) that will be used in the proof of Theorem 1.

2.1. Orbifolds and canonical singularities

We define orbifolds as a particular type of log pairs. The data (X, Δ) are a log pair if X is a normal algebraic variety (or a normal complex space) and $\Delta = \sum_i d_i D_i$ is an effective $\mathbb{Q}$-divisor, where the D_i's are distinct, irreducible divisors and $d_i \in \mathbb{Q}$.

For orbifolds, we need to consider only pairs (X, Δ) such that Δ has the form $\Delta = \sum_i (1 - \frac{1}{m_i}) D_i$, where the D_i's are prime divisors and $m_i \in \mathbb{N}^*$.

Definition 4
An *orbifold chart* on X compatible with Δ is a Galois covering $\varphi : U \to \phi(U) \subset X$ such that

(1) U is a domain in $\mathbb{C}^n$ and $\varphi(U)$ is open in X;

(2) the branch locus of φ is $\lceil \Delta \rceil \cap \varphi(U)$;

(3) for any $x \in U'' := U \setminus \varphi^{-1}(X_{\text{sing}} \cup \Delta_{\text{sing}})$ such that $\varphi(x) \in D_i$, the ramification order of φ at x verifies $\text{ord}_\varphi(x) = m_i$.

Definition 5
An orbifold $\mathcal{X}$ is a log pair (X, Δ) such that X is covered by orbifold charts compatible with Δ.

Definition 6
Let (X, Δ), $\Delta = \sum_i (1 - \frac{1}{m_i}) C_i$, be a pair where X is a normal surface and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\pi : \tilde{X} \to X$ be a resolution of the singularities of (X, Δ), so that the exceptional divisors E_i and the components of $\tilde{\Delta}$ (the strict transform of Δ) have normal crossings and $K_{\tilde{X}} + \tilde{\Delta} = \pi^*(K_X + \Delta) + \sum_i a_i E_i$.

We say that (X, Δ) has *canonical* singularities if $a_i \geq 0$ for every exceptional curve E_i.

If $\Delta = 0$, then canonical singularities of X are the same as Du Val singularities (or ADE singularities), which are quotient singularities (see [13] for details). As a consequence, X has an orbifold structure $\mathcal{X}$. Moreover, their minimal resolution $Y \to X$ is such that $K_Y = f^* K_X$.

2.2. Chern classes
Let $\pi : \mathcal{X} \to (X, \Delta)$ be a 2-dimensional orbifold for which $\Delta = 0$ and the singularities are ADE. Let a_n (resp., d_n, e_n) be the number of A_n (resp., D_n, E_n) singularities on X, and let $Y \to X$ be its minimal resolution.

PROPOSITION 7 (see [20])
The Chern numbers of $\mathcal{X}$ are $c_1^2(\mathcal{X}) = c_1^2(X) = c_1^2(Y)$ and

$$c_2(\mathcal{X}) = c_2(Y) - \sum (n+1)(a_n + d_n + e_n) + \sum \frac{a_n}{n+1} + \frac{d_n}{4(n-2)} + \frac{e_6}{24} + \frac{e_7}{48} + \frac{e_8}{120}. \tag{2.1}$$

The denominators $4(n-2)$, $24, 48$, and 120 are the order of the binary dihedral $BD_{4(n-2)}$, the binary tetrahedral, the binary octahedral, and the binary icosahedral group, respectively.

2.3. Orbifold Riemann–Roch
Let L be an orbifold line bundle on the orbifold $\mathcal{X}$ of dimension n. We will use Kawasaki's orbifold Riemann–Roch theorem (see [10]) or Toën's for Deligne–Mumford stacks (see [25]), using intersection theory on stacks.

