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Curves with low Harbourne constants on Kummer and abelian surfaces

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Abstract We construct and study curves with low H-constants on abelian and K3 surfaces. Using the Kummer (16_6)-configurations on Jacobian surfaces and some (16_{10})-configurations of curves on $(1, 3)$-polarized Abelian surfaces, we obtain examples of configurations of curves of genus > 1 on a generic Jacobian K3 surface with H-constants < -4.

Keywords Bounded negativity conjecture · Harbourne constants · Abelian surfaces · Kummer surfaces · Kummer configuration

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1 Introduction

The bounded negativity conjecture predicts that for a smooth complex projective surface X there exists a bound b_X such that for any reduced curve C on X one has

$$C^2 \geq b_X.$$

That conjecture holds in some cases, for instance if X is an abelian surface, but we do not know whether it remains true if one considers a blow-up of X. With that question in mind, the H-constants have been introduced in [1].

For a reduced (but not necessarily irreducible) curve C on a surface X and $\mathcal{P} \subset X$ a finite non empty set of points, let $\pi : \bar{X} \to X$ be the blowing-up of X at $\mathcal{P}$ and let $\bar{C}$ denotes the strict transform of C on $\bar{X}$. Let us define the number

$$H(C, \mathcal{P}) = \frac{\bar{C}^2}{|\mathcal{P}|},$$

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where $|\mathcal{P}|$ is the order of $\mathcal{P}$. We define the Harbourne constant of C (for short the H-constant) by the formula

$$H(C) = \inf_{\mathcal{P}} H(C, \mathcal{P}) \in \mathbb{R},$$

where $\mathcal{P} \subset X$ varies among all finite non-empty subsets of X (note that there is a slight difference with the definition of Hadean constant of a curve given in [1, Remark 2.4], which definition exists only for singular curves; see Remark 4 for the details). Singular curves tend to have low H-constants. It is in general difficult to construct curves having low H-constants, especially if one requires the curve to be irreducible. The (global) Harbourne constant of the surface X is defined by

$$H_X = \inf_{C} H(C) \in \mathbb{R} \cup \{-\infty\}$$

where the infimum is taken among all reduced curves $C \subset X$. Harbourne constants and their variants are intensively studied (see e.g. [1,11,12,14]); note that the finiteness of H_X implies the BNC conjecture. Using some elliptic curve configurations in the plane [15], it is known that

$$H_{\mathbb{P}^2} \leq -4,$$

and for any surface X one has $H_X \leq H_{\mathbb{P}^2} \leq -4$ (see [14]). However, the curves $(C_n)_{n\in\mathbb{N}}$ on $X \neq \mathbb{P}^2$ with H-constant tending to -4 used to prove that $H_X \leq -4$ are not very explicit and they all satisfy $H(C_n) > -4$.

The H-constant is an invariant of the isogeny class of an abelian surface. Using the classical (16_6) configuration R_1 of 16 genus 2 curves and 16 2-torsion points in a principally polarized abelian surface and a (16_{10}) configuration of 16 smooth genus 4 curves and 16 2-torsion points on a $(1, 3)$-polarized abelian surface, plus the dynamic of the multiplication by $n \in \mathbb{Z}$ map, we construct explicitly some curves with low H-constants on abelian surfaces:

Theorem 1 *Let A be a simple abelian surface. There exists a sequence of curves $(R_n)_{n\in\mathbb{N}}$ in A such that $R_n^2 \to \infty$ and $H(R_n) = -4$.*
If A is the Jacobian of a smooth genus 2 curve, the curve R_n can be chosen either as the union of 16 smooth curves or as an irreducible singular curve.

It is known that on two particular abelian surfaces with CM there exists a configuration C of elliptic curves with $H(C) = -4$. Moreover for any elliptic curve configuration C in an abelian surface A, one always has

$$H(C) \geq -4,$$

with equality if and only if the complement of the singularities of C is an open ball quotient surface (for these previous results see [14]). Thus elliptic curve configurations with $H(C) = -4$ are rather special, in particular these configurations are rigid. Indeed to an algebraic family $(A_t, C_t)_t$ of such surfaces A_t, each containing a configuration C_t of elliptic curves with H-constant equals to -4, such that C_t varies algebraically with A_t, one can associate a family of ball quotient surfaces. Since ball quotient surfaces are rigid, the family $(A_t, C_t)_t$ is trivial and the pairs (A_t, C_t) are isomorphic.

