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On finiteness of curves with high canonical degree on a surface

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Abstract The *canonical degree* of a curve C on a surface X is $K_X \cdot C$. Our main result, Theorem 1.1, is that on a surface of general type there are only finitely many curves with negative self-intersection and sufficiently large canonical degree. Our proof strongly relies on results by Miyaoka. We extend our result both to surfaces not of general type and to non-negative curves, and give applications, e.g., to finiteness of negative curves on a general blow-up of $\mathbb{P}^2$ at $n \geq 10$ general points (a result related to *Nagata's Conjecture*). We finally discuss a conjecture by Vojta concerning the asymptotic behaviour of the ratio between the canonical degree and the geometric genus of a curve varying on a surface. The results in this paper go in the direction of understanding the *bounded negativity* problem.

Keywords Bounded negativity conjecture · Nagata's conjecture · Votja's conjecture

Mathematics Subject Classification Primary: 14C17 · Secondary: 14G35, 14J29

1 Introduction

Let C be a projective curve on a smooth projective complex surface X. By *curve* we mean an irreducible, reduced 1-dimensional scheme. We denote by $g = g(C)$ its geometric genus and by $p = p_a(C)$ its arithmetic genus, i.e., $C^2 + K \cdot C = 2p_a - 2$, where $K = K_X$ is a canonical divisor of X. We set $\delta = \delta(C) = p - g$. We call a curve *negative* if $C^2 < 0$. The *canonical degree* of C is $k_C = K \cdot C$, often simply denoted by k. If $g(C) \neq 1$, we set

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$$\beta_C = \frac{k_C}{g(C)-1},$$

often simply denoted by β. For a surface X we set

$$a_X = 3c_2(X) - K_X^2,$$

often simply denoted by a. If the Kodaira dimension $\kappa = \kappa(X)$ is non-negative, one has $a \geq 0$.

The main result of this paper concerns negative curves with high canonical degree:

Theorem 1.1 *(A) Let C be a negative curve not isomorphic to $\mathbb{P}^1$ on a surface X with $\kappa \geq 0$. Then*

$$k_C \leq 3(g-1) + \frac{3}{4}a + \frac{1}{4}\sqrt{9a^2 + 24a(g-1)}, \tag{1.1}$$

(we remark that in case $a = 0$, there are no curve with $g = 0$). Furthermore, if $g > 1$ then

$$\beta_C \leq 3 + \frac{3}{4}a + \frac{1}{4}\sqrt{9a^2 + 24a} \leq 4 + \frac{3}{2}a. \tag{1.2}$$

If, in addition, $\beta = 3 + \epsilon > 3$, then

$$g \leq 1 + \frac{3a(\epsilon+1)}{2\epsilon^2}. \tag{1.3}$$

(B) Suppose $\kappa(X) = 2$. Then for each $\epsilon > 0$ there are at most finitely many negative curves C on X such that $k_C \geq (3+\epsilon)(g-1) > 0$.

From (B) it follows that:

Corollary 1.2 *Let X be a surface of general type. There is a function $B(\epsilon)$, defined for $\epsilon \in]0, \infty[$ such that for all negative curves C we have*

$$k_C \leq (3+\epsilon)(g-1) + B(\epsilon) \quad \textit{and} \quad -C^2 \leq (1+\epsilon)(g-1) + B(\epsilon).$$

A Shimura surface X is a quotient of the bidisk by a torsion free discrete cocompact lattice. It is of general type and on some Shimura surfaces there are infinitely many totally geodesic curves (see [4]). Such a curve C is also called a *Shimura curve* and satisfies $k_C = 4(g-1)$. We thus obtain the following corollary of Theorem 1.1 (B), which was one of the main results of [2]:

Corollary 1.3 *On a Shimura surface, there exist finitely many (may be none) negative Shimura curves.*

The proof of Theorem 1.1, contained in Sect. 2, strongly relies on a result by Miyaoka's (see [11, Cor 1.4] stated as Theorem 2.1 below). In particular, the inequality (1.1) is very similar to [11, formula (3)], which has a slightly lower growth in g, but applies only to minimal surfaces.

In Sect. 3 we make an extension of Theorem 1.1 which works also in the case $\kappa = -\infty$, and we prove a finiteness result for negative curves on a general blow-up of $\mathbb{P}^2$ at $n \geq 10$ general points. This is a bounded negativity result which is reminiscent of the famous *Nagata's Conjecture*, predicting that there is no negative curve on such a surface except for (-1)-rational curves.