THEOREM 8 (see [25])
Let $\mathcal{X}$ be a Deligne–Mumford stack with quasiprojective coarse moduli space and which has the resolution property (i.e., every coherent sheaf is a quotient of a vector bundle). Let E be a coherent sheaf on $\mathcal{X}$. Then

$$\chi(\mathcal{X}, E) = \int_{\mathcal{X}} \widetilde{ch}(E)\widetilde{Td}(T_{\mathcal{X}}).$$

From this, we obtain the asymptotic formula

$$\chi(\mathcal{X}, L^k) = \frac{c_1(L)^n}{n!}k^n + O(k^{n-1}),$$

using orbifold Chern classes.

We will apply this result to orbifold surfaces $\mathcal{X}$ of general type associated to canonical surfaces. Then

$$\chi(\mathcal{X}, S^m\Omega_{\mathcal{X}}) = \frac{m^3}{6}(c_1^2 - c_2) + O(m^2),$$

where c_1 and c_2 are the orbifold Chern classes of $\mathcal{X}$.

2.4. Vanishing theorems

In the case of smooth minimal surfaces of general type Y, thanks to the semistability of the cotangent bundle Ω_Y with respect to K_Y, we have Bogomolov's vanishing theorem (see [2]):

$$H^0(Y, S^m T_Y \otimes K_Y^p) = 0, \tag{2.2}$$

for $m - 2p > 0$.

Now, let us consider an orbifold surface $\mathcal{X}$ of general type associated to a canonical surface X. Then Bogomolov's vanishing theorem easily extends to this situation:

$$H^0(\mathcal{X}, S^m T_{\mathcal{X}} \otimes K_{\mathcal{X}}^p) = 0, \tag{2.3}$$

for $m - 2p > 0$.

Indeed, $\mathcal{X}$ can be equipped with an orbifold Kähler–Einstein metric (see [11], [24]), and the standard Bochner identities apply (see [12]).

As a corollary, if $s_2(\mathcal{X}) := c_1^2(\mathcal{X}) - c_2(\mathcal{X}) > 0$, then

$$H^0(\mathcal{X}, S^m\Omega_{\mathcal{X}}) \geq \frac{s_2(\mathcal{X})}{6}m^3 + O(m^2).$$

2.5. Logarithmic differentials and extension of sections

Let X be a complex manifold with a normal crossing divisor D.

The logarithmic cotangent sheaf $\Omega_X(\log D)$ is defined as the locally free subsheaf of the sheaf of meromorphic 1-forms on X, whose restriction to $X \setminus D$ is Ω_X and whose localization at any point $x \in D$ is given by

$$\Omega_X(\log D)_x = \sum_{i=1}^{l} \mathcal{O}_{X,x} \frac{dz_i}{z_i} + \sum_{j=l+1}^{n} \mathcal{O}_{X,x}\, dz_j,$$

where the local coordinates $z_1, \dots, z_n$ around x are chosen such that $D = \{ z_1, \dots, z_l = 0\}$.

Let X be a projective surface with canonical singularities, and let $Y \to X$ be the minimal resolution with E the exceptional divisor and $\mathcal{X} \to X$ the orbifold.

Sections of $H^0(\mathcal{X}, S^m\Omega_{\mathcal{X}})$ do not give a priori sections of $H^0(Y, S^m\Omega_Y)$; they give only sections of $H^0(Y \setminus E, S^m\Omega_Y)$. Nevertheless, in the case of quotient singularities, we have the following extension theorem of Miyaoka [17] (see also [9, Corollary 3.2]):

$$H^0(Y \setminus E, S^m\Omega_Y) \cong H^0\big(Y, S^m\Omega_Y(\log E)\big). \tag{2.4}$$

3. Proof of Theorem 1

In this section we prove Theorem 1, the main result of this paper.

THEOREM 9

Suppose that $s_2(Y) + s_2(\mathcal{X}) > 0$. Then

$$h^0(Y, S^m\Omega_Y) \geq \frac{s_2(Y) + s_2(\mathcal{X})}{12} m^3 + O(m^2);$$

in particular, the cotangent bundle of Y is big.