We observe that for our new examples of curves with $H(C) = -4$ there is no such links with ball quotient surfaces. Indeed the pairs (A, C) we give such that $H(C) = -4$ have deformations.

We then consider the images of the curves R_n in the associated Kummer surface X and we obtain:

Theorem 2 *Let X be a Jacobian Kummer surface. For any $n > 1$, there are configurations C_n of curves of genus > 1 such that $H(C_n) = -4\frac{n^4}{n^4-1} < -4$.*

The H-constants of curves (and some related variants such as the s-tuple Harbourne constants) on K3 surfaces have been previously studied, by example in [8] and [12]. Laface and Pokora [8] study transversal arrangements $\mathcal{C}$ of rational curves on K3 surfaces and they give examples of configurations $\mathcal{C}$ with a low Harbourne constant. In their examples, one has $H(\mathcal{C}) \geq -3.777$, with the exception of two examples on the Schur quartic and the Fermat quartic surfaces, both reaching

$$H(\mathcal{C}) = -8.$$

In the last section, we then turn our attention to irreducible curves with low H-constants in abelian and Kummer surfaces, which are more difficult to obtain, some of which have been recently constructed in [16].

2 Smooth hyperelliptic curves in abelian surfaces and H-constants

2.1 Preliminaries, Notations

By [4], an abelian surface A contains a smooth hyperelliptic curve C_0 of genus g if and only if it is a generic $(1, g-1)$-polarized abelian surface and $g \in \{2, 3, 4, 5\}$.

In this section, we study the configurations of curves obtained by translation of these hyperelliptic curves C_0 (of genus 2, 3, 4 or 5) by 2-torsion points and by taking pull-backs by endomorphisms of A. In the present sub-section, we recall some facts on the computation of the H-constants and some notations.

Let $C_1, \dots, C_t$ be smooth curves in a smooth surface X such that the singularities of $C = \sum_j C_j$ are *ordinary* (i.e. resolved after one blow-up). Let $\mathrm{Sing}(C)$ be the singularity set of C; we suppose that it is non-empty. Let $f : \bar{X} \to X$ be the blow-up of X at $\mathrm{Sing}(C)$. For each p in $\mathrm{Sing}(C)$, let m_p be the multiplicity of C (we say that such a singularity p is a *m_p-point*) and let E_p be the exceptional divisor in $\bar{X}$ above p. Let us recall the following notation:

$$H(C, \mathcal{P}) = \frac{\bar{C}^2}{|\mathcal{P}|},$$

where $\bar{C}$ is the strict transform of a curve C in the blowing-up surface at $\mathcal{P} \neq \emptyset$. The following formula is well known:

Lemma 3 *Let s be the cardinal of* $\mathrm{Sing}(C)$. *One has*

$$H(C, \mathrm{Sing}(C)) = \frac{C^2 - \sum_{p\in\mathcal{P}} m_p^2}{s} = \frac{\sum_{j=1}^{t} C_j^2 - \sum_{p\in\mathcal{P}} m_p}{s}.$$

Proof One can compute $\bar{C}^2$ in two ways, indeed

$$\bar{C} = f^*C - \sum m_p E_p,$$

thus $\bar{C}^2 = C^2 - \sum_{p\in\mathcal{P}} m_p^2$ (that formula is valid for any configurations). But $\bar{C} = \sum_{i=1}^{t} \bar{C}_i = \sum_{i=1}^{t}(f^*C_i - \sum_{p\in C_i} E_p)$, and since the singularities are ordinary, the curves $\bar{C}_i$ are disjoint, thus

$$\bar{C}^2=\sum_{i=1}^{t}\bar{C_i}^2=\sum_{i=1}^{t}C_i^2-\sum_{p\in \mathrm{Sing}(C)}m_p,$$

where we just use the fact that $\sum_{i=1}^{t}\sum_{p\in C_i}1=\sum_{p\in \mathrm{Sing}(C)}m_p$. □

Recall that we define the H-constant of a curve C by the formula

$$H(C):=\inf_{\mathcal{P}}H(C,\mathcal{P})\in\mathbb{R},$$

where $\mathcal{P}\subset X$ varies among all finite non-empty subsets of X.

