NEGATIVE CURVES ON ALGEBRAIC SURFACES

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Abstract

We study curves of negative self-intersection on algebraic surfaces. In contrast to what occurs in positive characteristics, it turns out that any smooth complex projective surface X with a surjective nonisomorphic endomorphism has bounded negativity (i.e., that C^2 is bounded below for prime divisors C on X). We prove the same statement for Shimura curves on quaternionic Shimura surfaces of Hilbert modular type. As a byproduct, we obtain that there exist only finitely many smooth Shimura curves on such a surface. We also show that any set of curves of bounded genus on a smooth complex projective surface must have bounded negativity.

1. Introduction

In recent years there has been a lot of progress in understanding various notions and concepts of positivity (see [18]). In the present article, we go in the opposite direction and study negative curves on complex algebraic surfaces. By a *negative curve* we will always mean a reduced, irreducible curve with negative self-intersection.

The results we present here were motivated by the study of an old folklore conjecture, sometimes referred to as the *Bounded Negativity conjecture*, which we state as follows.

CONJECTURE 1.1 (Bounded Negativity conjecture)
For each smooth complex projective surface X there exists a number $b(X) \geq 0$ such that $C^2 \geq -b(X)$ for every negative curve $C \subset X$.

DUKE MATHEMATICAL JOURNAL
Vol. 162, No. 10, © 2013 DOI 10.1215/00127094-2335368
Received 16 March 2012. Revision received 5 October 2012.
2010 *Mathematics Subject Classification*. Primary 14C17; Secondary 14G35.
Harbourne's work partially supported by National Security Agency grant H98230-11-1-0139.
Küronya's work partially supported by the Deutsche Forschungsgemeinschaft-Forschergruppe 790 "Classification of Algebraic Surfaces and Compact Complex Manifolds" and by Hungarian Academy of Sciences OTKA grants 77476, 77604, and 81203.
Roulleau's work as a member of the project Geometria Algebrica PTDC/MAT/099275/2008 partially supported by Fundação para a Ciência e a Tecnologia (Portugal) grant SFRH/BPD/72719/2010.
Tomasz Szemberg was partially supported by Narodowe Centrum Nauki (Poland) grant UMO-2011/01/B/ST1/04875.

The origins of this conjecture are unclear, but it has a long oral tradition. (Michael Artin mentioned it to the second author no later than about 1980, and we recently learned that Federigo Enriques had mentioned the conjecture to his last student, Alfredo Franchetta, who in turn mentioned it to his student, Ciro Ciliberto. Ciliberto also recalls Franchetta discussing the problem with Enrico Bombieri during a trip to Naples many years ago. For recent references to the conjecture, see [11], [14, Conjecture 1.2.1], and [15, Question, p. 24].) While the occurrence of smooth complex surfaces having curves of arbitrarily negative self-intersection still remains mysterious, we present here related results that arose from our attempts to decide the validity of the conjecture.

It has been known for a long time that there are algebraic surfaces with infinitely many negative curves, the simplest examples being the projective plane blown up in the base locus of a general elliptic pencil or certain elliptic K3 surfaces. In the first example all negative curves have self-intersection -1, and in the second example the self-intersection is -2, but in both cases all the negative curves are rational. In characteristic $p > 0$, surfaces with negative curves of arbitrarily negative self-intersection have also been known for some time (see [16, Exercise V.1.10]), but the curves in these examples all have the same genus and result from surjective endomorphisms (coming from powers of the Frobenius) of the surface containing them.

At this point, the following questions appear to be quite natural.

(1) Can one construct examples over the complex numbers of surfaces with surjective endomorphisms that result in negative curves of arbitrarily negative self-intersection?

(2) More generally, what happens if we replace endomorphisms by correspondences?

(3) Is it possible to have a surface X with infinitely many negative curves C of bounded genus such that C^2 is not bounded from below?

(4) For which $d < 0$ (or $g \geq 0$) is it possible to produce examples of surfaces X with infinitely many negative curves C such that $C^2 = d$ (or such that C has genus g)?

(5) If there is a lower bound for the self-intersections of negative curves on a given surface X, is there also a lower bound for the self-intersections of reduced but not necessarily irreducible curves C on X? If so, how are the bounds related?

In Section 2, we answer the first and third questions. We show that over the complex numbers a surface with a noninvertible surjective endomorphism must have bounded negativity. In addition, we point out that bounding the genus of a set of curves on a given complex surface of nonnegative Kodaira dimension immediately leads to a lower bound on their self-intersections. (The latter result was first proved in [6].)