For $m \in \mathbb{N}^*$, let us consider the following exact sequence:

$$0 \to S^m\Omega_Y \to S^m\Omega_Y(\log E) \to Q_m \to 0. \tag{3.1}$$

The quotient sheaf Q_m is supported by the divisor E that is the sum of the exceptional components of the map $Y \to X$. Since the singularities of X are ADE, there exists a neighborhood U of E such that the canonical sheaf of Y is trivial: $(K_Y)_{|U} \simeq (\mathcal{O}_Y)_{|U}$. Therefore multiplying by $\otimes K_Y^{\otimes(1-m)}$, we get the following exact sequence:

$$0 \to S^m\Omega_Y \otimes K_Y^{\otimes(1-m)} \to S^m\Omega_Y(\log E) \otimes K_Y^{\otimes(1-m)} \to Q_m \to 0. \tag{3.2}$$

The proof will distinguish two cases according to the value of $\overline{\lim}\, \frac{h^0(Q_m)}{m^3}$.

Let us first suppose that

$$\overline{\lim}\, \frac{h^0(Q_m)}{m^3} \leq \frac{s_2(\mathcal{X}) - s_2(Y)}{12}.$$

As explained above, the Riemann–Roch Theorem 8 and Bogomolov's vanishing property (2.3) give

$$\overline{\lim}\,\frac{1}{m^3}h^0(\mathcal{X}, S^m\Omega_{\mathcal{X}}) \geq \frac{s_2(\mathcal{X})}{6}.$$

Combined with the extension property (2.4), this implies that

$$\overline{\lim}\,\frac{1}{m^3}h^0\big(Y, S^m\Omega_Y(\log E)\big) \geq \frac{s_2(\mathcal{X})}{6}.$$

Then the exact sequence (3.1) implies that

$$\begin{aligned}\overline{\lim}\,\frac{h^0(S^m\Omega_Y)}{m^3} &\geq \overline{\lim}\,\frac{1}{m^3}h^0\big(Y, S^m\Omega_Y(\log E)\big) - \overline{\lim}\,\frac{h^0(Q_m)}{m^3}\\ &\geq \frac{s_2(\mathcal{X})}{6} - \frac{s_2(\mathcal{X}) - s_2(Y)}{12}\\ &= \frac{s_2(\mathcal{X}) + s_2(Y)}{12}.\end{aligned}$$

Let us suppose now that

$$\overline{\lim}\,\frac{h^0(Q_m)}{m^3} > \frac{s_2(\mathcal{X}) - s_2(Y)}{12}.$$

The extension property (2.4), combined with the triviality of K_Y on U, and Serre duality give

$$\begin{aligned}h^0\big(Y, S^m\Omega_Y(\log E) \otimes K_Y^{\otimes(1-m)}\big) &\cong h^0(Y \setminus E, S^m\Omega_Y \otimes K_Y^{\otimes(1-m)})\\ &\cong h^0(\mathcal{X}, S^m\Omega_{\mathcal{X}} \otimes K_{\mathcal{X}}^{\otimes(1-m)})\\ &\cong h^2(\mathcal{X}, S^m\Omega_{\mathcal{X}}).\end{aligned}$$

The latter dimension being zero by Bogomolov's vanishing property (2.3), we obtain

$$h^0\big(Y, S^m\Omega_Y(\log E) \otimes K_Y^{\otimes(1-m)}\big) = 0.$$

Thus, by the exact sequence (3.2), we obtain

$$h^0(Q_m) \leq h^1(Y, S^m\Omega_Y \otimes K_Y^{\otimes(1-m)}).$$

Serre duality again implies that

$$h^1(Y, S^m\Omega_Y \otimes K_Y^{\otimes(1-m)}) = h^1(Y, S^m\Omega_Y).$$

Since $h^2(Y, S^m\Omega_Y) = 0$ by Bogomolov's vanishing property (2.2), we get by Riemann–Roch

$$\overline{\lim}\,\frac{1}{m^3}h^0(Y, S^m\Omega_Y) = \overline{\lim}\,\frac{1}{m^3}\big(\chi(S^m\Omega_Y) + h^1(Y, S^m\Omega_Y)\big)$$
$$\geq \frac{s_2(Y)}{6} + \frac{s_2(\mathcal{X}) - s_2(Y)}{12},$$

and therefore

$$\overline{\lim}\,\frac{1}{m^3}h^0(Y, S^m\Omega_Y) \geq \frac{s_2(\mathcal{X}) + s_2(Y)}{12}.$$