In Sect. 4, again using Miyaoka's result, we prove a boundedness theorem for non-negative curves of high canonical degree. In Sect. 5 we discuss a conjecture by Vojta concerning the asymptotic behaviour of $k_C/(g-1)$ when C varies among all curves on a surface. We introduce an invariant related to Vojta's conjecture and we prove a bound for it.

The results in this paper go in the direction of understanding *bounded negativity* (see [2]). The *Bounded Negativity Conjecture* (BNC) predicts that on a surface of general type over $\mathbb{C}$ (and indeed on any smooth complex projective surface) the self-intersection of negative curves is bounded below. Nagata's conjecture, which we mentioned above, is also a sort of bounded negativity assertion. As a general reference on both bounded negativity and Nagata's conjecture, see [8]. Also Vojta's conjecture is related to bounded negativity, as we discuss in Sect. 5.

As a consequence of Theorem 1.1, we have the following information on negative curves for surfaces with $\kappa \geq 0$:

Corollary 1.4 *Suppose BNC fails for X with $\kappa(X) \geq 0$, so that there exists a sequence $(C_n)_{n\in\mathbb{N}}$ of negative curves of genus g_n with* $\lim C_n^2 = -\infty$. *Then* $\lim g_n = \infty$ *and*

$$\limsup_n \frac{K \cdot C_n}{g_n - 1} \leq 3.$$

Observe that if BNC fails for a surface Y with $\kappa(Y) < 0$, by taking a suitable (e.g., double) cover one obtain a surface X on which BNC fails with $\kappa(X) \geq 0$.
In conclusion, the authors would like to thank B. Harbourne and J. Roé for useful exchanges of ideas about the application to Nagata's Conjecture in Sect. 3.

2 The proof of the main theorem

Our proof relies on the following result by Miyaoka (see [11, Cor 1.4]):

Theorem 2.1 *Let C be curve on a surface X with $\kappa \geq 0$. Then for all $\alpha \in [0, 1]$, we have:*

$$\alpha^2(C^2 + 3k_C - 6g + 6) - 4\alpha(k_C - 3g + 3) + 2a \geq 0. \tag{2.1}$$

Suppose C is not isomorphic to $\mathbb{P}^1$ and $k_C > 3(g-1)$. Then

$$2(k_C - 3g + 3)^2 - a(C^2 + 3k_C - 6g + 6) \leq 0. \tag{2.2}$$

Suppose in addition $K^2 > 0$. Then

$$\left(\frac{c_2}{K^2} - 1\right) k_C^2 + (4(g-1)+a)k_C - 2(g-1)(3(g-1)+a) \geq \left(\frac{c_2}{K^2} - \frac{1}{3}\right)\left[k_C^2 - C^2K^2\right] \geq 0. \tag{2.3}$$

Proof Inequality (2.1) is [11, Thm 1.3, (i)] and (2.2) is [11, Thm 1.3, (ii)]. As for (2.3) this is [11, Thm 1.3, (iii)], which is stated there under the assumption that X is minimal of general type and $C \not\cong \mathbb{P}^1$. However Miyaoka's argument works more generally under the weaker assumption $K^2 > 0$. □

We are now ready for the:

Proof of Theorem 1.1 Let us prove (A). Let C be a negative curve on X not isomorphic to $\mathbb{P}^1$. Then $-aC^2 \geq 0$, with equality if and only if $a = 0$. If $k_C \leq 3(g-1)$, there is nothing to prove. Let us suppose $k = k_C > 3(g-1)$ and set $\mathfrak{g} = g - 1$. By (2.2), one has

$$P(k) := 2(k - 3\mathfrak{g})^2 - a(3k - 6\mathfrak{g}) \leq 0, \tag{2.4}$$

with strict inequality if $a > 0$.

If $a = 0$ then Eq. 2.4 implies $k = 3(g-1)$, against the assumption that $k > 3(g-1)$, thus this cannot occur. In the remaining cases, k_C is less than or equal to the largest root of P, whence we get (1.1).

Suppose $g > 1$. We obtain (1.2) directly from (1.1) by dividing by $g-1$. For a curve C with $\epsilon > 0$, one has from (1.1)

$$\epsilon(g-1) \leq \frac{3}{4}a + \frac{1}{4}\sqrt{9a^2 + 24a(g-1)}. \tag{2.5}$$

This gives

$$4\epsilon(g-1) - 3a \leq \sqrt{9a^2 + 24a(g-1)},$$

and by squaring one gets (1.3), finishing the proof of (A).