In Section 3, we study the second question in the arithmetic setting. Recall that a Hilbert modular surface is a quotient of the second Cartesian power of the upper half-plane $\mathbb{H}$ by a Hilbert modular group. Such surfaces, as is well known, carry a large infinite algebra of Hecke correspondences. In this article, we look at certain variants of Hilbert modular surfaces which arise from quaternionic algebras (see Section 3.1 for details). We call these surfaces *quaternionic Shimura surfaces of Hilbert modular type*. We prove that the negativity of Shimura curves on quaternionic Shimura surfaces of Hilbert modular type is bounded and that there exist only finitely many negative Shimura curves. This implies immediately a result that seems to have escaped general attention so far, namely, that there are only finitely many smooth Shimura curves on any such Shimura surface (see Theorem 3.5).

In Section 4, we address the fourth question above; specifically, we verify that for each integer $m > 0$ there is a smooth projective complex surface containing infinitely many smooth irreducible curves of self-intersection $-m$ (see Theorem 4.1, whose genus can be prescribed when $m \geq 2$; see also Theorem 4.3).

Finally, in Section 5, we address the fifth question by giving a sharp lower bound on the self-intersections of reduced curves for surfaces for which the self-intersections of negative curves are bounded below.

2. Bounded negativity

In positive characteristic there exist surfaces carrying a sequence of irreducible curves with self-intersection tending to negative infinity (see [16, Exercise V.1.10]). These curves are constructed by taking iterative images of a negative curve under a surjective endomorphism of the surface.

To give it more detail, the construction goes as follows. Let C be a curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic p, and let $X = C \times C$ be the product surface with $\Delta \subset X$ the diagonal. Furthermore, let $F : C \to C$ be the Frobenius homomorphism defined by taking coordinates of a point on C to their pth powers. Then $G = \mathrm{id} \times F$ is a surjective endomorphism of X. The self-intersections in the sequence of irreducible curves $\Delta, G(\Delta), G^2(\Delta), \ldots$ tend to negative infinity.

We now show that in characteristic zero it is not possible to construct a sequence of curves with unbounded negativity using endomorphisms as above. In fact, we prove an even stronger statement: the existence of a nontrivial surjective endomorphism implies a bound on the negativity of self-intersections of curves on the surface.

PROPOSITION 2.1
Let X be a smooth projective complex surface admitting a surjective endomorphism that is not an isomorphism. Then X has bounded negativity; that is, there is a bound

$b(X)$ such that

$$C^2 \geq -b(X)$$

for every reduced irreducible curve $C \subset X$.

Proof

It is a result of Fujimoto and Nakayama (see [12] and [24]) that a surface X satisfying our hypothesis is of one of the following types:

(1) X is a toric surface;

(2) X is a $\mathbb{P}^1$-bundle;

(3) X is an abelian surface or a hyperelliptic surface;

(4) X is an elliptic surface with Kodaira dimension $\kappa(X) = 1$ and topological Euler number $e(X) = 0$.

In cases (1) and (2), the assertion is clear as the effective cone of X is finitely generated. This is trivial for a $\mathbb{P}^1$-bundle as the Picard number is 2 and was proved by Cox [9] for toric varieties. Consequently X carries only finitely many negative curves. In case (3), bounded negativity follows from the adjunction formula (see [6, Proposition 3.3.2]).

Case (4) requires a little bit more care. First, we establish the following general fact.

CLAIM

Let $f : X \to Y$ be a finite morphism between two smooth surfaces X and Y. If Conjecture 1.1 holds on X, then it holds on Y.

Indeed, let C be an arbitrary curve on Y. Then

$$C^2 = \frac{1}{\deg(f)}(f^*C)^2 \geq \frac{-b(X)}{\deg(f)},$$

so the claim holds with $b(Y) = b(X)/\deg(f)$.

Next we observe that *if $X = F \times G$ is a product of two smooth curves with F an elliptic curve, then X contains no negative curves at all.* To this end, note that F operates on X in the obvious manner (i.e., by translation on the first factor and identity on the second factor). If C is a curve in X invariant under this operation, then it is a fiber of the second projection; otherwise it sweeps out the whole surface X and cannot be negative again.

We conclude showing that all surfaces in case (4) are dominated by a product (the elliptic fibration must be isotrivial in that case) and the assertion follows then from the claim above. More specifically, let $\pi : X \to B$ be an elliptic fibration, where

B is a smooth curve, and let F be the class of a fiber of π. By the properties of $e(X)$ of a fibered surface (see [2, Chapter III, Proposition 11.4 and Remark 11.5]), the only singular fibers of X are possible multiple fibers, and the reduced fibers are always smooth elliptic curves. In particular, X must be minimal, and its fibers do not contain negative curves. After a finite base change, we can resolve all multiple fibers and obtain an elliptic fibration with smooth fibers only (see [8, Lemma VI.7]). Taking another finite cover if necessary, we obtain a fibration with smooth fibers and level n structure ($n \geq 3$). Such a fibration is trivial since the moduli space of elliptic curves with level n structure is affine (see [8, Proposition VI.8]). $\square$

We now consider question 3 of the introduction. The first general result known to us that answers this question is due to Bogomolov. It says that, on a surface X of general type with $c_1^2(X) > c_2(X)$, curves of a fixed geometric genus lie in a bounded family. This implies, of course, that their numeric invariants, in particular their self-intersections, are bounded. An effective version of Bogomolov's result was obtained by Lu and Miyaoka in [19, Theorem 1(1)]. Their proof relies on Corollary 2.3. We state here a more general result due to Miyaoka [21, Theorem 1.3(i)–(ii)], as we need it anyway in the next section. In fact, Theorem 3.5 is a nice application of this most general known version of Miyaoka's theorem. We do not know of a proof of Theorem 3.5 in which Corollary 2.3 would suffice.