In any of the two above cases, we get

$$\overline{\lim}\,\frac{1}{m^3}h^0(Y, S^m\Omega_Y) \geq \frac{s_2(\mathcal{X}) + s_2(Y)}{12},$$

and therefore the cotangent sheaf of Y is big.

4. Applications

4.1. Surfaces with A_k singularities

As a corollary of Theorem 1 we obtain the following.

THEOREM 10

Let $X \subset \mathbb{P}^3$ be a hypersurface of degree d with ℓ singularities A_k, and let $Y \to X$ be its minimal resolution. If

$$\ell > \frac{4(k+1)}{k(k+2)}(2d^2 - 5d),$$

then Y has big cotangent bundle.

Proof

Applying Proposition 7 and the Brieskorn resolution theorem (see [4]), easy computations give

$$s_2(Y) = 10d - 4d^2, \qquad s_2(\mathcal{X}) = 10d - 4d^2 + \Big(k + 1 - \frac{1}{k+1}\Big)\ell.$$

We apply Theorem 1 with these values. □

COROLLARY 11

If $d \geq 13$, then there exist nodal surfaces in $\mathbb{P}^3$ of degree d whose minimal resolution has big cotangent bundle.

Proof
The condition on the number ℓ of nodes is $\ell > \frac{8}{3}(2d^2 - 5d)$. In [22], Segre constructed nodal hypersurfaces with $\ell \geq \frac{1}{4}d^2(d-1)$ nodes. For $d \geq 20$, we have $\frac{1}{4}d^2(d-1) > \frac{8}{3}(2d^2 - 5d)$, and thus we obtain examples of hypersurfaces of degree $d \geq 20$ with symmetric differentials.

Chmutov [6] (see also [14, p. 58]) constructed surfaces of degree d with the number $\mu(d)$ of A_1 singularities as follows:

d	13	14	15	16	17	18	19
$\mu(d)$	732	949	1155	1450	1728	2097	2457
$[\frac{8}{3}(2d^2 - 5d)] + 1$	729	859	1001	1153	1315	1489	1673

Therefore we also obtain examples for d in the range $13 \leq d \leq 19$. □

Remark 12
In [3], the result of Corollary 11 was claimed for hypersurfaces of degree $d \geq 6$. However, the proof uses the results of [3, Lemma 2.2], which turns out to be false. Let us explain this briefly in more details, using the notations of the proof of Theorem 1. In [3, p. 94, Lemma 2.2], it is claimed that

$$\dim\big(H^0(Y \setminus E, S^m\Omega_Y)/H^0(Y, S^m\Omega_Y)\big) = \frac{1}{4}\ell m^3 + O(m^2),$$

where ℓ is the number of nodal singularities on X.

To compute this dimension, the authors exhibit symmetric differentials of $Y \setminus E$ nonzero in the quotient but do not verify the linear independence. In fact, one can verify that

$$\dim\big(H^0(Y \setminus E, S^m\Omega_Y)/H^0(Y, S^m\Omega_Y)\big) = \frac{11}{108}\ell m^3 + O(m^2).$$

This computation is done independently in [23], deriving slightly better bounds in the case of nodes. In particular, the existence of a surface of degree 10 with big cotangent sheaf is obtained.

