Journal of Algebra 413 (2014) 1–14

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On the $PSL_2(\mathbb{F}_{19})$-invariant cubic sevenfold

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ARTICLE INFO

Article history:
Received 7 January 2013
Available online 2 June 2014
Communicated by Michel Van den Bergh

MSC:
11G10
14J50
14J70

Keywords:
Cubic sevenfold
Automorphism group
Intermediate Jacobian

ABSTRACT

It has been proved by Adler that there exists a unique cubic hypersurface X^7 in $\mathbb{P}^8$ which is invariant under the action of the simple group $PSL_2(\mathbb{F}_{19})$. In the present note we study the intermediate Jacobian of X^7 and in particular we prove that the subjacent 85-dimensional torus is an Abelian variety. The symmetry group $G = PSL_2(\mathbb{F}_{19})$ defines uniquely a G-invariant Abelian 9-fold $A(X^7)$, which we study in detail and describe its period lattice.

Introduction

Let $PSL_2(\mathbb{F}_q)$ be the projective special linear group of order 2 matrices over the finite field with q element $\mathbb{F}_q$. There exist exactly two non-trivial irreducible complex representations $W_{\frac{q-1}{2}}, \bar{W}_{\frac{q-1}{2}}$ of $PSL_2(\mathbb{F}_q)$ on a space of dimension $\frac{q-1}{2}$, each one complex conjugated to the other, see [8]. In [1], Adler proved that for any q a power of a prime $p \geq 7$,

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http://dx.doi.org/10.1016/j.jalgebra.2014.05.004
0021-8693/

with $p = 3 \bmod 8$, there exists a unique (up to a constant multiple) $PSL_2(\mathbb{F}_q)$-invariant cubic form f_q of $W_{\frac{q-1}{2}}$. We call the corresponding unique $PSL_2(\mathbb{F}_q)$-invariant cubic hypersurface

$$X_{\frac{q-5}{2}} = \{f_q = 0\} \hookrightarrow \mathbb{P}(W_{\frac{q-1}{2}})$$

the *Adler cubic for q*.

In the smallest non-trivial case, $q = 11$, the Adler cubic in $\mathbb{P}^4 = \mathbb{P}(W_5)$ coincides with the Klein cubic threefold

$$X_3 = \left\{x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_1 = 0\right\}.$$

This threefold has been introduced by Felix Klein when he studied the well known Klein quartic curve, the unique curve in the projective plane with symmetry group $PSL_2(\mathbb{F}_7)$, see [11]. In [15], the second author has proven that there exists a unique Abelian fivefold $A(X_3)$ with a $PSL_2(\mathbb{F}_{11})$-invariant principal polarization and he has explicitly described the period lattice of $A(X_3)$. In this case, $A(X_3)$ coincides with the Griffiths intermediate Jacobian $J(X_3)$ of the cubic threefold X_3, with the principal polarization coming from the intersection of real 3-cycles on X_3.

In this paper we study the next case – the Adler cubic for $q = 19$. By using the general descriptions from Theorem 4 of [1], one can find an equation of the Adler cubic X_7:

$$\begin{aligned} f_{19} = {} & x_1^2x_6 + x_6^2x_2 + x_2^2x_7 + x_7^2x_4 + x_4^2x_5 + x_5^2x_8 + x_8^2x_9 + x_9^2x_3 + x_3^2x_1 \\ & - 2(x_1x_7x_8 + x_2x_3x_5 + x_4x_6x_9). \end{aligned}$$

In the first section, we study a similar invariant principally polarized Abelian ninefold $A(X_7)$ defined uniquely by the Adler cubic sevenfold, and compute the period lattice and the first Chern class of the polarization of $A(X_7)$.

In the second section, we study the 85-dimensional Griffiths intermediate Jacobian

$$J(X_7) = \left(H^{5,2} \oplus H^{4,3}\right)^* / H_7(X_7, \mathbb{Z})$$

of X_7. We prove that forgetting the natural polarization on $J(X_7)$, one can introduce another $PSL_2(\mathbb{F}_{19})$-invariant polarization which provides the complex torus subjacent to $J(X_7)$ with a structure of an Abelian variety.

In the third section, we study invariant properties of the Adler–Klein pencil of cubic sevenfolds, an analog of the Dwork pencil of quintics threefolds see e.g. [6]. Notice that from the point of view of variation of Hodge structure, the cubic sevenfolds can be considered as higher dimensional analogs of Calabi–Yau threefolds, see [2] and [10].

1. The invariant Abelian 9-fold of the Adler cubic 7-fold

We begin by some notations. Let $e_1, \dots, e_9$ be a basis of a 9-dimensional vector space V. Let $\tau \in GL(V)$ be the order 19 automorphism defined by

$$\tau : e_j \to \xi^{j^2} e_j,$$

where $\xi = e^{2i\pi/19}$, $i^2 = -1$. The order 9 automorphism $\sigma \in GL(V)$ is defined by its action on the coordinates e_j, $1 \leq j \leq 9$, as the permutation $(6, 7, 1, 5, 8, 2, 4, 9, 3)$ of the indices. We denote by μ the order 2 automorphism given in the basis $e_1, \cdots, e_9$ by the matrix:

$$\mu = \left(\frac{i}{\sqrt{19}} \left(\frac{kj}{19} \right) \left(\xi^{kj} - \xi^{-kj} \right) \right)_{1 \leq k,j \leq 9}$$

where $(\frac{kj}{19})$ is the Legendre symbol. The group generated by τ, σ and μ is isomorphic to $PSL_2(\mathbb{F}_{19})$ and defines a representation of $PSL_2(\mathbb{F}_{19})$, see [8]. Let us define

$$\begin{aligned} v_k &= \tau^k (e_1 + \ldots + e_9) \\ &= \xi^k e_1 + \xi^{4k} e_2 + \xi^{9k} e_3 + \xi^{16k} e_4 + \xi^{6k} e_5 + \xi^{17k} e_6 + \xi^{11k} e_7 + \xi^{7k} e_8 + \xi^{5k} e_9 \end{aligned}$$