THEOREM 2.2
Let X be a surface of nonnegative Kodaira dimension, and let C be an irreducible curve of geometric genus g on X. Then

$$\frac{\alpha^2}{2}(C^2 + 3CK_X - 6g + 6) - 2\alpha(CK_X - 3g + 3) + 3c_2 - K_X^2 \geq 0 \qquad (1)$$

for all $\alpha \in [0, 1]$.

Moreover, if $C \not\cong \mathbb{P}^1$ and $K_X C > 3g - 3$, then

$$2(K_X C - 3g + 3)^2 - (3c_2 - K_X^2)(C^2 + 3CK_X - 6g + 6) \leq 0. \qquad (2)$$

Putting $\alpha = 1$ in (1), we recover the classical logarithmic Miyaoka–Yau inequality (see also [6, Appendix] for a complete direct proof).

COROLLARY 2.3 (Logarithmic Miyaoka–Yau inequality)
Let X be a smooth projective surface of nonnegative Kodaira dimension, and let C be a smooth curve on X. Then

$$c_1^2\big(\Omega_X^1(\log C)\big) \leq 3c_2\big(\Omega_X^1(\log C)\big),$$

equivalently $(K_X + C)^2 \leq 3(c_2(X) - 2 + 2g(C))$.

We recall here a statement that is numerically slightly weaker than the result of Lu and Miyaoka [19, Theorem 1(1)] but which has a simpler proof. This result appeared first in [6, Proposition 3.5.3], and we refer to that article for a more detailed exposition.

THEOREM 2.4 ([6, Proposition 3.5.3])
Let X be a smooth projective surface with $\kappa(X) \geq 0$. Then for every reduced, irreducible curve $C \subset X$ of geometric genus $g(C)$, we have

$$C^2 \geq c_1^2(X) - 3c_2(X) + 2 - 2g(C). \tag{3}$$

The proof is a combination of Corollary 2.3 and the following simple lemma on the behavior of (3) under blowups.

LEMMA 2.5
Let X be a smooth projective surface, let $C \subset X$ be a reduced, irreducible curve of geometric genus $g(C)$, and let $P \in C$ be a point with $m := \operatorname{mult}_P C \geq 2$. Let $\sigma : \widetilde{X} \to X$ be the blowup of X at P with the exceptional divisor E. Let $\widetilde{C} = \sigma^*(C) - mE$ be the proper transform of C. Then the inequality

$$\widetilde{C}^2 \geq c_1^2(\widetilde{X}) - 3c_2(\widetilde{X}) + 2 - 2g(\widetilde{C})$$

implies that

$$C^2 \geq c_1^2(X) - 3c_2(X) + 2 - 2g(C).$$

Proof
This follows by direct computation using the facts that $C^2 = \widetilde{C}^2 + m^2$, $c_1^2(X) = c_1^2(\widetilde{X}) + 1$, $c_2(X) = c_2(\widetilde{X}) - 1$, and $g(C) = g(\widetilde{C})$. □

Proof of Theorem 2.4
Taking an embedded resolution $f : \widetilde{X} \to X$ of C and applying Lemma 2.5 to every step, we reduce to proving the assertion for C smooth.

The latter case easily follows from Corollary 2.3. Indeed, our assumption $\kappa(X) \geq 0$ implies that $K_{\widetilde{X}} + \widetilde{C}$ is $\mathbb{Q}$-effective. Hence we have

$$\begin{aligned} c_1^2(X) + 2C \cdot (K_X + C) - C^2 &= c_1^2\big(\Omega_X^1(\log C)\big) \\ &\leq 3c_2\big(\Omega_X^1(\log C)\big) \\ &= 3c_2(X) - 6 + 6g(C). \end{aligned}$$

Rearranging terms and using the adjunction formula, we arrive at (3). □

A closer analysis of Corollary 2.3 allows one to ease the assumption of X being of nonnegative Kodaira dimension by the assumption of X being of nonnegative logarithmic Kodaira dimension (see [20, Corollary 1.2]).