Remark 4 (a) If C is smooth one has $H(C)=\min(-1,C^2-1)$.
(b) In [1, Remark 2.4] the Hadean constant of a singular curve C on a surface X is defined by the formula

$$H_{ad}(C):=\min_{\mathcal{P}\subset \mathrm{Sing}(C),\,\mathcal{P}\neq\emptyset}H(C,\mathcal{P}).$$

Let C be an arrangement of $n\geq 2$ smooth curves intersecting transversally (with at least one intersection point). In [7] is defined and studied the quantity $H(X,C):=H(C,\mathrm{Sing}(C))$. An advantage of our definition of H-constant is that it is defined for any curves. Moreover with our definition, it is immediate that the global H-constant of the surface X satisfies $H(X)=\inf H(C)$, where the infimum is taken over reduced curves C in X.

Let $m\in\mathbb{N}^*$ and let $C\hookrightarrow X$ be a singular curve having singularities of multiplicity m only (this will be the case for most of the curves in this paper). Let s be the order of $\mathrm{Sing}(C)$.

Lemma 5 *One has* $H(C,\mathrm{Sing}(C))=\frac{C^2}{s}-m^2$.
The H-constant of C is

$$H(C)=\min(-1,\ C^2-\ m^2,\ H(C,\mathrm{Sing}(C))).$$

Proof For integers $0\leq a\leq s$, $b\geq 0$, $c\geq 0$ such that $a+b+c>0$, let $\mathcal{P}_{a,b,c}$ be a set of a m-points, b smooth points of C and c points in $X\setminus C$. Let

$$H_{a,b,c}=H(C,\mathcal{P}_{a,b,c})=\frac{C^2-am^2-b}{a+b+c}.$$

The border cases are $H_{1,0,0}=C^2-m^2$, $H_{0,1,0}=C^2-1$ and $H_{0,0,1}=C^2$. If $a<\frac{c+C^2}{m^2-1}$ (case which occurs when c is large) the function $b\to H_{a,b,c}$ is decreasing and converging to -1 when $b\to\infty$.

If $a\geq\frac{c+C^2}{m^2-1}$, the function $b\to H_{a,b,c}$ is increasing, thus if $a\neq 0$, one has

$$\inf_{b\geq 0}H_{a,b,c}=H_{a,0,c}=\frac{C^2-am^2}{a+c},$$

(note that even if $a<\frac{C^2}{m^2-1}$, one still has $\frac{C^2-am^2}{a+c}\geq -1$). If $C^2-am^2>0$, $H_{a,0,c}$ is a decreasing function of c, with limit 0, otherwise this is an increasing function and the infimum is attained for $c=0$, which gives $\frac{C^2-am^2}{a}$ (if $a=0$, one gets C^2). Then taking the minimum over a, one obtains the result. □

Let us recall (see [6]) that for $a,b,n,m\in\mathbb{N}^*$, a (a_n,b_m)*-configuration* is the data of two sets A,B of order a and b, respectively, and a relation $R\subset A\times B$, such that $\forall\alpha\in A$, $\#\{(\alpha,x)\in R\}=n$ and $\forall\beta\in B$, $\#\{(y,\beta)\in R\}=m$. One has $an=bm=\#R$. If

$a = b$ and $n = m$, it is called a (a_n)-*configuration*. If for $\alpha \neq \alpha'$ in A the cardinality λ of $\{(\alpha, x) \in R\} \cap \{(\alpha', x) \in R\}$ does not depend on $\alpha \neq \alpha'$, this is called a (a_n, b_m)-*design* and $m(n-1) = \lambda(a-1)$; λ is called the *type of the design*.