(thus $v_k = v_{k+19}$) and

$$w'_k = \frac{1}{1 + 2\nu} (v_k - 5v_{k+1} + 10v_{k+2} - 10v_{k+3} + 5v_{k+4} - v_{k+5}),$$

where

$$\nu = \sum_{k=1}^{k=9} \xi^{k^2} = \frac{-1 + i\sqrt{19}}{2}.$$

An endomorphism h of a torus A acts on the tangent space $T_{A,0}$ by its differential dh, which we call the *analytic representation* of h. For the ease of the notations, we will use the same letter for h and its analytic representation.

In the present section, we will prove the following theorem:

Theorem 1. *There exist:*

(1) a 9-dimensional torus $A = V/\Lambda$, $T_{A,0} = V$ such that the elements τ, σ of $GL(V)$ are the analytic representations of automorphisms of A. The torus A is isomorphic to the Abelian variety E^9, where E is the elliptic curve $\mathbb{C}/\mathbb{Z}[\nu]$ ($\nu = \frac{-1+i\sqrt{19}}{2}$).

(2) a unique principal polarization Θ on A which is invariant by the automorphisms σ, τ. The period lattice of A is then:

$$H_1(A, \mathbb{Z}) = \frac{\mathbb{Z}[\nu]}{1 + 2\nu} w'_0 + \frac{\mathbb{Z}[\nu]}{1 + 2\nu} w'_1 + \frac{\mathbb{Z}[\nu]}{1 + 2\nu} w'_2 + \frac{\mathbb{Z}[\nu]}{1 + 2\nu} w'_3 + \bigoplus_{k=4}^{8} \mathbb{Z}[\nu] v_k,$$

and $c_1(\Theta) = \frac{2}{\sqrt{19}} \sum_{k=1}^{k=9} dx_k \wedge d\bar{x}_k$, where $x_1, \dots, x_9$ is the dual basis of the e_j's.

In order to prove Theorem 1, we first suppose that such a torus $A = V/\Lambda$ exists and find the necessary conditions for its existence. We then check immediately that these conditions are also sufficient, and that they give us the uniqueness of A.

Let $\Lambda \subset V$ be a lattice such that τ and σ of $GL(V)$ are analytic representations of automorphisms of the torus $A = V/\Lambda$. Let

$$\ell_k = \xi^k x_1 + \xi^{4k} x_2 + \xi^{9k} x_3 + \xi^{16k} x_4 + \xi^{6k} x_5 + \xi^{17k} x_6 + \xi^{11k} x_7 + \xi^{7k} x_8 + \xi^{5k} x_9.$$

Let q be the endomorphism $q = \sum_{j=0}^{j=8} \sigma^j$. For $z = \sum_{j=1}^{j=9} x_j e_j$, we have $q(z) = \ell_0(z) v_0$, thus the image of q is an elliptic curve $\mathbb{E}$ contained in A.

The restriction of $q \circ \tau : A \to \mathbb{E}$ to $\mathbb{E} \hookrightarrow A$ is the multiplication by $\nu = \ell_0(\tau v_0)$. Thus the endomorphism group $\operatorname{End} \mathbb{E}$ of $\mathbb{E}$ contains the ring $\mathbb{Z}[\nu]$; since this is a maximal order, we have $\operatorname{End} \mathbb{E} = \mathbb{Z}[\nu]$. We remark that the ring $\mathbb{Z}[\nu]$ is one of the 9 rings of integers of quadratic fields that are Principal Ideal Domains, see [13]. Since $\mathbb{Z}[\nu]$ is a PID and $H_1(A, \mathbb{Z}) \cap \mathbb{C} v_0$ is a rank one $\mathbb{Z}[\nu]$-module, there exists a constant $c \in \mathbb{C}^*$ such that

$$H_1(A, \mathbb{Z}) \cap \mathbb{C} v_0 = \mathbb{Z}[\nu] c v_0.$$

Up to normalization of the e_j's, we can suppose that $c = 1$.

Let $\Lambda_0 \subset V$ be the $\mathbb{Z}$-module generated by the v_k, $k \in \mathbb{Z}/19\mathbb{Z}$. The group Λ_0 is stable under the action of τ and $\Lambda_0 \subset H_1(A, \mathbb{Z}) = \Lambda$.

Lemma 2. *The $\mathbb{Z}$-module $\Lambda_0 \subset H_1(A, \mathbb{Z})$ is equal to the lattice:*

$$R_0 = \sum_{k=0}^{k=8} \mathbb{Z}[\nu] v_k.$$

Proof. We have $\nu v_0 = \sum_{k=1}^{k=9} v_{k^2}$ hence νv_0 is an element of Λ_0. This implies that the vectors $\nu v_k = \tau^k \nu v_0$ are elements of Λ_0 for all k, hence: $R_0 \subset \Lambda_0$. Conversely, we have:

$$v_9 = v_0 + (1 + \nu) v_1 - 2 v_2 + (1 - \nu) v_3 + (3 + \nu) v_4 + (-2 + \nu) v_5 - (2 + \nu) v_6 + 2 v_7 + \nu v_8$$

and similar formulas for $v_{10}, \ldots, v_{17}$. This proves that the lattice R_0 contains the vectors $v_k = \tau^k v_0$ generating Λ_0, thus: $R_0 = \Lambda_0$. $\square$