ASIDE 2.6 (Strong Logarithmic Miyaoka–Yau inequality)
Let X be a smooth projective surface, and let C be a smooth curve on X such that the adjoint line bundle $K_X + C$ is $\mathbb{Q}$-effective (i.e., there is an integer $m > 0$ such that $h^0(m(K_X + C)) > 0$). Then

$$c_1^2\big(\Omega_X^1(\log C)\big) \le 3c_2\big(\Omega_X^1(\log C)\big),$$

and equivalently

$$(K_X + C)^2 \le 3\big(c_2(X) - 2 + 2g(C)\big).$$

So one gets the same bound (3) as in Theorem 2.4, for all curves C such that $K_X + C$ is $\mathbb{Q}$-effective.

3. Negativity of Shimura curves on quaternionic Shimura surfaces of Hilbert modular type

3.1. Smoothness of Shimura curves and Hecke translates

A Hilbert modular surface is a quotient of a product of two copies of the upper half-plane $\mathbb{H}$ by a lattice group. Here we are interested in quotients of $\mathbb{H} \times \mathbb{H}$ by a cocompact arithmetic subgroup of $\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R})$. Such quotients are compact surfaces (there are no cusps) and they are called *Shimura surfaces of Hilbert modular type*.

In this section, we give a criterion for a Shimura curve on a Shimura surface of Hilbert modular type to be smooth. We indicate also why its Hecke translates might fail to remain smooth. In the next section, we will show that the worst scenario actually happens.

We recall first how quaternionic Shimura surfaces of Hilbert modular type are defined. (For a complete reference on their construction, see [10]; for particularly interesting examples, see [13] and [25].) Let A be a ramified quaternion algebra over a totally real number field k. Let $\mathcal{O}_A$ be a maximal order of A, and let

$$\Gamma(1) = \big\{\gamma \in \mathcal{O}_A : \mathrm{nr}(\gamma) = 1\big\},$$

where nr denotes the reduced norm. Suppose that A splits over exactly two places in $\mathbb{R}$—that is, there exist two embeddings $\sigma_i : k \to \mathbb{R}$ such that the tensor products

$A \otimes_{k^{\sigma_i}} \mathbb{R}$ over these places are isomorphic to $M_2(\mathbb{R})$, while for all the other embeddings the tensor product is isomorphic to the Hamiltonian quaternions.

We fix such isomorphisms, giving rise to a representation,

$$\begin{aligned} \rho : A &\to M_2(\mathbb{R}) \times M_2(\mathbb{R}) \\ \gamma &\to \qquad (\gamma_1, \gamma_2). \end{aligned}$$

The morphism ρ maps $A^\times$ into $\mathrm{GL}_2(\mathbb{R})^2$. Let A^+ be the subgroup of elements γ of A such that $\det(\gamma_i) > 0$ for $i = 1, 2$. The group A^+ acts on $\mathbb{H} \times \mathbb{H}$ by

$$\gamma \cdot (z_1, z_2) = (\gamma_1 \cdot z_1, \gamma_2 \cdot z_2),$$

where, for $\gamma_i = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we have

$$\gamma_i \cdot z = \frac{az + b}{cz + d}.$$

Let us denote by Γ a subgroup of A^+ commensurable to $\Gamma(1)$ (i.e., $\Gamma(1) \cap \Gamma$ has finite index in both $\Gamma(1)$ and Γ). With these hypotheses, the quotient $X = \mathbb{H} \times \mathbb{H}/\Gamma$ is a compact algebraic surface (see [10]).

Let us suppose, in addition, that Γ is torsion-free, or equivalently, that X is smooth. The surface X is then minimal of general type with $c_1^2 = 2c_2, q = 0$. We denote by $\pi : \mathbb{H} \times \mathbb{H} \to X$ the quotient map.

A Shimura curve is, in particular, a totally geodesic curve in X. Let C_1' be such a Shimura curve on X, and let

$$\mathbb{H}_1 \subset \pi^{-1} C_1' \subset \mathbb{H} \times \mathbb{H}$$

be a subspace isomorphic to $\mathbb{H}$ so that

$$\Lambda_1 = \{\gamma \in \Gamma : \gamma \mathbb{H}_1 = \mathbb{H}_1\}$$

is a lattice in $\mathrm{Aut}(\mathbb{H}_1)$. Then $C_1 = \mathbb{H}_1/\Lambda_1$ is a smooth compact curve whose image under the generically one-to-one map $C_1 \to X$ we call C_1'.

PROPOSITION 3.1
The Shimura curve C_1' is smooth if and only if $\mathbb{H}_1 \cap \gamma\mathbb{H}_1 = \emptyset$ for all $\gamma \in \Gamma \setminus \Lambda_1$.