Next we prove (B). Let $\beta_0 > 3$. By (1.2) and (1.3), negative curves with $\beta > \beta_0$ have bounded genus g, therefore by (1.1) also k_C is bounded, hence the arithmetic genus p is bounded.

Suppose K is big. By [10, Cor.2.2.7] there exist $m \in \mathbb{N}^*$, an ample divisor A and an effective divisor Z such that

$$mK \equiv A + Z.$$

Since Z is effective, the set of integers $Z \cdot C$, when C varies among negative curves, is bounded from below, therefore the degree $A \cdot C = (mK - Z) \cdot C$ of these curves with respect to the ample divisor A is bounded. Hence, by results of Chow–Grothendieck [7], [12, Lecture 15], one has only finitely many components of the Hilbert scheme containing points corresponding to such curves. Since they are negative, these components contain only one curve, proving the assertion. □

For the proof of Corollary 1.4, since $KC_n + C_n^2 \geq -2$, one get $\lim KC_n = \infty$. Since we are on a surface with $\kappa \geq 0$, we can apply (1.1), $\lim g_n = \infty$, thus the result.

3 Surfaces not of general type

We want to deduce from Theorem 1.1 a result valid for any smooth surface. Let Y be any smooth projective surface. Let $\eta \in \mathrm{Pic}(Y)$ be such that $\eta^2 > 0$, $|K_Y + \eta|$ is big and $|2\eta|$ contains a base point free linear system of positive dimension. Let be $\beta_0 > 3$.

Theorem 3.1 *Then there are at most finitely many negative curves D on Y such that*

$$k_D \geq \beta_0(g-1) + \frac{\beta_0 - 2}{2} D \cdot \eta. \tag{3.1}$$

Proof Under the hypotheses there is a smooth curve $B \equiv 2\eta$ intersecting every negative curves of X only at smooth points with intersection multiplicity 1. Let us make a double cover $f : X \to Y$ branched along B. Then for every negative curve D of Y, $C = f^*(D)$ is irreducible, negative and $g(C) = 2g(D) - 1 + \eta \cdot D$ by the Hurwitz formula. Now, if one has any sequence of distinct curves D_n such that $\eta \cdot D_n$ is bounded, then the curves D_n belong to finitely many components of the Hilbert scheme. Since the curves D_n are negative, they are isolated, so they belong to different components of the Hilbert scheme, and therefore $\eta \cdot D$ goes to $+\infty$. By Hurwitz formula, $g(C)$ goes to infinity as well. Since $f_*(K_X) = K_Y \oplus (K_Y + \eta)$, then $\kappa(X) = 2$ and we finish by applying (B) of Theorem 1.1 to X and to $C = f^*(D)$. □

As an application, we take Y_n to be the plane blown up at n general points. Then $\mathrm{Pic}(Y_n) \cong \mathbb{Z}^{n+1}$ is generated by the classes of the pull-back L of a line and of minus the exceptional divisors $E_1, \ldots, E_n$ over the blown up points. We write $D = (d, m_1, \ldots, m_n)$ to denote the class of a curve with components $d, m_1, \ldots, m_n$ with respect to this basis. We may use exponential notation to denote repeated m_i's. Thus $-K = (3, 1^n)$.

Proposition 3.2 *Fix $\beta_0 > 3$. There are at most finitely many irreducible curves of class $D = (d, m_1, \ldots, m_n)$ on Y_n such that*

$$\frac{D^2}{d} \leq \frac{2-\beta_0}{\beta_0}\left(1+\frac{M}{d}\right), \quad \text{where} \quad M = \sum_{i=1}^{n} m_i. \tag{3.2}$$

Proof We apply Theorem 3.1, by taking $\eta = 4L$. Indeed $K + \eta = (1, -1^n)$ is big. Moreover (3.2) is equivalent to $k_D \geq \beta_0(p-1) + \frac{\beta_0-2}{2} D \cdot \eta$ (where p is the arithmetic genus of D), which implies (3.1) (where g is the geometric genus of D). □

Recall that

$$\epsilon_n = \inf\left\{\frac{d}{M}, \quad \text{for all effective} \quad D = (d, m_1, \ldots, m_n), \quad \text{such that} \quad M > 0\right\}$$

is the *Seshadri constant* of Y_n. Nagata's Conjecture (see [13]) is equivalent to say that $\epsilon_n = 1/\sqrt{n}$ if $n \geq 10$ (see [9]).