Now we construct a lattice that contains $H_1(A, \mathbb{Z})$. Let be $k \in \mathbb{Z}/19\mathbb{Z}$. The image of $z \in V$ by the endomorphism $q \circ \tau^k : V \to V$ is

$$q \circ \tau^k(z) = \ell_k(z) v_0.$$

Let be $\lambda \in H_1(A, \mathbb{Z})$. Since $H_1(A, \mathbb{Z}) \cap \mathbb{C} v_0 = \mathbb{Z}[\nu] v_0$, the scalar $\ell_k(\lambda)$ is an element of $\mathbb{Z}[\nu]$. Let

$$\Lambda_8 = \{ z \in V \mid \ell_k(z) \in \mathbb{Z}[\nu],\ 0 \leq k \leq 8 \}.$$

Lemma 3. *The $\mathbb{Z}$-module $\Lambda_8 \supset H_1(A,\mathbb{Z})$ is the lattice:*

$$\sum_{k=0}^{k=7} \frac{\mathbb{Z}[\nu]}{1+2\nu}(v_k - v_{k+1}) + \mathbb{Z}[\nu]v_0.$$

Moreover τ stabilizes Λ_8.

Proof. Let $\ell_0^*, \dots, \ell_8^*$ be the basis dual to $\ell_0, \dots, \ell_8$. By definition, the $\mathbb{Z}[\nu]$-module Λ_8 is $\bigoplus_{i=0}^{i=8} \mathbb{Z}[\nu]\ell_i^*$. By expressing the ℓ_i^* in the basis $v_0, \dots, v_8$ we obtain the lattice Λ_8. Using the formula:

$$\begin{aligned} v_9 - v_8 = {} & v_0 + (1+\nu)v_1 - 2v_2 + (1-\nu)v_3 + (3+\nu)v_4 \\ & + (\nu - 2)v_5 - (2+\nu)v_6 + 2v_7 + (\nu - 1)v_8, \end{aligned}$$

one can check that $v_9 - v_8$ is a $\mathbb{Z}[\nu]$-linear combination of $(2\nu+1)v_0$ and the $v_k - v_{k+1}$ for $k = 0, \dots, 7$. Therefore $\tau(\frac{1}{2\nu+1}(v_8 - v_7)) = \frac{1}{2\nu+1}(v_9 - v_8)$ is in Λ_8 and Λ_8 is stable by τ. $\square$

We denote by $\phi : \Lambda_8 \to \Lambda_8/\Lambda_0$ the quotient map. The ring $\mathbb{Z}[\nu]/(1+2\nu)$ is the finite field with 19 elements. The quotient Λ_8/Λ_0 is a $\mathbb{Z}[\nu]/(1+2\nu)$-vector space with basis $t_1, \dots, t_8$ such that $t_i = \frac{1}{1+2\nu}(v_{i-1} - v_i) + \Lambda_0$.

Let R be a lattice such that: $\Lambda_0 \subset R \subset \Lambda_8$. The group $\phi(R)$ is a vector subspace of Λ_8/Λ_0 and:

$$\phi^{-1}\phi(R) = R + \Lambda_0 = R.$$

The set of such lattices R corresponds bijectively to the set of vector subspaces of Λ_8/Λ_0.

Because the automorphism τ preserves Λ_0, it induces an automorphism $\widehat{\tau}$ on the quotient Λ_8/Λ_0 such that $\phi \circ \tau = \widehat{\tau} \circ \phi$. As τ stabilizes $H_1(A,\mathbb{Z})$, the vector subspace $\phi(H_1(A,\mathbb{Z}))$ is stable by $\widehat{\tau}$. Let

$$w_{8-k} = (-1)^k \sum_{j=0}^{j=k} \binom{k}{j} (-1)^j t_{j+1}, \quad k = 0 \dots 7.$$

Then $w_{i-1} = \widehat{\tau}w_i - w_i$ for $i > 1$ and $\widehat{\tau}w_1 = w_1$. The matrix of $\widehat{\tau}$ in the basis $w_1, \dots, w_8$ is the size 8×8 matrix:

$$\begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}$$

The sub-spaces stable by $\widehat{\tau}$ are the spaces W_j, $1 \leq j \leq 8$, generated by $w_1, \dots, w_j$ and $W_0 = \{0\}$. Let Λ_j be the lattice $\phi^{-1} W_j$. It is easy to check that, as a lattice in $\mathbb{C}^9$, the Λ_j are all isomorphic to $\mathbb{Z}[\nu]^9$. Since $\Lambda = H_1(A, \mathbb{Z})$ is stable by τ, we have proved that:

Lemma 4. *The torus $A = V/\Lambda$ exists, it is an Abelian variety isomorphic to E^9. There exists $j \in \{0, 1, \cdots, 8\}$ such that $\Lambda = \Lambda_j$.*

Let $w'_0, \dots, w'_3$ be the vectors defined by:

$$w'_0 = \frac{1}{1+2\nu}(v_0 - 5v_1 + 10v_2 - 10v_3 + 5v_4 - v_5) = \frac{1}{1+2\nu}(1-\tau)^5 v_0$$

and $w'_k = \tau^k w'_0$ (we have $\phi(w'_0) = w_4$). Let us suppose that $A = V/\Lambda$ has moreover a principal polarization Θ that is invariant under the action of τ and σ. Then:

Lemma 5. *The lattice $H_1(A, \mathbb{Z})$ is equal to Λ_4, and*

$$\Lambda_4 = \frac{\mathbb{Z}[\nu]}{1+2\nu} w'_0 + \frac{\mathbb{Z}[\nu]}{1+2\nu} w'_1 + \frac{\mathbb{Z}[\nu]}{1+2\nu} w'_2 + \frac{\mathbb{Z}[\nu]}{1+2\nu} w'_3 + \bigoplus_{k=4}^{8} \mathbb{Z}[\nu] v_k.$$

The Hermitian matrix associated to Θ is equal to $\frac{2}{\sqrt{19}} I_9$ in the basis $e_1, \dots, e_9$ and $c_1(\Theta) = \frac{i}{\sqrt{19}} \sum_{k=1}^{k=9} dx_k \wedge d\bar{x}_k$.