Proof
The map $C_1 \to X$ is an immersion because the map $\mathbb{H}_1 \to X$ is so. Thus singularities on C_1' can occur if and only if there are two distinct points $\Lambda_1 t, \Lambda_1 u$ on C_1 (with $u, t \in \mathbb{H}_1$) mapped onto the same point by the generically one-to-one map $C_1 \to C_1'$. For such points, we have $\Gamma t = \Gamma u$ (i.e., there exist $\gamma \in \Gamma$ such that $t = \gamma u$). As

$\Lambda_1 t \neq \Lambda_1 u$, we have $\gamma \in \Gamma - \Lambda_1$, and the intersection of the upper half-planes $\mathbb{H}_1$ and $\gamma\mathbb{H}_1$ is not empty. Conversely, if the intersection of the upper half-planes $\mathbb{H}_1$ and $\gamma\mathbb{H}_1$ is not empty, there are two distinct points on C_1 that have the same image on C_1', and thus there is a singularity on C_1'. □

For $h \in A^+$, set $\mathbb{H}_h := h(\mathbb{H}_1)$ and let

$$\Lambda_h = \{\lambda \in \Gamma : \lambda\mathbb{H}_h = \mathbb{H}_h\}.$$

The group Λ_h is equal to the lattice $h\Lambda_1 h^{-1} \cap \Gamma$. Let $C_h = \mathbb{H}_h/\Lambda_h$, and let C_h' be the image of C_h in X under the natural map. Again, C_h' is a Shimura curve.

PROPOSITION 3.2
Suppose that the curve C_1' is smooth. Then the Shimura curve C_h' is smooth if and only if $\mathbb{H}_1 \cap \gamma\mathbb{H}_1 = \emptyset$ for all $\gamma \in h^{-1}\Gamma h \setminus \Gamma$.

Proof
We apply Proposition 3.1 to C_h'. The curve C_h' is smooth if and only if $\mathbb{H}_h \cap \gamma\mathbb{H}_h = \emptyset$ for all $\gamma \in \Gamma \setminus \Lambda_h$. Suppose that the curve C_h' is singular. Then there exist $z_1, z_2 \in \mathbb{H}_1$ (whence $hz_1, hz_2 \in \mathbb{H}_h$) and $\gamma \in \Gamma \setminus \Lambda_h$ such that $hz_1 = \gamma(hz_2)$. Then $z_1 = h^{-1}\gamma h z_2$. As C_1' is smooth, we have two possibilities: either $h^{-1}\gamma h \in \Lambda_1$ and $h^{-1}\gamma h \notin \Gamma$, or for $\gamma' = h^{-1}\gamma h \in h^{-1}\Gamma h \setminus \Gamma$ we have $\mathbb{H}_1 \cap \gamma\mathbb{H}_1 \neq \emptyset$. The first possibility is impossible because $\Lambda_h = h\Lambda_1 h^{-1} \cap \Gamma$ and we assumed that $\gamma \in \Gamma \setminus \Lambda_h$. Therefore, the second possibility holds. For the converse statement, we remark that all the above arguments are in fact equivalences. □

As we will remark below, each element h of A^+ defines a Hecke correspondence T_h, and the curve C_h' is an irreducible component of the image of C_1' by T_h. We have $T_h = T_{h'}$ if and only if $\Gamma h = \Gamma h'$. When varying Γh in $\Gamma \setminus A^+$, we see by the above Proposition 3.2 that in order to keep C_h' smooth, the half-plane $\mathbb{H}_1$ must avoid more and more half-planes $\gamma\mathbb{H}_1$. Our next result (Theorem 3.5) shows that this is only possible in finitely many cases.

Let us now explain how Hecke correspondences come into the game here. For $h \in A^+$, let

$$\Gamma_h = \Gamma \cap h^{-1}\Gamma h,$$

which is a subgroup of finite index m in Γ. Let $t_1 = 1, t_2, \ldots, t_m$ be a full set of coset representatives of G with respect to Γ_h. Denote by X_h the Shimura surface $X_h = \mathbb{H} \times \mathbb{H}/\Gamma_h$.

There are two étale maps of degree m,

$$\begin{tikzcd} X_h \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X \\ X & \end{tikzcd}$$

where $\pi_1(\Gamma_h.z) = \Gamma.z$ and $\pi_2(\Gamma_h.z) = \Gamma h.z$. We need to check that π_2 is well defined. Let $\tau := h^{-1}\gamma h \in \Gamma_h$, with $\gamma \in \Gamma$ and $z' := \tau z$. Then

$$\Gamma h.z' = \Gamma h\tau.z = \Gamma hh^{-1}\gamma h.z = \Gamma\gamma h.z = \Gamma h.z,$$

and therefore the map π_2 does not depend on the choice of a representative in $\Gamma_h.z$. The Hecke operator T_h is defined by $T_h = \pi_{2*}\pi_1^*$. We have $\pi_1^{-1}\Gamma z = \Gamma_h t_1.z + \cdots + \Gamma_h t_m.z$ and

$$T_h(\Gamma.z) = \Gamma h t_1.z + \cdots + \Gamma h t_m.z.$$

It follows that $T_h C_1' = C_h' + Y_2 + \cdots + Y_t$ for some irreducible curves $Y_2, \ldots, Y_t$.