2.2 Construction of configuration from genus 2 curves

Let A be a principally polarized abelian surface such that the principal polarization C_0 is a smooth genus 2 curve. One can choose an immersion such that $0 \in A$ is a Weierstrass point of C_0. The configuration of the 16 translates

$$C_t = t + C_0, \ t \in A[2]$$

of C_0 by the 2 torsion points of A is the famous (16_6) Kummer configuration: there are 6 curves through each point in $A[2]$, and each curve contains 6 points in $A[2]$ (since $C_t C_{t'} = 2$ for $t \neq t'$ in $A[2]$, it is even a (16_6)-design of type 2).

Let now $n > 0$ be an integer and let $[n] : A \to A$ be the multiplication by n map on A. For $t \in A[2]$, let us define $D_t = [n]^* C_t$, in other words

$$D_t = \{x \mid nx \in C_t\} = \{x \mid nx + t \in C_0\}.$$

Since $[n]$ is étale, the curve D_t is a smooth curve, thus it is irreducible since its components are the pull back of an ample divisor. By [9, Proposition 2.3.5], since C_t is symmetric (i.e. $[-1]^* C_t = C_t$), one has $D_t \sim n^2 C_t$ (in particular $D_t^2 = 2n^4$). The curve

$$W_n = [n]^* \sum_{t \in A[2]} C_t = \sum_{t \in A[2]} D_t$$

has 16 irreducible components and $16n^4$ ordinary singularities of multiplicity 6 (6-points), which are the torsion points $A[2n] := \mathrm{Ker}\,[2n]$. Each curve D_t contains $6n^4$ 6-points; the configuration of curves D_t and singular points of W_n is a $(16_{6n^4}, 16n^4{}_6)$-configuration. Using Lemma 3, we get:

Lemma 6 *One has $H(W_n, \mathrm{Sing}(W_n)) = -4$.*

The Harbourne constant H_A of a surface A is an invariant of the isogeny class of A (see [14]). Thus if A is generic, it is isogeneous to the Jacobian of a smooth genus 2 curve, and we thus obtain the following:

Proposition 7 *On a generic abelian surface A, one has:*

$$H_A \leq -4.$$

Note that when A is isogeneous to the product of 2 elliptic curves E, E' (thus non generic in our situation), the H-constant of A verifies that $H_A \leq -2$, and $H_A \leq -3$ if E and E' are isogeneous (see [14]). Moreover, there are two examples of surfaces with CM for which $H_A \leq -4$.

Remark 8 (1) Suppose that n is odd, then

$$D_t = \{x \mid n(x + t) \in C_0\} = D_0 + t.$$

Moreover, if u is a 2-torsion point one has $2u = 0 \Leftrightarrow 2nu = 0$, thus D_0 and each curve D_t contains 6 points of 2-torsion.

(2) Suppose that n is even. Let $u \in A[2]$ be a 2-torsion point. One has $u \in D_t \Leftrightarrow nu + t \in C_0 \Leftrightarrow t \in C_0$. Therefore the 6 curves D_t with t in $A[2] \cap C_0$ contain $A[2]$, and the remaining curves do not contain any points from $A[2]$.

2.3 Genus 3 curves

Let A be an abelian surface containing a hyperelliptic genus 3 curve C_0 such that 0 is a Weierstrass point. Then the 8 Weierstrass points of C_0 are contained in the set of 2-torsion points of A. Let $\mathcal{O}$ be the orbit of C_0 under the action of $A[2]$ by translation and let a be the cardinal of $\mathcal{O}$. The stabilizer S_t of C_0 acts as a fix-point free automorphism group of C_0. Thus considering the possibilities for the genus of C_0/S_t it is either trivial or an involution, therefore $a = 16$ or 8. By [5, Remark 1], the curve C_0 is stable by translation by a 2-torsion point, therefore $a = 8$. Let m be the number of curves in $\mathcal{O}$ through one point in $A[2]$ (this is well defined because $A[2]$ acts transitively on itself). The sets of 8 genus 3 curves and $A[2]$ form a $(8_8, 16_m)$-configuration, thus $m = 4$. Moreover, since they are translates, two curves $C, C' \in \mathcal{O}$ satisfy $CC' = C^2 = 4$, thus