Proof. Let H be the matrix of the Hermitian form of the polarization Θ in the basis $e_1, \dots, e_9$. Since τ preserves the polarization Θ, this implies that:

$${}^t\tau H \bar{\tau} = H$$

where $\bar{\tau}$ is the matrix whose coefficients are complex conjugates of τ. The only Hermitian matrices that verify this equality are the diagonal matrices. By the same reasoning with σ instead of τ, we obtain that these diagonal coefficients are equal, and:

$$H = a \frac{2}{\sqrt{19}} I_9,$$

where a is a positive real (H is a positive definite Hermitian form). As H is a polarization, the alternating form $c_1(\Theta) = \Im m(H)$ takes integer values on $H_1(A, \mathbb{Z})$, hence $\Im m({}^t v_2 H \bar{v}_1) = a$ is an integer.

Let $c_1(\Theta) = \Im m(H) = i \frac{a}{\sqrt{19}} \sum dx_k \wedge d\bar{x}_k$ be the alternating form of the principal polarization Θ. Let $\lambda_1, \dots, \lambda_{18}$ be a basis of a lattice Λ'. By definition, the square of the Pfaffian $Pf_\Theta(\Lambda')$ of Λ' is the determinant of the matrix

$$M_{\Lambda'} = \big(c_1(\Theta)(\lambda_j, \lambda_k)\big)_{1 \leq j,k \leq 18}.$$

Since Θ is a principal polarization on A, we have $Pf_\Theta(H_1(A, \mathbb{Z})) = 1$.

For $j \in \{0, \dots, 8\}$, it is easy to find a basis of the lattice Λ_j. We compute that

$$P_j^2 = a^{18} 19^{8-2j}$$

where P_j its Pfaffian of Λ_j. As a is positive, the only possibility that P_j equals 1 is $j = 4$ and $a = 1$. Moreover since all coefficients of the matrix M_{Λ_4} are integers, the alternating form $\frac{i}{\sqrt{19}} \sum dx_k \wedge d\bar{x}_k$ defines a principal polarization on A, see [5, Chap. 4.1]. □

Theorem 1 follows from Lemmas 4 and 5.

Remark 6. One can compute that the involution μ acts on the principally polarized Abelian variety (A, Θ), and therefore that $\mathrm{Aut}(A, \Theta)$ contains the group $PSL_2(\mathbb{F}_{19})$.

Remark 7. In [4], Beauville proves that the intermediate Jacobian of the S_6-symmetric quartic threefolds F is not a product of Jacobians of curves (see also [3]). That implies, using Clemens–Griffiths results for threefolds, the irrationality of F. Using the arguments at the end of [4], one can see that the principally polarized Abelian ninefold (A, Θ) is not a product of Jacobians of curves.

2. Intermediate Jacobians of the Adler cubic sevenfold

2.1. Intermediate Jacobian

For a smooth algebraic complex manifold X of dimension $n = 2k + 1$, let $H^{p,q}(X)$ be the Hodge (p, q)-cohomology space $H^{p,q}(X) = H^q(X, \Omega^p)$, and $h^{p,q} = \dim H^{p,q}(X)$ be the Hodge numbers of X, $0 \le p + q \le 2n = 4k + 2$. The intersection of real $(2k + 1)$-dimensional chains in X defines an integer valued quadratic form Q on the integer cohomology $H_{2k+1}(X, \mathbb{Z})$, and yields an embedding of $H_{2k+1}(X, \mathbb{Z})$ as an integer lattice in the dual space of $H^{2k+1,0}(X) + H^{2k,1}(X) + \dots + H^{k+1,k}(X)$. The quotient compact complex torus:

$$J(X) = \left(H^{2k+1,0}(X) + H^{2k,1}(X) + \dots + H^{k+1,k}(X)\right)^* / H_{2k+1}(X, \mathbb{Z})$$

is the *Griffiths intermediate Jacobian* of the n-fold X, see [7, p. 123]. By the Riemann–Hodge bilinear relations, the quadratic form Q is definite on any space $H^{p,n-p}(X)$, and has opposite signs on $H^{p,n-p}(X)$ and $H^{p',n-p'}(X)$ if and only if $p - p'$ odd. Moreover $\Im m(Q)$ takes integral values on $H_{2k+1}(X, \mathbb{Z})$. We call the pair $(J(X), Q)$ the polarized Griffiths intermediate Jacobian.

The quadratic form Q gives the polarized torus $(J(X), Q)$ a structure of a polarized Abelian variety if and only if p has the same parity for all non-zero spaces $H^{p,n-p}(X)$ [7, p. 123].

2.2. Griffiths formulas

The Hodge structure on the middle cohomology of a smooth cubic sevenfold can be computed by the following Griffiths formulas, which we shall use below in order to determine the representation of the action of the symmetry group $PSL_2(\mathbb{F}_{19})$ on the middle cohomology of the Adler cubic sevenfold.

Let $X = (f(x) = 0)$ be a smooth hypersurface of degree $m \geq 2$ in the complex projective space $\mathbb{P}^{n+1}(x)$, $(x) = (x_1, ..., x_{n+2})$. Let $S = \bigoplus_{d\geq 0} S_d$ be the graded polynomial ring $S = \mathbf{C}[x_1, ..., x_{n+2}]$, with S_d being the space of polynomials of degree d. Let $I = \bigoplus I_d \subset S$ be the graded ideal generated by the $n+2$ partials $\frac{\partial f}{\partial x_j}$, $j = 1, ..., n+2$, with $I_d = I \cap S_d$.