Remark 3.3

Let X be a smooth Picard surface (i.e., $X = \mathbb{B}_2/\Gamma$ is a quotient of the unit complex two-dimensional ball $\mathbb{B}_2$ by a cocompact torsion-free group $\Gamma \subset PU(2,1)$). It is possible to obtain the same results (smoothness criteria, smoothness of the Hecke translates) for a Shimura curve $C = \mathbb{B}_1/\Lambda$ on X. Again, the main idea is that, in order for an irreducible component of the translate $T_h C$ of a Shimura curve to be smooth, the ball $h\mathbb{B}_1$ must avoid more and more balls when h varies.

3.2. *Finiteness of smooth Shimura curves*

Let X be a quaternionic Shimura surface of Hilbert modular type. As we will see, the self-intersection of a smooth Shimura curve C on X is very negative, in particular, $C^2 = -(2g(C) - 2) < 0$. On the other hand, the set of Shimura curves on X is preserved by Hecke correspondences. It is therefore very natural to hope to obtain a counterexample to the bounded negativity conjecture by taking the images of a Shimura curve by Hecke correspondences. We will see however that there is only a finite number of Shimura curves (smooth or not) with $C^2 < 0$.

Let C be a curve on X of geometric genus g. The difference

$$\delta = \frac{1}{2}(K_X \cdot C + C^2 - 2g + 2), \tag{4}$$

where K_X is the canonical divisor of X, is a positive integer. If the curve is nodal, then this equals the number of nodes on C. We recall the following important result from [4], for which we refer to the discussion on pages 265–266 (Section B.3.D) of [4].

THEOREM 3.4 (Hirzebruch–Höfer proportionality theorem)
For a Shimura curve C on a quaternionic Shimura surface of Hilbert modular type X, we have

$$K_X C = 4(g-1) \qquad \textit{and} \qquad K_X C + 2C^2 = 4\delta.$$

Although we will not use this fact, it is interesting to notice that the curve C in Theorem 3.4 is nodal.

The main result of this section is the following.

THEOREM 3.5
For a Shimura curve C on X, we have the following inequalities:

$$g \leq 1 + c_2 + \sqrt{c_2^2 + c_2\delta} \qquad \textit{and} \qquad C^2 \geq -6c_2.$$

In particular, if C is smooth, then $g \leq 1 + 2c_2$.

Moreover, there is only a finite number of Shimura curves with $C^2 < 0$, since for $\delta \geq 3c_2$, the curve C satisfies $C^2 \geq 0$.

Proof
The idea is to show that for $\delta \geq 3c_2$, the Shimura curve C satisfies $C^2 \geq 0$. Computing g from (4) and inserting it into (1), we obtain

$$P(\alpha) = \alpha^2(3\delta - C^2) + \alpha(CK_X + 3C^2 - 6\delta) + 3c_2 - K_X^2 \geq 0$$

for $0 \leq \alpha \leq 1$. Using the second equality in Theorem 3.4, and since $K_X^2 = 2c_2$ for compact Hilbert modular surfaces, we get

$$P(\alpha) = \alpha^2(3\delta - C^2) + \alpha(C^2 - 2\delta) + c_2 \geq 0.$$

If $C^2 \geq 2\delta$, then obviously $C^2 \geq 0$. If $C^2 < 2\delta$, then the minimum of $P(\alpha)$ is attained for

$$\alpha_0 = \frac{2\delta - C^2}{2(3\delta - C^2)}.$$

Note that $0 < \alpha_0 < 1$. Evaluating the condition $P(\alpha_0) \geq 0$, we obtain

$$2c_2 + 2\sqrt{c_2^2 + \delta c_2} \geq 2\delta - C^2 \geq 2c_2 - 2\sqrt{c_2^2 + \delta c_2}. \tag{5}$$

For $C^2 < 2\delta$, we get the lower bound

$$C^2 \geq 2\delta - 2c_2 - 2\sqrt{c_2^2 + \delta c_2}.$$

Hence, if $\delta \geq 3c_2$, we indeed have $C^2 \geq 0$.

Suppose now that $C^2 < 0$. Then $\delta < 3c_2$, and therefore $-2\sqrt{c_2^2 + \delta c_2} > -4c_2$. We get from (5) that

$$C^2 \geq 2\delta - 2c_2 - 2\sqrt{c_2^2 + \delta c_2} \geq 2(\delta - 3c_2)$$

and, consequently, $C^2 \geq -6c_2$.

Miyaoka's formula (2) with $C^2 + K_X C = 2g - 2 + 2\delta$ implies that

$$(K_X C - 3g + 3)^2 - c_2(K_X C + \delta - 2g + 2) \leq 0.$$

As $K_X C = 4g - 4$, we get

$$(g-1)^2 - 2c_2(g-1) - c_2\delta \leq 0$$

and therefore

$$g - 1 \leq c_2 + \sqrt{c_2^2 + c_2\delta}.$$

Now for $C^2 < 0$, we know that $\delta < 3c_2$, and thus we have $g \leq 3c_2 + 1$. Since $K_X C = 4g - 4$, the intersection number $K_X C$ is bounded from above. An infinite number of Shimura curves with bounded geometric genus g and bounded intersection with K_X must be in a finite number of families of curves, and thus these Shimura curves must deform and satisfy $C^2 \geq 0$, and therefore the number of Shimura curves with $C^2 < 0$ must be finite. □

COROLLARY 3.6

There are only finitely many smooth Shimura curves on a quaternionic Shimura surface of Hilbert modular type.