$$C \sum_{C' \in \mathcal{O},\, C' \neq C} C' = 7 \cdot 4.$$

If the singularities of the union of the curves in $\mathcal{O}$ were only at the points in $A[2]$ and ordinary, one would have

$$C \sum_{C' \in \mathcal{O},\, C' \neq C} C' = 8 \cdot 3.$$

The configuration $\mathcal{C} = \sum_{C \in \mathcal{O}} C$ contains therefore other singularities than the points in $A[2]$ or the singularities are non ordinary. It seems less interesting from the point of view of H-constants. Observe that if the singularities at $A[2]$ are ordinary, one has $H(\mathcal{C}, A[2]) = -2$. If there are other singularities, since the configuration is stable by translations by $A[2]$, there are at least 16 more singularities.

2.4 Construction of configurations from genus 4 curves

Traynard in [17], almost one century later Barth, Nieto in [3], and Naruki in [10] constructed (16_{10}) configurations of lines lying on a 3-dimensional family of quartic K3 surfaces X in $\mathbb{P}^3$: there exist two sets $\mathcal{C}$, $\mathcal{C}'$ of 16 disjoint lines in X such that each line in $\mathcal{C}$ meets exactly 10 ten lines in $\mathcal{C}'$, and vice versa.

By the famous results of Nikulin characterizing Kummer surfaces, there exists a double cover $\pi : \tilde{A} \to X$ branched over $\mathcal{C}$. That cover contains 16 (-1)-curve over $\pi^{-1}\mathcal{C}$. The contraction $\mu : \tilde{A} \to A$ of these 16 exceptional divisors is an abelian surface and the image of these 16 curves is the set $A[2]$ of two torsion points of A.

We denote by $C_1, \dots, C_{16}$ the 16 smooth curves images by $\mu_* \pi^*$ of the 16 disjoint lines in $\mathcal{C}'$. By [3, Section 6], the 16 curves $C_1, \dots, C_{16}$ are translates of each other by the action by the group $A[2]$ of 2-torsion points; the argument is that if C_i' is a translate of C_i by a 2-torsion point, then $\pi_* \mu^* C_i'$ is a line in the quartic X, but a such a generic quartic has exactly 32 lines.

Proposition 9 *The curves $C_1, \dots, C_{16}$ in A are smooth of genus 4. The 16 2-torsion points $A[2]$ and these 16 curves form a (16_{10})-design of type 6: 10 curves though one point in $A[2]$, a curve contains 10 points in $A[2]$ and two curves meet at 6 points in $A[2]$.*
The H-constant of that configuration $\sum C_i$ is $H = -4$.

Proof The 10 intersection points between the lines in $\mathcal{C}$ and $\mathcal{C}'$ are transverse, therefore by the Riemann–Hurwitz Theorem, the genus of the 16 irreducible components of $\pi^* \mathcal{C}'$ is 4.

The intersections of the 16 components in $\mu_*\pi^*\mathcal{C}'$ are transverse (since $\pi^*\mathcal{C}'$ is a union of disjoint curves) and that intersection holds over points in $A[2]$ (which is the image of the exceptional divisors of $\tilde{A}$).

Since the curves in $\mathcal{C}$ and $\mathcal{C}'$ form a (16_{10}) configuration, the 16 curves $C_1, \dots, C_{16}$ and the 2 torsion points in A have the described (16_{10}) configuration.

Since the strict transform in $\tilde{A}$ of the curves $C_i \neq C_j$ are two disjoint curves, the 6 intersection points of $C_i \neq C_j$ are 2-torsion points, the configuration is therefore a (16_{10})-design of type 6.