4.2. Ramified covers of the plane
In this section we give applications of Theorem 1 for cyclic covers of the plane.

Let $D = \bigcup_{j=1}^{j=k} D_j \subset \mathbb{P}^2$, where D_j is a smooth curve of degree d_j and such that D has nodal singularities. For $n > 1$ dividing $d = \sum_{j=1}^{j=k} d_j$, there exists an n-cyclic covering $X \to \mathbb{P}^2$ branched along D.

Since locally a singularity s of D has equation $x^2 + y^2 = 0$, the singularity in X above s has equation $z^n = x^2 + y^2$ and is a A_{n-1} singularity.

The Chern numbers of the desingularization Y of X are

$$c_1^2 = n\Big(-3 + \Big(1 - \frac{1}{n}\Big)d\Big)^2,$$
$$c_2 = 3n + (n-1)(d^2 - 3d),$$

and Y has general type unless $(d,n) = (2,2), (4,2), (6,2), (3,3), (4,4)$, cases we disregard from now on. We remark that the surface Y is minimal. Such ramified coverings provide a family where the number of symmetric differentials may jump.

First, one should note that in the smooth case there is no symmetric differentials at all.

PROPOSITION 13
Suppose that X is smooth. Then

$$H^0(X, S^m\Omega_X) = 0,$$

for all $m > 0$.

Proof
Let us denote by $W \to \mathbb{P}^2$ the cyclic degree d cover branched over the smooth curve D. There is a cyclic degree $v = \frac{d}{n}$ cover $g : W \to X$ making the diagram

$$\begin{array}{ccc} W & \overset{g}{\dashrightarrow} & X \\ & \searrow & \downarrow \\ & & \mathbb{P}^2 \end{array}$$

commute. Since W is a smooth hypersurface in $\mathbb{P}^3$, the space $H^0(W, S^m\Omega_W)$ is 0 (see [21]). That implies $H^0(X, S^m\Omega_X) = 0$. □

4.2.1. Criteria for $n = d$ and arbitrary D_j
Let us consider the case when the cover has degree $n = d$. This gives us a hypersurface $X \subset \mathbb{P}^3$ of degree d with A_{d-1} singularities over the singularities of D, that is, the intersection points of the D_j's.

THEOREM 14
Suppose that $d_i \geq c$ for $i = 1, \dots, k$ and

$$k(k-1) > \frac{8d^2(2d-5)}{c^2(d^2-1)}.$$

Then the minimal resolution $Y \to X$ has big cotangent bundle.

Proof

The Chern numbers of the minimal desingularization Y of X are $c_1^2 = d(d-4)^2$ and $c_2 = d(d^2 - 4d + 6)$. The Chern numbers of the orbifold $\mathcal{X}$ are $c_1^2(\mathcal{X}) = c_1^2$ and

$$c_2(\mathcal{X}) = c_2 - \Big(d - \frac{1}{d}\Big)\Big(\sum_{i<j} d_i d_j\Big).$$

Thus

$$\begin{aligned} s_2(Y) + s_2(\mathcal{X}) &= 4d(5-2d) + \Big(d - \frac{1}{d}\Big)\Big(\sum_{i<j} d_i d_j\Big) \\ &> 4d(5-2d) + \frac{k(k-1)}{2}\Big(d - \frac{1}{d}\Big)c^2. \end{aligned}$$

□

As a corollary, we obtain many examples of surfaces in $\mathbb{P}^3$ with big cotangent bundle.

Example 4.1

For every $d \geq 15$, the degree d covering of d lines in $\mathbb{P}^2$ has big cotangent sheaf.