Proof

This follows immediately from Theorem 3.5, as smooth Shimura curves have a negative self-intersection by the second equality in Theorem 3.4. Indeed, the canonical divisor is in this situation ample by [23, Proposition 3.4(b)] (see also [1]). □

Remark 3.7

It is easy to see that the compactness of X was not used in the course of the proof of Theorem 3.5. The same statement holds therefore for Hilbert modular surfaces with cusps. We refrain from stating the precise formula to avoid becoming repetitive. The main point, however, is the *proportionality theorem* (Theorem 3.4). In fact, this theorem holds also for modular curves (i.e., those passing through the cusps of a Hilbert modular surface; see [22, Theorems 0.1 and 0.2]); hence the statement of Theorem 3.5 remains valid for such curves.

Since the numerics are different for ball quotients, such as $K_X \cdot C = 3(g-1)$ in that case, our method using Miyaoka's theorem does not give any numerical bounds. We do not know whether there are also only finitely many smooth Shimura curves, but we suspect that is the case.

4. Surfaces with infinitely many negative curves of fixed self-intersection

The well-known example of $\mathbb{P}^2$ blown-up at nine points shows that there are surfaces containing infinitely many (-1)-curves. Along similar lines, we point out here that one can exhibit surfaces with infinitely many negative curves of any given (fixed) negative self-intersection.

THEOREM 4.1
For every integer $m > 0$ there are smooth projective complex surfaces containing infinitely many smooth irreducible curves of self-intersection $-m$.

Proof
Let E be an elliptic curve without complex multiplication, and let A be the abelian surface $E \times E$. We denote by F_1 and F_2 the fibers of the projections and by Δ the diagonal in A. It is shown in [7, Proposition 2.3] that every elliptic curve on A that is not a translate of F_1, F_2, or Δ has numerical equivalence class of the form

$$E_{c,d} := c(c+d)F_1 + d(c+d)F_2 - cd\Delta,$$

where c and d are suitable coprime integers, and, conversely, that every such numerical class corresponds to an elliptic curve $E_{c,d}$ on A. In our construction we will make use of a sequence (E_n) of such curves, for instance, taking $E_n = E_{n,1}$ for $n \geq 2$. No two of the curves E_n are then translates of each other.

Fix a positive integer t such that $t^2 \geq m$. For each of the elliptic curves E_n, the number of t-division points on E_n is t^2, and these points are among the t-division points of A. (Actually, the latter is only true if E_n is a subgroup of A, but this can be achieved by using a translate of E_n passing through the origin.) Since the number of t-division points on A is finite—there are exactly t^4 of them—there must exist a subsequence of (E_n) having the property that all curves E_n in the subsequence have the same set of t-division points, say $\{e_1, \ldots, e_{t^2}\}$.

Consider now the blowup $f : X \to A$ at the set $\{e_1, \ldots, e_m\}$. The proper transform C_n of E_n is then a smooth irreducible curve on X with

$$C_n^2 = E_n^2 - m = -m,$$

as claimed. $\square$

Remark 4.2
Note that the proof yields a one-dimensional family of surfaces and that the constructed surfaces are of Picard number $m + 3$.

For each $m \geq 1$, the proof above gives a surface X with infinitely many curves of genus 1 of self-intersection $-m$. This raises the question of whether, for each $m \geq 1$ and each $g \geq 0$, there is a surface X with infinitely many curves of genus g of self-intersection $-m$. We now show that the answer is yes at least for $m > 1$. This is probably well known to specialists, but we enclose the proof for the lack of a reference.

THEOREM 4.3
For each $m > 1$ and each $g \geq 0$ there exists a smooth projective complex surface containing infinitely many smooth irreducible curves of self-intersection $-m$ and genus g.

Proof
Let $f : X \to B$ be a smooth complex projective minimal elliptic surface with section, fibered over a smooth base curve B of genus $g(B)$. Then X can have no multiple fibers, so that by Kodaira's well-known result (see [2, Chapter V, Corollary 12.3]), K_X is a sum of a specific choice of $2g(B) - 2 + \chi(\mathcal{O}_X)$ fibers of the elliptic fibration. Let C be any section of the elliptic fibration f. By adjunction, $C^2 = -\chi(\mathcal{O}_X)$.

Take X to be rational, and take f to have infinitely many sections; for example, blow up the base points of a general pencil of plane cubics. Then $\chi(\mathcal{O}_X) = 1$, so that $C^2 = -1$ for any section C.