Remark 3.3 Proposition 3.2 can be seen as a weak form of Nagata's Conjecture. Indeed, let us look at the *homogeneous case* $D = (d, m^n)$ with $n \geq 10$. Nagata's Conjecture predicts that, if the n blown up points are in very general position, there is no irreducible such curve with $D^2 < 0$ (see [5,13]), i.e., with $d < \sqrt{n}m$. The conclusion of Proposition 3.2 is not absence of curves, but finiteness of their set, under a stronger assumption than Nagata's. Let us look at the difference between the two assumptions. In the (m, d)-plane (3.2) applies to pairs (m, d) in the first quadrant below the hyperbola with equation

$$\beta_0 d^2 + d(\beta_0 - 2) - n\beta_0 m^2 + (\beta_0 - 2)nm = 0 \tag{3.3}$$

drawn in black in Fig. 1. One of its asymptotes (the lower line of Fig. 1) is parallel to the *Nagata line* $d = \sqrt{n}m$ (the upper line of Fig. 1).

Since for all effective divisor $D = (d, m_1, \ldots, m_n)$ one has $d/M \geq \epsilon_n$, one has approximations $d/M \geq e_n$ to Nagata's conjecture for any lower approximation e_n of ϵ_n. The best known in general is the one in [9]

$$\epsilon_n \geq e_n = \sqrt{\frac{1}{n}\left(1 - \frac{1}{f(n)}\right)} \tag{3.4}$$

where $f(n)$ is, for most n, an explicitly given quadratic function of n (see [9, Corollary 1.2.3]). For $n = 10$ in the homogeneous case the best result is $e_{10} = 228/721$ (see [14]).

The hyperbola (3.3) meets the line $d = e_n m$, therefore Proposition 3.2 gives some information in an unlimited region where the above approximations to Nagata do not work.

Remark 3.4 Proposition 3.2 implies that there are finitely many irreducible curves of class $D = (d, m_1, \ldots, m_n)$ on Y_n such that

$$\frac{D^2}{d} \leq \frac{2-\beta_0}{\beta_0}\left(1+\frac{1}{\epsilon_n}\right), \tag{3.5}$$

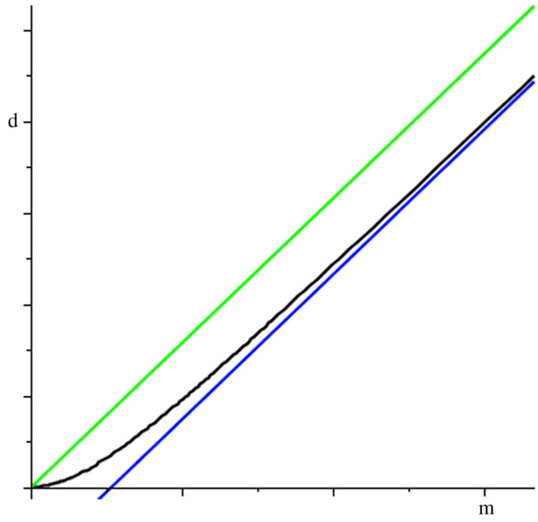

Fig. 1 The hyperbola, its asymptote and the Nagata line

where ϵ_n can be replaced by e_n in Harbourne–Roé's approximation (3.4). This result is not surprising. Indeed, J. Roé pointed out to us an easy argument which shows that there is no irreducible curve of class $D = (d, m_1, \ldots, m_n)$ on Y_n such that

$$\frac{D^2}{d} < -\frac{1}{n\epsilon_n}$$

which is better than (3.5), and the difference

$$\frac{\beta_0 - 2}{\beta_0}\left(1 + \frac{1}{\epsilon_n}\right) - \frac{1}{n\epsilon_n}$$

tends to $\frac{\beta_0-2}{\beta_0} \sim \frac{1}{3}$ for $n \to \infty$.

4 A boundedness result for non-negative curves

With the usual notation, for a curve C on the surface X with $C^2 \neq 0$, we set $x_C := \frac{\delta_C}{C^2}$, with the usual convention that the index C can be dropped if there is no ambiguity.

Theorem 4.1 *Consider real numbers $x_0 > \frac{1}{2}$ and $\beta_0 > 3$. Let C be a curve on X, with $\kappa(X) \geq 0$, satisfying the following conditions:*

(1) $C^2 > 0$, $k_C = \beta(g-1)$ *with* $\beta > \beta_0$ *and* $g > 1$;
(2) $x_C > x_0$.