4.2.2. Criteria when the D_j are lines and for n dividing d

Let us consider the case when all the curves D_i are lines and the degree n of the cover divides d. Thus $d = k$ and the number ℓ of A_{n-1} singularities is $d(d-1)/2$. Since $s_2(Y) + s_2(\mathcal{X}) = 2s_2(X) + \ell(n - \frac{1}{n})$, we get

$$s_2(Y) + s_2(\mathcal{X}) = 2n\big[6 - (n-1)(3v + v^2)\big] + \frac{1}{2}(nv-1)(n^2-1)v,$$

where $d = nv$. For a cover of degree $n = 2$ and $n = 3$, we get, respectively, $s_2(X) + s_2(\mathcal{X}) = 24 - \frac{27}{2} - v^2$ and $s_2(X) + s_2(\mathcal{X}) = 36 - 40v$; this is always negative (for $v \geq 2$) and we cannot apply Theorem 1. For the remaining cases, a simple computation gives the following.

THEOREM 15

For a cyclic cover of degree $n \geq 4$ branched over the union of $d = vn > 4$ lines in general position, we have $s_2(Y) + s_2(\mathcal{X}) > 0$, except for the following finite number of cases for the couples (v, n):

v	1	2	3	$4 \leq v \leq 6$	$7 \leq v \leq 12$
n	≤ 14	≤ 8	≤ 6	$4, 5$	4

4.3. Remarks when $s_2(Y) > 0$

We close this section by remarking that Theorem 1 also has an application to surfaces with $s_2(Y) > 0$. Let us consider a surface Y with ample cotangent bundle. For $m \gg 0$, we have $h^i(Y, S^m\Omega_Y) = 0, i = 1, 2$, and thus

$$h^0(Y, S^m\Omega_Y) = \frac{s_2(Y)}{6}m^3 + O(m^2).$$

Suppose that Y has a deformation Y_0 which is a surface containing one (-2)-curve (examples of such surfaces can be obtained, for instance, as a complete intersection of ample divisors in an Abelian variety). Then the space of symmetric differentials jumps:

$$h^0(Y_0, S^m\Omega_{Y_0}) \geq \Big(\frac{s_2(Y)}{6} + \frac{1}{8}\Big)m^3 + O(m^2) > h^0(Y, S^m\Omega_Y).$$

This is another illustration of the importance of the presence or absence of (-2)-curves for the geometry of a surface.

5. On the geography of the surfaces with big cotangent bundle

5.1. A Chern classes inequality

As mentioned in the Introduction, surfaces of general type with ample cotangent bundle are known to satisfy the Chern classes inequality $c_1^2 > c_2$ (see [8]). A natural question to ask is if surfaces of general type with big cotangent bundle satisfy a Chern classes inequality. We investigate here the case of a surface that satisfies the hypothesis of Theorem 1.

PROPOSITION 16

If a surface Y satisfies $s_2(Y) + s_2(\mathcal{X}) > 0$, then we have

$$c_1^2(Y) > \frac{3}{5}c_2(Y). \tag{5.1}$$

Proof

Since $s_2(Y) + s_2(\mathcal{X}) > 0$, dividing this inequality by $c_1^2(\mathcal{X}) = c_1^2(Y)$, we obtain

$$2 - \frac{c_2(Y)}{c_1^2(Y)} - \frac{c_2(\mathcal{X})}{c_1^2(Y)} > 0.$$

One of the main results of [17], translated into the language of orbifolds, is the orbifold Bogomolov–Miyaoka–Yau inequality for surfaces with quotient singularities:

$$c_1^2(\mathcal{X}) \leq 3c_2(\mathcal{X}).$$

Applying this inequality, we obtain

$$\frac{c_2(\mathcal{X})}{c_1^2(Y)} > \frac{1}{3}$$

and

$$\frac{c_2(X)}{c_1^2(Y)} < \frac{5}{3}.$$

□

Remark 17

As mentioned above, the existence of a surface of degree 10 in $\mathbb{P}^3$ with big cotangent sheaf is obtained in [23]. This shows that the inequality (5.1) is not satisfied by all such surfaces.

Remark 18

Recall that for minimal surfaces of general type, we have the Noether inequality:

$$c_1^2(Y) \geq \frac{1}{5}(c_2 - 36).$$

The surfaces that are on the Noether line $c_1^2 = \frac{1}{5}(c_2 - 36)$ are called *Horikawa surfaces*. As a consequence, it is hopeless to get in that way the proof of the existence of (higher) symmetric forms for Horikawa surfaces with canonical singularities.