It is then immediate to compute the H-constant of $\mathcal{C} = C_1 + \cdots + C_{16}$. □

Remark 10 Since the 16 curves are the orbit of a curve by the group $A[2]$ of torsion points, one can change the notations and define $C_t = C_0 + t$ for $t \in A[2]$, for a chosen curve C_0 containing 0. As in Sect. 2.2, let us define $D_t = [n]^*C_t$; this is a smooth curve. It is then immediate to check that the H-constant of the curve $W_n = \sum D_t$ equals -4. We will use these configurations of curves in Sect. 3.

2.5 Genus 5 curves

By [4], a generic (1, 4)-polarized abelian surface contains a smooth genus 5 curve C which is hyperelliptic, the set of Weierstrass points in C is 12 2-torsion points, and C is stable by a sub-group of $A[2]$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Thus the orbit of C by the translations by elements of $A[2]$ is the union of 4 genus 5 curves.

The intersection of two of these curves equals $C^2 = 2g - 2 = 8$. Since each of these two curves contains 12 points in $A[2]$, the intersections are transverse and are on 8 points in $A[2]$. The 4 curves and the 16 2-torsion points form a $(4_{12}, 16_3)$ configuration. The H-constant of that configuration is $H = \frac{4\cdot 8 - 16\cdot 3}{16} = -1$.

3 Configurations of curves with low H-constant in Kummer surfaces

In this Section, we study the images in the Kummer surface $\mathrm{Km}(A)$ of the various curve configurations studied in Sect. 2 in abelian surfaces A.

3.1 The genus 2 case

We keep the notations and hypothesis of Sect. 2.2. In particular, A is the Jacobian of a genus 2 curve. Let $\mu : \tilde{A} \to A$ be the blow-up of A at the 16 2-torsion points. We denote by $\bar{D}$ the strict transform in $\tilde{A}$ of a curve $D \hookrightarrow A$. Let $\pi : \tilde{A} \to X$ be the quotient map by the automorphism $[-1]$. Since on A one has $[-1]^*C_t = [-1]^*(t + C_0) = C_t$, one obtains

$$[-1]^*D_t = D_t$$

and the map $\bar{D}_t \to D'_t = \pi(\bar{D}_t)$ has degree 2, thus $D_t'^2 = \frac{1}{2}(\bar{D}_t)^2$.

Proposition 11 *Let be $n > 1$. The configuration $\mathcal{D}$ of the 16 curves D'_t with $t \in A[2]$ in the Kummer surface X has Harbourne constant*

$$H\left(\sum_{t\in A[2]} D'_t\right) = -4\frac{n^4}{n^4-1}.$$

Proof If n is even, a curve D_t contains 16 or 0 points of 2 torsion depending if $t \in C_0$ or not (thus there are 10 curves without points of 2 torsion, and 6 with). If n is odd, each curve D_t contains 6 points of 2-torsion and then one has:

$$D_t'^2 = \frac{1}{2}(2n^4 - 6) = n^4 - 3.$$

If n is even, one has:

$$D_t'^2 = \frac{1}{2}(2n^4 - 16) = n^4 - 8 \quad \text{or} \quad D_t'^2 = n^4,$$

according if $t \in A[2]$ is in C_0 or not. The configuration $\mathcal{D}$ contains

$$\frac{1}{2}(16n^4 - 16) = 8(n^4 - 1)$$

6-points and no other singularities. If n is even, then the configuration has 10 curves with self-intersection n^4 and 6 curves with self-intersection $n^4 - 8$. Thus if n is even one has

$$H(\mathcal{D}) = \frac{10n^4 + 6(n^4 - 8) - 8(n^4 - 1)6}{8(n^4 - 1)} = -4\frac{n^4}{n^4 - 1} \sim -4,$$

which for $n = 2$ gives $H = -64/15 \simeq -4.2\bar{6}$.

If n is odd, one has 16 curves with self-intersection $n^4 - 3$, and we get the same formula:

$$H(\mathcal{D}) = \frac{\sum D_t'^2 - 8(n^4 - 1)6}{8(n^4 - 1)} = \frac{16(n^4 - 3) - 8(n^4 - 1)6}{8(n^4 - 1)} = -4\frac{n^4}{n^4 - 1}.$$

□

Remark 12 (a) The H-constants of the various configurations are < -4.
(b) For $n = 1$, the H-constant is -2.