Then

$$g \leq a\frac{(\beta-2)}{(\beta-3)^2}\frac{3x_0-1}{2x_0-1} + 1, \tag{4.1}$$

$$k_C \leq a\frac{(\beta-2)}{\beta(\beta-3)^2}\frac{3x_0-1}{2x_0-1}, \tag{4.2}$$

$$k_C \leq 2(g-1) + a\frac{(\beta-2)^2}{(\beta-3)^2} \cdot \frac{3x_0-1}{2x_0-1}. \tag{4.3}$$

If $\kappa(X) = 2$, then the Hilbert scheme of curves on X satisfying (1) and (2) has finitely many irreducible components.

Proof One has

$$k_C - 2(g-1) = 2\delta - C^2 = (\beta - 2)(g-1) > 0.$$

Hence by (2.1), we have

$$P(\alpha) := \alpha^2 \left(3\delta - C^2\right) + \alpha \frac{2(\beta-3)}{\beta-2} \left(C^2 - 2\delta\right) + a \geq 0 \tag{4.4}$$

for $\alpha \in [0, 1]$. Since the coefficient of the leading term of P is positive, the minimum of $P(\alpha)$ is attained for

$$\alpha_0 = \frac{(\beta-3)\left(2\delta - C^2\right)}{(\beta-2)\left(3\delta - C^2\right)}.$$

Since $\beta > 3$, we have $\alpha_0 \in]0, 1[$, and, by (4.4) we have

$$P(\alpha_0) = -\frac{(\beta-3)^2 \left(2\delta - C^2\right)^2}{(\beta-2)^2 \left(3\delta - C^2\right)} + a \geq 0$$

Thus

$$\frac{a}{\mu} \geq \frac{\left(2\delta - C^2\right)^2}{\left(3\delta - C^2\right)} \quad \text{where} \quad \mu = \frac{(\beta-3)^2}{(\beta-2)^2},$$

hence

$$\frac{a}{\mu} \cdot \frac{3\delta - C^2}{2\delta - C^2} \geq 2\delta - C^2.$$

We have

$$\frac{3\delta - C^2}{2\delta - C^2} = \frac{3x-1}{2x-1} < \frac{3x_0 - 1}{2x_0 - 1}$$

because $\frac{3x-1}{2x-1}$ is decreasing for $x > x_0 > \frac{1}{2}$, hence

$$(\beta-2)(g-1) = k_C - 2(g-1) = 2\delta - C^2 \leq \frac{a}{\mu} \cdot \frac{3x_0 - 1}{2x_0 - 1},$$

which implies (4.1), (4.2) and (4.3). Moreover both g and k_C are bounded from above and, if $\kappa(X) = 2$, we conclude with the same argument at the end of the proof of Theorem 1.1. $\square$

Corollary 4.2 *Let β_0 be greater than 3 and let $(C_n)_{n\in\mathbb{N}}$ be a sequence of curves on X with $\kappa \geq 0$ such that $k_{C_n} > \beta_0(g(C_n) - 1)$, $C_n^2 > 0$ and $\lim g(C_n) = \infty$. Then*

$$\lim_n \frac{\delta_{C_n}}{C_n^2} = \frac{1}{2}, \tag{4.5}$$

moreover $\lim_n \frac{g(C_n)}{\delta_{C_n}} = \lim_n \frac{K \cdot C_n}{\delta_n} = 0$.

Proof Let C be a curve with $k_C = (3+\epsilon)(g-1)$, $\epsilon > 0$. Since $(1+\epsilon)(g-1) = 2\delta - C^2$, we get $\frac{\delta}{C^2} - \frac{1}{2} = (1+\epsilon)\frac{g-1}{2C^2} \geq 0$. Therefore $\liminf_n \frac{\delta_n}{C_n^2} \geq \frac{1}{2}$. On the other hand, using inequality (4.1) of Theorem 4.1 which holds for surface with $\kappa \geq 0$, we obtain $\limsup_n \frac{\delta_n}{C_n^2} \leq \frac{1}{2}$. The remaining limits are readily computed, using inequalities (4.1) and (4.2), respectively, and the fact that $\delta \sim C^2/2$. □

Example 4.3 For Shimura curves on Shimura surfaces, we have $K \cdot C = 4(g-1)$ and, if there is one, there are infinitely many of them (see [1]). We suppose that this is the case. By Theorem 1.1 (B), all but a finite number of Shimura curves C satisfy $C^2 > 0$. By Theorem 4.1, the geometric genus of a sequence of such curves goes to ∞. For such a sequence, one gets $\lim \frac{\delta_C}{C^2} = \frac{1}{2}$ from Corollary 4.2 (this example is used in [15, Section 4.1]).