Remark 19

Since we suppose that $s_2(Y) + s_2(\mathcal{X}) > 0$, an immediate computation gives

$$\frac{c_2(Y) + c_2(\mathcal{X})}{c_1^2(Y)} = \frac{c_2(Y)}{c_1^2(Y)} + \frac{c_2(\mathcal{X})}{c_1^2(\mathcal{X})} < 2.$$

Therefore a ratio $\frac{c_1^2(Y)}{c_2(Y)}$ close to $\frac{3}{5}$ forces the ratio $\frac{c_1^2(\mathcal{X})}{c_2(\mathcal{X})}$ to be close to the Miyaoka bound 3. Examples of orbifolds with $s_2(Y) < 0$ and $\frac{c_1^2(\mathcal{X})}{c_2(\mathcal{X})}$ close to 3 are rather rare (see [20] for some references).

Acknowledgments. The second author thanks Fedor Bogomolov, Bruno De Oliveira, and Jordan Thomas for several interesting and fruitful discussions and remarks. He also thanks the Centre national de la recherche scientifique (CNRS) for the opportunity to spend a semester in Montreal at UMI CNRS-CRM, as well as the hospitality of the Université du Québec à Montréal–CIRGET, where part of this work was done.

References

[1] F. A. BOGOMOLOV, *Families of curves on surfaces of general type*, Dokl. Akad. Nauk SSSR **236** (1977), 1041–1044. MR 0457450. *(1337)*

[2] ———, *Holomorphic tensors and vector bundles on projective manifolds* (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978), 1227–1287; English translation in Math. USSR-Izv. **13** (1979), 499–555. MR 522939. *(1341)*

[3] F. A. BOGOMOLOV and B. DE OLIVEIRA, *Hyperbolicity of nodal hypersurfaces*, J. Reine Angew. Math. **596** (2006), 89–101. MR 2254806. DOI 10.1515/CRELLE.2006.053. *(1337, 1338, 1339, 1345)*

[4] E. BRIESKORN, "Singular elements of semi-simple algebraic groups" in *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, Gauthier-Villars, Paris, 1971, 279–284. MR 0437798. *(1344)*

[5] F. CAMPANA, *Orbifoldes spéciales et classification biméromorphe des variétés kählériennes compactes*, J. Inst. Math. Jussieu **10** (2011), 809–934. MR 2831280. DOI 10.1017/S1474748010000101. *(1339)*

[6] S. V. CHMUTOV, *Examples of projective surfaces with many singularities*, J. Algebraic Geom. **1** (1992), 191–196. MR 1144435. *(1345)*

[7] O. DEBARRE, *Hyperbolicity of complex varieties*, notes from the PRAGMATIC conference, Catania, Italy, 2004. *(1337)*

[8] W. FULTON and R. LAZARSFELD, *Positive polynomials for ample vector bundles*, Ann. of Math (2) **118** (1983), 35–60. MR 0707160. DOI 10.2307/2006953. *(1337, 1348)*

[9] D. GREB, S. KEBEKUS, and S. KOVÁCS, *Extension theorems for differential forms and Bogomolov-Sommese vanishing on log canonical varieties*, Compos. Math. **146** (2010), 193–219. MR 2581247. DOI 10.1112/S0010437X09004321. *(1342)*

[10] T. KAWASAKI, *The Riemann-Roch theorem for complex V-manifolds*, Osaka J. Math. **16** (1979), 151–159. MR 0527023. *(1340)*

[11] R. KOBAYASHI, *Einstein-Kähler V-metrics on open Satake V-surfaces with isolated quotient singularities*, Math. Ann. **272** (1985), 385–398. MR 0799669. DOI 10.1007/BF01455566. *(1341)*