Let $R = S/I = \bigoplus R_d$ be the graded Jacobian ring of X (or the Jacobian ring of the polynomial $f(x)$), with graded components $R_d = S_d/I_d$. Then for $p + q = n$, the primitive cohomology space $H^{p,q}_{prim}(X)$ is isomorphic to the graded piece $R_{m(q+1)-n-2}$, where $R_d = 0$ for $d < 0$ [7, p. 169]. For odd $n = 2k+1$ all the middle cohomology of X are primitive, and in this case

$$H^{p,q}(X) \cong R_{m(q+1)-n-2}, \qquad p + q = 2k + 1 = \dim X, \qquad m = \deg X.$$

In particular, for a smooth cubic sevenfold $X = (f(x) = 0) \subset \mathbb{P}^8$ one has

$$H^{7-q,q}(X) = R_{3(q+1)-9}, \quad 0 \leq q \leq 7,$$

which yields

$$H^{7,0}(X) = R_{-6} = 0, \qquad H^{6,1}(X) = R_{-3} = 0,$$
$$H^{5,2}(X) = R_0 \cong \mathbb{C}, \qquad H^{4,3}(X) = R_3 \cong \mathbf{C}^{84}.$$

2.3. Character table of $PSL_2(\mathbb{F}_{19})$

In Proposition 8 below, we shall use the known description of the irreducible representations of the automorphism group $PSL_2(\mathbb{F}_{19}) \subset \mathrm{Aut}(X)$, which we state here in brief:

By [8], the group $PSL_2(\mathbb{F}_{19})$ has 12 conjugacy classes:

$$1, \{w_1\}, \{w_2\}, \{x\}, \{x^2\}, \{x^3\}, \{x^4\}, \{y\}, \{y^2\}, \{y^3\}, \{y^4\}, \{y^5\}$$

where x has order 9, y has order 10, $w_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $w_2 = w_1^2$ (we observe that $w_1^2, w_1^3 \in \{w_2\}$, $w_2^2, w_2^3 \in \{w_1\}$). Correspondingly, the 12 irreducible representations of $PSL_2(\mathbb{F}_{19})$ are

$$T_1, W_9, \overline{W}_9, W_{18}^1, W_{18}^2, W_{18}^3, W_{18}^4, W_{20}^1, W_{20}^2, W_{20}^3, W_{20}^4, W_{19}$$

	1	w_1	w_2	x	x^2	x^3	x^4	y	y^2	y^3	y^4	y^5
1	1	1	1	1	1	1	1	1	1	1	1	1
W_9	9	ν	$\bar{\nu}$	0	0	0	0	1	-1	1	-1	1
$\overline{W}_9$	9	$\bar{\nu}$	ν	0	0	0	0	1	-1	1	-1	1
W_{18}^1	18	-1	-1	0	0	0	0	b_1	b_2	b_3	b_4	2
W_{18}^2	18	-1	-1	0	0	0	0	b_2	b_4	b_6	b_8	-2
W_{18}^3	18	-1	-1	0	0	0	0	b_3	b_6	b_9	b_{12}	2
W_{18}^4	18	-1	-1	0	0	0	0	b_4	b_8	b_{12}	b_{16}	-2
W_{20}^1	20	1	1	a_1	a_2	a_3	a_4	0	0	0	0	0
W_{20}^2	20	1	1	a_2	a_4	a_6	a_8	0	0	0	0	0
W_{20}^3	20	1	1	a_3	a_6	a_9	a_{12}	0	0	0	0	0
W_{20}^4	20	1	1	a_4	a_8	a_{12}	a_{16}	0	0	0	0	0
W_{19}	19	0	0	1	1	1	1	-1	-1	-1	-1	-1

Fig. 2.1. Character table of $PSL_2(\mathbb{F}_{19})$.

where the notations are uniquely defined by the character table of $PSL_2(\mathbb{F}_{19})$ below for which $a_k = 2\cos(\frac{2k\pi}{9})$, $b_k = -2\cos(\frac{k\pi}{5})$ and $\nu = \frac{-1+i\sqrt{19}}{2}$ (see Fig. 2.1). Note that the representation W_9 is described otherwise in Section 1.

2.4. Periods of the Adler cubic

Let $X \subset \mathbb{P}^8 = \mathbb{P}(W_9)$ be the Adler cubic sevenfold:

$$\{f_{19} = x_1^2x_6 + x_6^2x_2 + x_2^2x_7 + x_7^2x_4 + x_4^2x_5 + x_5^2x_8 + x_8^2x_9 + x_9^2x_3 + x_3^2x_1 - 2(x_1x_7x_8 + x_2x_3x_5 + x_4x_6x_9) = 0\},$$

and let $H^{p,q} = H^{p,q}(X)$ be the Hodge cohomology spaces of X.

Proposition 8. *The representation of the group $PSL_2(\mathbb{F}_{19})$ on the 84-dimensional space $H^{4,3}(X)^*$ of the Adler cubic X is:*

$$H^{4,3}(X)^* = \overline{W}_9 \oplus W_{18}^1 \oplus W_{18}^3 \oplus W_{19} \oplus W_{20}^3.$$

The group $PSL_2(\mathbb{F}_{19})$ acts trivially on the one-dimensional space $H^{5,2}(X)^*$.

Remark 9. Let a group G act on a vector space V on the left. Then the group G acts on the right on the dual space V^*: for $\ell \in V^*$ and $g \in G$, $\ell \cdot g = \ell \circ g$. Since the traces of the action of g on V and on V^* are equal, the two representations V and V^* are isomorphic. The representation V^* should not be confused with the dual representation of G defined such that the pairing between V and V^* is G-invariant (see [9]).