5 On a conjecture by Vojta

The results in Sect. 4 are reminiscent of the following conjecture (see [1]), which predicts that curves of bounded geometric genus on a surface of general type form a bounded family:

Conjecture 5.1 *Let X be a smooth projective surface. There exist constants A, B such that for any curve C we have*

$$k_C \leq A(g-1) + B.$$

If this conjecture is satisfied for X with $\kappa(X) = 2$, then X contains finitely many curves of genus 0 or 1. This is known to hold for minimal surfaces with big cotangent bundle (see [3,6]).

A stronger version of Conjecture 5.1 is the following conjecture by Vojta (see again [1]):

Conjecture 5.2 *For any real number $\epsilon > 0$, we can take $A = 4 + \epsilon$ in Conjecture 5.1 (and $B = B(\epsilon)$ a function of ϵ).*

An even stronger, more recent version, predicts that $A = 2 + \epsilon$ (see [1]).

Remark 5.3 If C is a smooth curve on X, then $k_C = 2(g-1) - C^2$, therefore if BNC holds, then Vojta's conjecture holds for smooth curves with $A = 2$. This suggests a close relationship between Vojta's conjecture and BNC.

Miyaoka proves in [11] that Conjecture 5.1 also holds if $K^2 > c_2$ and he gives explicit values for A and B, but they are far away from the ones predicted by Conjecture 5.2. Moreover Miyaoka proves that $k_C \leq 3(g-1)$ for (smooth) compact ball quotient surfaces on which the equality is attained by an infinite number of curves, i.e., Shimura curves, if they exist.

In [1] one proves that for surfaces whose universal cover is the bi-disk, one has

$$k_C \leq 4(g-1).$$

This is sharp since for Shimura curves on Shimura surfaces, one has $k_C = 4(g-1)$.

For X a surface, we define

$$\Lambda_X = \sup_{(C_n)_{n\in\mathbb{N}}} \left\{ \limsup_n \frac{K \cdot C_n}{g_n - 1} \right\}$$

where $(C_n)_{n\in\mathbb{N}}$ varies among all sequences of curves C_n in X of genus $g_n = g(C_n) > 1$ with $\lim_n g_n = \infty$. Conjecture 5.1 says that $\Lambda_X < \infty$.

If X has trivial canonical bundle, then $\Lambda_X = 0$. Apart from this case, and the aforementioned cases studied in [1,11], nothing is known about Λ_X. The following result gives us a piece of information:

Theorem 5.4 *Let X be a surface of Kodaira dimension $\kappa > 0$. Let L be a very ample divisor on X, and let γ be the arithmetic genus of curves in $|L|$. Then*

$$\Lambda_X \geq \frac{K \cdot L}{\gamma - 1 + L^2} > 0.$$

Proof Look at the surface X embedded in $\mathbb{P}^r$, with $r \geq 3$, via L. Then take a general projection $\pi : X \to \mathbb{P}^2$. Consider a general rational curve of degree n in $\mathbb{P}^2$ and let C_n be its pull-back via π. Then $C_n \in |nL|$.

By Hurwitz formula, the ramification divisor R of π is such that $K \equiv \pi^*(K_{\mathbb{P}^2}) + R$, thus $R \equiv K + 3L$. The ramification divisor of the restriction of π to C_n is $R \cdot C_n$. So Hurwitz formula for curves implies that the geometric genus g_n of C_n satisfies

$$2g_n - 2 = nL \cdot K + (3n - 2)L^2.$$

Therefore

$$\frac{K \cdot C_n}{g_n - 1} = \frac{2nL \cdot K}{n(L \cdot K + L^2) + (2n - 2)L^2} = \frac{L \cdot K}{\gamma - 1 + \left(1 - \frac{1}{n}\right) L^2}$$

and this proves the left-hand side inequality. Since X has Kodaira dimension $\kappa > 0$, we get $K \cdot L > 0$ and $\Lambda_X > 0$. □

Example 5.5 Suppose that, in the setting of Theorem 5.4, one has $K = mL$, with $m > 0$. Then

$$\Lambda_X \geq \frac{2m}{m + 3}.$$

So there are sequences $(X_n)_{n \in \mathbb{N}}$ of surfaces, e.g., complete intersections of increasing degree in projective space, with $m \to \infty$, and therefore $\Lambda_{X_n} \to 2$ (from below).

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