3.2 The genus 4 case

Let us consider the configuration (16_{10}) considered in Sect. 2.4 of 16 genus 4 curves C_t, $t \in A[2]$ in a generic (1, 3)-polarized abelian surface A. Let $X = \mathrm{Km}(A)$ be the Kummer surface associated to A. Let $\mu : \tilde{A} \to A$ the blow-up at the points in $A[2]$, and $\pi : \tilde{A} \to X$ be the quotient map. Let us consider as in Remark 10 the 16 curves $D_t = [n]^* C_t$, $t \in A[2]$ in A. Let be $\bar{D}_t$ the strict transform in $\tilde{A}$ of D_t and $D_t' = \pi(\bar{D}_t)$.

Proposition 13 *For $n > 1$, the configuration $\mathcal{C} = \sum_{t \in A[2]} D_t'$ in the Kummer surface X has Harbourne constant*

$$H(\mathcal{C}) = -4\frac{n^4}{n^4 - 1}.$$

Proof The involution $[-1] : A \to A$ fixes the set $A[2]$ and stabilizes the configuration $\mathcal{C} = \sum_{t \in A[2]} C_t$, since a curve C_t in $\mathcal{C}$ is determined by the 2-torsion points it contains, $[-1]$ stabilizes each curve C_t, $t \in A[2]$, and thus also one has $[-1]^* D_t = D_t$. Thus the restriction $\bar{D}_t \to D_t'$ of π has degree 2. Numerically, one has $D_t = n^2 C_t$ and $C_t^2 = 6$.
Since D_t as a set is $\{x \in A \mid nx + t \in C_0\}$, a point $t' \in A[2]$ is in D_t if and only if $nt' + t \in C_0$. Thus if n is even, the curve D_t contains 16 or 0 points of 2 torsion depending if $t \in C_0$ or not (thus there are 6 curves without points of 2 torsion, and 10 with), moreover one has:

$$D_t'^2 = \frac{1}{2}(\bar{D}_t)^2 = \frac{1}{2}(6n^4 - 16) = 3n^4 - 8 \quad \text{or} \quad D_t'^2 = 3n^4,$$

according if $t \in A[2]$ is in C_0 or not. If n is odd, each curve D_t contains 10 points of 2-torsion and

$$D_t'^2 = \frac{1}{2}(6n^4 - 10) = 3n^4 - 5.$$

The configuration $\mathcal{C} = \sum D_t'$ has $\frac{1}{2}(16n^4 - 16) = 8(n^4 - 1)$ 10-points and no other singularities. If n is even, then the configuration contains 6 curves with self-intersection $3n^4$ and 10 curves with self-intersection $3n^4 - 8$, thus

$$H(\mathcal{C}) = \frac{6 \cdot 3n^4 + 10(3 \cdot n^4 - 8) - 8(n^4 - 1)10}{8(n^4 - 1)} = -4\frac{n^4}{n^4 - 1} \sim -4.$$

If n is odd, one has 16 curves with self-intersection $3n^4 - 5$, and one gets the same formula. □

Remark 14 The multiplication by n map $[n]$ on A induces a rational map $[\mathfrak{n}] : X \dashrightarrow X$. The configurations $\sum D_t$ we are describing are the pull back by $[\mathfrak{n}]$ of a configuration in $X = \mathrm{Km}(A)$ of 16 disjoint rational curves.