Proof of Proposition 8. In order to decompose the representation of $PSL_2(\mathbb{F}_{19}) \subset \mathrm{Aut}(X)$ on the dual cohomology space $H^{4,3^*}$, we shall use the identification as representation space

$$H^{4,3}(X) \cong R_3 = S_3/I_3 = \mathrm{Sym}^3 W_9/I_3$$

between $H^{4,3}(X)$ and the graded component of degree 3 in the quotient polynomial ring $S = \mathbf{C}[x_1, ..., x_9]$ by the Jacobian ideal I spanned by the 9 partials of f_{19}, see above.

The space $\mathbb{P}^8$ containing the Adler cubic X is the projectivization of the representation space $W_9 = (S_1)^*$. We have $S_3 = \mathrm{Sym}^3(W_9^*) \cong \mathrm{Sym}^3(W_9)$. Let us decompose $\mathrm{Sym}^3 W_9$ into irreducible representations of $PSL_2(\mathbb{F}_{19})$. The character of the third symmetric power of a representation V is:

$$\chi_{\mathrm{Sym}^3 V}(g) = \frac{1}{6}\big(\chi_V(g)^3 + 3\chi_V\big(g^2\big)\chi_V(g) + 2\chi_V\big(g^3\big)\big),$$

see [14]. Therefore the traces of the action of the elements $1, w_1, w_2$ etc. on $\mathrm{Sym}^3 W_9$ are

$$v = {}^t(165, 3-\nu, 3-\bar{\nu}, 0, 0, 3, 0, 0, 0, 0, 0, 5).$$

Using the character table of $PSL_2(\mathbb{F}_{19})$, we obtain:

$$\mathrm{Sym}^3 W_9 = T_1 \oplus \overline{W}_9 \oplus W_{18}^1 \oplus W_{18}^3 \oplus W_{20}^1 \oplus W_{20}^2 \oplus \big(W_{20}^3\big)^{\oplus 2} \oplus W_{20}^4 \oplus W_{19}.$$

The graded component I_2 of the Jacobian ideal I of X is generated by the 9 derivatives $\frac{df_{19}}{dx_k}$, $k = 1, \dots, 9$. The space I_2 is a representation of $PSL_2(\mathbb{F}_{19})$. The action of w_1 on x_j is the multiplication by ξ^{j^2}, where $\xi = e^{2i\pi/19}$, see Section 1. The action of w_1 on I_2 is then easy to compute. By example, since

$$\frac{df_{19}}{dx_1} = 2x_1x_6 + x_3^2 - 2x_7x_8,$$

we get $w_1 \cdot \frac{df_{19}}{dx_1} = \xi^{-1}\frac{df_{19}}{dx_1}$. By looking at the character table, we obtain that the representation I_2 is $\overline{W}_9$. Therefore $I_3 = W_9 \otimes \overline{W}_9$. Using the fact that for two representations V_1, V_2, their characters satisfy the relation $\chi_{V_1 \otimes V_2} = \chi_{V_1}\chi_{V_2}$, we obtain:

$$I_3 = W_9 \otimes \overline{W}_9 = T_1 \oplus W_{20}^1 \oplus W_{20}^2 \oplus W_{20}^3 \oplus W_{20}^4.$$

Since

$$H^{4,3}(X)^* \cong (S_3/I_3)^* \cong \mathrm{Sym}^3 W_9/I_3$$

then

$$H^{4,3}(X)^* = \overline{W}_9 \oplus W_{18}^1 \oplus W_{18}^3 \oplus W_{19} \oplus W_{20}^3.$$

The action of the simple group $PSL_2(\mathbb{F}_{19}) \subset \mathrm{Aut}(X)$ on $H^{5,2}(X)^* \cong R_0^* \cong \mathbf{C}$ is trivial. □

Let V be a representation of a finite group G and let W be an irreducible representation of G with character χ_W. We know that there exist a uniquely determined integer $a \geq 0$ and a representation V' of G such that W is not a sub-representation of V' and V is (isomorphic to) $W^{\oplus a} \oplus V'$. By [9, formula (2.31), p. 23], the linear endomorphism

$$\psi_W = \frac{\dim W}{|G|} \sum_{g\in G} \overline{\chi_W(g)} g \, : \, V \to V$$

is the projection of V onto the factor consisting of the sum of all copies of W appearing in V i.e. is the projection onto $W^{\oplus a}$. For the trivial representation T, $\chi_T = 1$ and ψ_T is the projection of V onto the invariant space V^G.

Recall that an endomorphism h of a torus Y acts on the tangent space TY of Y by a linear endomorphism dh called the analytic representation of h. The kernel of the map $\mathrm{End}(Y) \to \mathrm{End}(TY)$, $h \to dh$ is the group of translations (see [5]); in the following we will work with endomorphisms up to translation.

Suppose that for some irreducible representations $W_1, \dots, W_k$ the sum

$$\overline{\chi_{W_1}(g)} + \ldots + \overline{\chi_{W_k}(g)}$$

is an integer for every g. Then the endomorphism

$$h = \sum_{g\in G} \big(\overline{\chi_{W_1}(g)} + \ldots + \overline{\chi_{W_k}(g)}\big) g \in \mathrm{End}\big(J(X)\big)$$

is well defined and its analytic representation is

$$dh = \sum_{g\in G} \big(\overline{\chi_{W_1}(g)} + \ldots + \overline{\chi_{W_k}(g)}\big) dg \in \mathrm{End}\big(TJ(X)\big).$$

The tangent space of the image of h (translated to 0) is the image of dh.