[12] S. KOBAYASHI, *The first Chern class and holomorphic symmetric tensor fields*, J. Math. Soc. Japan **32** (1980), 325–329. MR 0567422. DOI 10.2969/jmsj/03220325. *(1341)*

[13] J. KOLLÁR and S. MORI, *Birational Geometry of Algebraic Varieties*, Cambridge Tracts in Math. **134**, Cambridge Univ. Press, Cambridge, 1998. MR 1658959. DOI 10.1017/CBO9780511662560. *(1338, 1340)*

[14] O. LABS, *Hypersurfaces with many singularities—History, constructions, algorithms, visualization*, Ph.D. thesis, Johannes Gutenberg-Universität, Mainz, Germany, 2005, http://www.oliverlabs.net/data/phdthesis_oliver_labs.pdf. *(1345)*

[15] M. MCQUILLAN, *Diophantine approximations and foliations*, Inst. Hautes Études Sci. Publ. Math. **87** (1998), 121–174. MR 1659270. *(1337, 1338)*

[16] Y. MIYAOKA, “Algebraic surfaces with positive indices” in *Classification of Algebraic and Analytic Manifolds (Katata, 1982)*, Progr. Math. **39**, Birkhäuser, Boston, 1983, 281–301. MR 0728611. *(1337)*

[17] ———, *The maximal number of quotient singularities on surfaces with given numerical invariants*, Math. Ann. **268** (1984), 159–171. MR 0744605. DOI 10.1007/BF01456083. *(1342, 1348)*

[18] ———, *The orbibundle Miyaoka-Yau-Sakai inequality and an effective Bogomolov-McQuillan theorem*, Publ. Res. Inst. Math. Sci. **44** (2008), 403–417. MR 2426352. DOI 10.2977/prims/1210167331. *(1337)*

[19] M. REID, “Young person’s guide to canonical singularities” in *Algebraic Geometry (Brunswick, Maine, 1985)*, Proc. Sympos. Pure Math. **46**, Part 1, Amer. Math. Soc., Providence, 1987, 345–414. MR 0927963. *(1338)*

[20] X. ROULLEAU and E. ROUSSEAU, *On the hyperbolicity of surfaces of general type with small* c_1^2, J. Lond. Math. Soc. (2) **87** (2013), 453–477. MR 3046280. DOI 10.1112/jlms/jds053. *(1339, 1340, 1349)*

[21] F. SAKAI, “Symmetric powers of the cotangent bundle and classification of algebraic varieties” in *Algebraic Geometry (Copenhagen, 1978)*, Lecture Notes in Math. **732**, Springer, Berlin, 1979, 545–563. MR 0555717. *(1337, 1346)*

[22] B. SEGRE, *Sul massimo numero di nodi delle superficie di dato ordine*, Boll. Un. Mat. Ital. (3) **2** (1947), 204–212. MR 0025178. *(1345)*

[23] J. THOMAS, *Hyperbolicity of nodal surfaces in* $\mathbb{P}^3$*: Effective bounds*, preprint, 2013. *(1345, 1349)*

[24] G. TIAN and S.-T. YAU, “Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry” in *Mathematical Aspects of String Theory (San Diego, Calif., 1986)*, Adv. Ser. Math. Phys. **1**, World Sci. Publishing, Singapore, 1987, 574–628. MR 0915840. *(1341)*

[25] B. TÖEN, *Théorèmes de Riemann-Roch pour les champs de Deligne Mumford*, K-Theory **18** (1999), 33–76. MR 1710187. DOI 10.1023/A:1007791200714. *(1340, 1341)*

Roulleau

Laboratoire de Mathématiques et Applications, Université de Poitiers, 86000 Poitiers, France; roulleau@math.univ-poitiers.fr

Rousseau

Laboratoire d’Analyse, Topologie, Probabilités, Université d’Aix-Marseille et CNRS, 13453 Marseille CEDEX 13, France; erwan.rousseau@cmi.univ-mrs.fr