Pick any $g \geq 0$ and any $m \geq 2$. Then, as is well known (see [17]), there is a smooth projective curve C of genus g and a finite morphism $h : C \to B$ of degree m that is not ramified over points of B over which the fibers of f are singular. Let $Y = X \times_B C$ be the fiber product. Then the projection $p : Y \to C$ makes Y into a minimal elliptic surface, and each section of f induces a section of p. By the property of the ramification of h, the surface Y is smooth and each singular fiber of f pulls back to m isomorphic singular fibers of p. Since $e(Y)$ is the sum of the Euler characteristics of the singular fibers of p (see, e.g., [2, Chapter III, Proposition 11.4], we obtain from Noether's formula that $\chi(\mathcal{O}_Y) = e(Y)/12 = me(X)/12 = m\chi(\mathcal{O}_X) = m$. Therefore, for any section D of p, we have $D^2 = -m$; that is, Y has infinitely many smooth irreducible curves of genus g and self-intersection $-m$. $\square$

Remark 4.4
Somewhat more abstractly, one can prove Theorem 4.3 using the fact that, given an elliptic rational surface with infinite Mordell–Weil group, one can perform a base

change of degree m with a curve of genus g to obtain a surface with $1-g+p_g=m$. Then there are also infinitely many elements in the Mordell–Weil group of the new surface, and they all satisfy the numerical requirements of the theorem.

Question 4.5
Is there for each $g>1$ a surface with infinitely many (-1)-curves of genus g?

5. Negativity of reducible curves

When asking for bounded negativity of curves, it is necessary to restrict attention to reduced curves. Irreducibility, however, is not an essential hypothesis, since by [6, Proposition 3.8.2], bounded negativity holds for the set of reduced, irreducible curves on a surface X if and only if it holds for the set of reduced curves on X. Here we improve this result by obtaining a sharp bound on the negativity for reducible curves, given a bound on the negativity for reduced, irreducible curves.

PROPOSITION 5.1
Let X be a smooth projective surface (over an arbitrary algebraically closed ground field) for which there is a constant $b(X)$ such that $C^2 \geq -b(X)$ for every reduced, irreducible curve $C \subset X$. Then

$$C^2 \geq -\big(\rho(X)-1\big)\cdot b(X)$$

for every reduced curve $C \subset X$, where $\rho(X)$ is the Picard number of X.

Proof
Consider the Zariski decomposition $C=P+N$ of the reduced divisor C. Then we have $C^2=P^2+N^2 \geq N^2$, as P is nef and P and N are orthogonal. So the issue is to bound N^2. The negative part N is of the form $N=a_1C_1+\cdots+a_rC_r$, where the curves C_i are among the components of C and the coefficients a_i are positive rational numbers. Note that $a_i \leq 1$ for all i, because C is reduced. Since the intersection matrix of N is negative definite, we have $r \leq \rho(X)-1$. Thus

$$C^2 \geq N^2 \geq a_1^2C_1^2+\cdots+a_r^2C_r^2 \geq -r\cdot b(X) \geq -\big(\rho(X)-1\big)\cdot b(X),$$

as claimed. □

Example 5.2
Here is an example of a surface of higher Picard number, for which equality holds in the inequality $C^2 \geq -(\rho(X)-1)\cdot b(X)$ that was established above. Consider a smooth Kummer surface $X \subset \mathbb{P}^3$ with 16 disjoint lines (or with 16 disjoint smooth rational curves of some degree) as in [3] or in [5]. The generic such surface has

$\rho(X) = 17$, we have $b(X) = -2$, and if C is the union of the 16 disjoint curves, then $C^2 = 16 \cdot (-2)$.

Example 5.3
A more elementary example is given by the blowup X of $\mathbb{P}^2$ at $n \leq 8$ general points, so $\rho(X) = n + 1$. Since $-K_X$ is ample, it follows by adjunction for any reduced, irreducible curve C that $C^2 \geq -1$, so $b(X) = 1$. But if E is the union of the exceptional curves of the n blown-up points, then $E^2 = -n = -(\rho(X) - 1) \cdot b(X)$.

Acknowledgments. We would like to thank Jan Hendrik Bruinier, Lawrence Ein, Friedrich Hirzebruch, Annette Huber-Klawitter, Jun-Muk Hwang, Nick Shepherd-Barron, and Kang Zuo for useful discussions. Special thanks go to Fritz Hörmann, who discovered that an earlier version of this manuscript relied on a published result that turns out to be false. This collaboration grew out of interactions during the Mathematisches Forschungsinstitut Oberwolfach mini-workshop "Linear Series on Algebraic Varieties" (October 2010), and we thank the institute for excellent working conditions. We also thank Sonderforschungsbereich/Transregio 45 for facilitating visits and for the 2009 summer school in Kraków, where some initial consideration of these problems occurred. Last but not least, we thank the referees for helpful remarks.

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