Corollary 10. *The torus $J(X)$ has the structure of an Abelian variety and is isogenous to:*

$$E \times A_9 \times A_{36} \times A_{19} \times A_{20}$$

where A_k is a k-dimensional Abelian subvariety of $J_G X$ and $E \subset J_G X$ an elliptic curve. The group $PSL_2(\mathbb{F}_{19})$ acts nontrivially on each factors $A_9, A_{36}, A_{19}, A_{20}$.

Proof. The character of the trivial representation is $\chi_0 = 1$. Let $\chi_1, \chi_2, \chi_3, \chi_4$ be respectively the characters of the representations $W_9 \oplus \overline{W}_9$, $W_{18}^1 \oplus W_{18}^2 \oplus W_{18}^3 \oplus W_{18}^4$, W_{19} and

$W_{20}^1 \oplus W_{20}^2 \oplus W_{20}^3 \oplus W_{20}^4$. By the character table, the numbers $\chi_k(g)$, $g \in PSL_2(\mathbb{F}_{19})$ are integers, therefore we can define

$$q_k = \sum_{g \in PSL_2(\mathbb{F}_{19})} \chi_k(g) g$$

for $k = 0, \ldots, 4$. The analytic representation of q_k is a multiple of the projection onto the subspace $(H^{5,2})^*, W_9, \ldots$ of $TJ(X)$. The images of q_k, $k = 0, \cdots, 4$, are therefore respectively $1, 9, 36, 19, 20$-dimensional sub-tori of $J(X)$, stable by the action of the group ring $\mathbb{Z}[PSL_2(\mathbb{F}_{19})]$ and denoted respectively by $E, A_9, \ldots, A_{20}$.

The image of the endomorphism $q_1 + \ldots + q_4$ (resp. q_0) is a sub-torus whose tangent space is $H^{4,3}(X)^*$ (resp. $H^{5,2}(X)^*$), therefore $H_7(X,\mathbb{Z}) \cap H^{4,3}(X)^*$ is a lattice in $H^{4,3}(X)^*$ (resp. $H_7(X,\mathbb{Z}) \cap H^{5,2}(X)^*$ is a lattice in $H^{5,2}(X)^*$).

Let Q be the Hodge–Riemann form on the tangent space $(H^{5,2} \oplus H^{4,3})^*$. It is positive definite on $H^{5,2}(X)^*$, negative definite on $H^{4,3}(X)^*$, the space $H^{4,3}(X)^*$ is orthogonal to $H^{5,2}(X)^*$ with respect to Q and Q takes integral values on $H_7(X,\mathbb{Z}) \subset (H^{5,2} \oplus H^{4,3})^*$ (see [7, 114]).

Let us define the quadratic form Q' on $(H^{5,2} \oplus H^{4,3})^*$ by $Q' = -Q$ on $H^{4,3}(X)^*$ and $Q' = Q$ on $H^{5,2}(X)^*$. This Q' is a definite quadratic form such that Q' takes integral values on the lattice $\Lambda \subset H^{5,2}(X)^*$ generated by $H_7(X,\mathbb{Z}) \cap H^{4,3}(X)^*$ and $H_7(X,\mathbb{Z}) \cap H^{5,2}(X)^*$. We thus see that $J' = (H^{5,2} \oplus H^{4,3})^*/\Lambda$ is an Abelian variety, and since $J(X)$ is isogenous to J', $J(X)$ is also an Abelian variety. $\square$

3. On the Adler–Klein pencil of cubics

Here we study the Adler–Klein pencil of cubics $X_\lambda = \{f_\lambda = 0\}$,

$$\begin{aligned} f_\lambda = {} & x_1^2 x_6 + x_6^2 x_2 + x_2^2 x_7 + x_7^2 x_4 + x_4^2 x_5 + x_5^2 x_8 + x_8^2 x_9 + x_9^2 x_3 + x_3^2 x_1 \\ & + \lambda(x_1 x_7 x_8 + x_2 x_3 x_5 + x_4 x_6 x_9), \end{aligned}$$

where X_{-2} is the Adler cubic, X_0 is the Klein cubic. Since X_{-2} and X_0 are smooth, the general member of the pencil is smooth.

The automorphism group of X_λ contains the group

$$H = \mathbb{Z}/9\mathbb{Z} \ltimes \mathbb{Z}/19\mathbb{Z}$$

whose law is defined multiplicatively by:

$$(a,b)(c,d) = \big(a + c, 4^c \cdot b + d\big).$$

Remark. The group H is a subgroup of $PSL_2(\mathbb{F}_{19})$: it is the stabilizer of a point in the projective line $\mathbb{P}^1(\mathbb{F}_{19})$ for the action of the simple group $PSL_2(\mathbb{F}_{19})$. Since $\mathbb{P}^1(\mathbb{F}_{19})$ has 20 points, there are therefore 20 such subgroups in $PSL_2(\mathbb{F}_{19})$.