4 Irreducible curves with low H-constant in abelian and Kummer surfaces

Obtaining irreducible curves with low Harbourne constant is in general a difficult problem. Let $k > 0$ be an integer. In [16], we prove that in a generic abelian surface polarized by M with $M^2 = k(k+1)$ there exists a hyperelliptic curve Γ_k numerically equivalent to $4M$ such that Γ_k has a unique singularity of multiplicity $4k + 2$. Thus:

Proposition 15 *The H-constant of Γ_k is*

$$H(\Gamma_k) = \Gamma_k^2 - (4k + 2)^2 = -4.$$

Let us study the case $k = 1$ and define $T_1 = \Gamma_1$. This is a curve of geometric genus 2 in an abelian surface A with one 6-point singularity, which we can suppose in 0. In [13] such a curve T_1 is constructed: A is the Jacobian of a genus 2 curve C_0 and T_1 is the image by the multiplication by 2 map of C_0 in $A = J(C_0)$. The self-intersection of T_1 is $T_1^2 = 32$ and the singularity of T_1 is a 6-point. Let $n \in \mathbb{N}$ be odd. The curve $T_n = [n]^* T_1 \sim n^2 T_1$ has 6-points singularities at each points of $A[n]$, the set of n-torsion points of A. Let $[n]$ be the multiplication by n map. The following diagram of curve configurations in A is commutative

$$\begin{array}{ccc} \sum_{t \in A[2]} D_t & \overset{[2]}{\to} & T_n \\ \downarrow [n] & & \downarrow [n] \\ \sum_{t \in A[2]} C_t & \overset{[2]}{\to} & T_1 \end{array}$$

Since the multiplication by 2 map [2] has degree 16 and the curves D_t are permuted by translations by elements of $A[2]$, the map $D_t \overset{[2]}{\to} T_n$ is birational, thus T_n is irreducible. Its singularities are 6-points over each n-torsion points.

Theorem 16 *Let $n \in \mathbb{N}$ be odd. The Harbourne constant of the irreducible curve $T_n \hookrightarrow A$ is:*

$$H(T_n) = \frac{32n^4 - 36n^4}{n^4} = -4.$$

Let $\bar{T_n}$ be the strict transform of T_n under the blowing-up map $\tilde{A} \to A$ at the points from $A[2]$. Since n is odd, among points in $A[2]$, T_n contains only 0, thus $\bar{T_n}^2 = 32n^4 - 36$. Moreover $[-1]^*T_n = T_n$ (it can be seen using the map $C^{(2)} \to A$ that $[-1]^*T_1 = T_1$, and therefore $[-1]^*T_n = T_n$). The image of $\bar{T_n}$ on the Kummer surface $X = \tilde{A}/[-1]$ is an irreducible curve W_n with $\frac{1}{2}(n^4 - 1)$ 6-points if n is odd. The map

$$\bar{T_n} \to W_n$$

has degree 2 and

$$W_n^2 = 16n^4 - 18,$$

thus

Proposition 17 *Let $n \in \mathbb{N}$ be odd. The H-constant of the irreducible curve W_n in the Kummer surface X is*

$$H(W_n) = \frac{-4n^4}{n^4 - 1}.$$

In particular, for $n = 3$ one has $H(W_3) = -\frac{81}{20}$.

5 Some remarks on H-constants of abelian surfaces

Let $\phi : C \hookrightarrow A$ be an irreducible curve of geometric genus g in an abelian surface A. Let $m_p = m_p(C)$ be the multiplicity of C at a point p. One has

$$C^2 = 2g - 2 + 2\gamma,$$

where

$$\gamma \geq \sum_p \frac{1}{2} m_p(m_p - 1),$$

with equality if all singularities are ordinary. Thus

$$H(C, \mathrm{Sing}(C)) = \frac{1}{\#\mathrm{Sing}(C)}\left(C^2 - \sum_{p \in \mathrm{Sing}(C)} m_p^2\right) \geq \frac{1}{\#\mathrm{Sing}(C)}\left(2g - 2 - \sum_{p \in \mathrm{Sing}(C)} m_p\right)$$

with equality if all singularities are ordinary. From the previous construction in Sect. 2, one can ask the following

Problem 18 Does there exists an abelian surface containing a curve C of geometric genus g such that

$$-4 > \frac{1}{\#\mathrm{Sing}(C)}\left(2g - 2 - \sum_{p \in \mathrm{Sing}(C)} m_p\right) ?$$

We were not able to find any example of such a curve. If the answer is no, it would imply that the bounded negativity conjecture holds true for surfaces which are blow-ups of abelian surfaces.

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