Let us study the representation of H on the tangent space of the intermediate Jacobian of X_λ. Let be $a = (1, 0)$ and $b = (0, 1) \in H$. Let us denote by C_g the conjugacy class of an element $g \in H$. The conjugacy classes of the group H are the 11 classes: C_b, C_{b^2} and C_{a^k}, $k = 0, \cdots, 8$. Let μ be a 9th-primitive root of unity. The group $\mathbb{Z}/9\mathbb{Z}$ is the quotient of H by $\mathbb{Z}/19\mathbb{Z}$, thus the irreducible one-dimensional representation

$$\chi_k : \left| \begin{array}{ccc} \mathbb{Z}/9\mathbb{Z} & \to & \mathbb{C}^* \\ a & \to & \mu^{ka} \end{array} \right. , \quad k \in \{0, \dots, 8\}$$

induces an irreducible representation V_k of H. Let $V_9, \overline{V}_9$ be the restrictions of the representations $W_9, \overline{W}_9$ to $H \subset PSL_2(\mathbb{F}_{19})$. As $171 = 1^2 + \ldots + 1^2 + 9^2 + 9^2$ is equal to the order of H, the representations $V_0, \dots, V_8, V_9, \overline{V}_9$ are the 11 irreducible non-isomorphic representations of H (see [12]). Using the character table of H, we obtain:

Proposition 11. *The restrictions of the representations W_{18}^3, W_{19}, W_{20} of $PSL_2(\mathbb{F}_{19})$ to $H \subset PSL_2(\mathbb{F}_{19})$ are decomposed as follows:*

$$W_{18}^3 = V_9 + \overline{V}_9$$
$$W_{19} = V_0 + V_9 + \overline{V}_9$$
$$W_{20} = V_3 + V_6 + V_9 + \overline{V}_9.$$

Let X_λ be a smooth cubic in the Klein–Adler pencil and let JX_λ be the Griffiths intermediate Jacobian of X_λ.

Corollary 12. *The representation of H on the tangent space to the Griffiths intermediate Jacobian of X_λ is:*

$$TJX_\lambda = V_0 + H^{4,3}(X)^* = 2V_0 + V_3 + V_6 + 4V_9 + 5\overline{V}_9.$$

There exist subtori A_2, B_2, B_{81} of JX_λ of dimension respectively 2, 2 , 81, such that B_2 and B_{81} are Abelian varieties and such that there is an isogeny of complex tori

$$JX_\lambda \to A_2 \times B_2 \times B_{81}.$$

Proof. For the Adler cubic X_{Ad}, the representation of $PSL_2(\mathbb{F}_{19})$ on $H^{4,3}(X_{Ad})$ is

$$H^{4,3}(X_{Ad}) = \overline{W}_9 \oplus W_{18}^1 \oplus W_{18}^3 \oplus W_{19} \oplus W_{20}^3.$$

From Proposition 11, we know the representation of H on each of the factors of $H^{4,3}$. Since H acts also on each X_λ, the representation of H on $H^{4,3}(X_\lambda)$ is the same for all cubics X_λ in the pencil, and we have:

$$R_3 = \mathrm{S}^3 V_9 / I_3 = V_0 + V_3 + V_6 + 4V_9 + 5\overline{V}_9.$$

Thus $TJX_\lambda = (V_0)^{\oplus 2} \oplus V_{83}$. The image of the endomorphism $\sum_{g\in G} g$ is therefore a 2-dimensional torus A_2. By considering the quotient map $JX_\lambda \to JX_\lambda/A_2$, we see that $V_{83} \cap H_7(X_\lambda, \mathbb{Z})$ is a lattice. There is moreover on it a positive definite integral valued form (the restriction of the Hodge–Riemann form), therefore V_{83} is the tangent space of a 83-dimensional Abelian subvariety A_{83} of JX_λ. By using the same arguments as in Section 2, we see that A_{83} is isogenous to a product $B_2 \times B_{81}$ of two Abelian subvarieties. $\square$

Acknowledgments

We thank Bert van Geemen for his comments on a preliminary version of this paper. The second author acknowledges the hospitality of the Seoul National University where part of this work was done. He was supported by the project Geometria Algebrica PTDC/MAT/099275/2008 and FCT grant SFRH/BPD/72719/2010.

References

[1] A. Adler, On the automorphism group of certain hypersurfaces, J. Algebra 72 (1) (1981) 146–165.
[2] A. Albano, A. Collino, On the Griffiths group of the cubic sevenfold, Math. Ann. 299 (1994) 715–726.
[3] A. Beauville, Non-rationality of the symmetric sextic Fano threefold, in: Geometry and Arithmetic, EMS Congress Reports, 2012, pp. 57–60.
[4] A. Beauville, Non-rationality of the S_6-symmetric quartic threefolds, arXiv:1212.5345.
[5] C. Birkenhake, H. Lange, Complex Abelian Varieties, 2nd edition, Grundlehren Math. Wiss., vol. 302, Springer, 2004.
[6] P. Candelas, X. de la Ossa, B. van Geemen, D. van Straten, Lines on the Dwork pencil of quintic threefolds, arXiv:1206.4961.
[7] J. Carlson, M. Green, P. Griffiths, J. Harris, Infinitesimal variation of Hodge structure I, Compos. Math. 50 (2–3) (1983) 109–205.
[8] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, Atlas of Finite Groups, Oxford University Press, 1985, xxxiv+252 pp.
[9] W. Fulton, J. Harris, Representation Theory. A First Course, Grad. Texts in Math., vol. 129, Springer-Verlag, New York, 1991.
[10] A. Iliev, L. Manivel, Fano manifolds of Calabi–Yau type, arXiv:1102.3623.
[11] F. Klein, Uber die Transformation elfter Ordnung der elliptischen Functionen, Math. Ann. 15 (3) (1879) 533–555.
[12] I.G. Macdonald, Symmetric Functions and Hall Polynomials, second edition, Oxford University Press, New York, ISBN 0-19-853489-2, 1995, x+475 pp. With contributions by A. Zelevinsky.
[13] J. Masley, Solution of the class number two problem for cyclotomic fields, Invent. Math. 28 (1975) 243–244.
[14] P. Pulay, X.-F. Zhou, Characters for symmetric and antisymmetric higher powers of representations: Application to the number of anharmonic force constants in symmetrical molecules, J. Comput. Chem. 10 (1989) 935–938.
[15] X. Roulleau, The Fano surface of the Klein cubic threefold, J. Math. Kyoto Univ. 49 (1) (2009) 113–129.