Journal of Number Theory 133 (2013) 2320–2323

Contents lists available at SciVerse ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On the Tate conjecture for the Fano surfaces of cubic threefolds

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ARTICLE INFO

Article history:
Received 12 June 2012
Revised 22 November 2012
Accepted 6 December 2012
Available online 6 March 2013
Communicated by J.-L. Colliot-Thélène

Keywords:
Tate conjecture
Surface of general type
Fano surfaces of lines

ABSTRACT

A Fano surface of a smooth cubic threefold $X \hookrightarrow \mathbb{P}^4$ parametrizes the lines on X. In this note, we prove that a Fano surface satisfies the Tate conjecture over a field of finite type over the prime field and characteristic not 2.

1. Introduction

Let k be a field of finite type over the prime field and let ℓ be a prime integer, prime to the characteristic. We denote by $\bar{k}$ an algebraic closure of k and by G the Galois group $\mathrm{Gal}(\bar{k}/k)$. Let X be a geometrically connected smooth projective variety over k and $\bar{X} := X \times_k \bar{k}$. We denote by $A^1(X)$ the $\mathbb{Q}_\ell$-span of the images of the divisor classes defined over k in the (twisted) étale cohomology group $H^2(\bar{X}, \mathbb{Q}_\ell(1))$. The group G acts on $H^2(\bar{X}, \mathbb{Q}_\ell(1))$ and fixes the subspace $A^1(X)$. The Tate conjecture for divisors is:

Conjecture *(Tate Conjecture). (See [11].) We have $A^1(X) = H^2(\bar{X}, \mathbb{Q}_\ell(1))^G$.*

The Tate conjecture for Abelian varieties has been proved by Tate [12] for finite fields, by Zarhin [14] in characteristic > 2, by Mori [9] in characteristic 2, and by Faltings [7] for fields of characteristic 0. We know a few cases of surfaces that satisfy the Tate conjecture (K3 surfaces, product of two curves, some Picard modular surfaces…).

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0022-314X/$ – see front matter
http://dx.doi.org/10.1016/j.jnt.2012.12.014

In this note, we prove the Tate conjecture for another family of surfaces. Let us suppose that the field k has moreover characteristic $\neq 2$ and let $X \hookrightarrow \mathbb{P}^4_{/k}$ be a smooth cubic hypersurface. The variety that parametrizes the lines on X is a smooth projective surface defined over k called the *Fano surface of lines* of X. This surface S is minimal of general type and has invariants:

$$c_1^2 = 45, \qquad c_2 = 27, \qquad b_1 = 10, \qquad b_2 = 45.$$

We obtain the following result:

Theorem 1. *The Tate conjecture holds for the surface S.*

For the proof we use the fact that the Fano surface is contained in its 5 dimensional Albanese variety A and has class $\frac{1}{3!}\Theta^3$ where Θ is a principal polarization. Using then the Hard Lefschetz Theorem, the Poincaré Duality, and the equality $b_2(\bar{S}) = b_2(\bar{A})$ of Betti numbers, we obtain that the natural map

$$H^2\big(\bar{A}, \mathbb{Q}_\ell(1)\big) \to H^2\big(\bar{S}, \mathbb{Q}_\ell(1)\big)$$

is an isomorphism. Therefore the second étale cohomology group of S is essentially the same as the second étale cohomology group of the Abelian variety A, for which the Tate conjecture is known.

To the knowledge of the author, Abelian surfaces and Fano surfaces are the only known surfaces S such that there is an isomorphism between the second étale cohomology groups of S and its Albanese variety.

2. The proof

Let k be a field finitely generated over its prime field, and let $\bar{k}$ be the algebraic closure of k. Recall [8, Theorem 11.1] that for a smooth n-dimensional projective variety Z over $\bar{k}$, there exists a canonical isomorphism $\eta_Z : H^{2n}(Z, \mathbb{Q}_\ell(n)) \to \mathbb{Q}_\ell$ sending the class of a closed point to 1. Let A be an Abelian variety of dimension $n \geqslant 2$ defined over k.

Definition 2. We say that a 2-dimensional cycle W on A is *non-degenerate* if the $\mathbb{Q}_\ell$-bilinear form:

$$\begin{aligned} Q_W: \quad H^2\big(\bar{A}, \mathbb{Q}_\ell(1)\big) \times H^2\big(\bar{A}, \mathbb{Q}_\ell(1)\big) &\to \mathbb{Q}_\ell \\ (x, y) &\to \eta_{\bar{A}}(x \cdot W \cdot y) \end{aligned}$$

is non-degenerate, where we consider the cycle W in $H^{2n-4}(\bar{A}, \mathbb{Q}_\ell(n-2))$ and $\cdot$ denotes the cup product.

An example of a non-degenerate cycle is:

Proposition 3. *Let Θ be an ample divisor on A. The cycle $\frac{1}{(n-2)!}\Theta^{n-2}$ is non-degenerate.*

Proof. By the Hard Lefschetz Theorem of Deligne [6, Théorème 4.1.1], the cup product induced by Θ^{n-2} induces an isomorphism between $H^2(\bar{A}, \mathbb{Q}_\ell(1))$ and $H^{2n-2}(\bar{A}, \mathbb{Q}_\ell(n-1))$. Moreover, by the Poincaré Duality [5, Chap. VI], the cup-product pairing

$$H^2\big(\bar{A}, \mathbb{Q}_\ell(1)\big) \times H^{2n-2}\big(\bar{A}, \mathbb{Q}_\ell(n-1)\big) \to H^{2n}\big(\bar{A}, \mathbb{Q}_\ell(n)\big) \simeq \mathbb{Q}_\ell$$

is perfect. Combining these two assumptions and the fact that cohomology class Θ^{n-2} is divisible by $(n-2)!$, we get that the cycle $\frac{1}{(n-2)!}\Theta^{n-2}$ is non-degenerate. $\square$

Let S be a smooth surface over k with a k-rational point s_0. Let A be the Albanese variety of S and let $\vartheta : S \to A$ be the Albanese map such that $\vartheta(s_0) = 0$.

Proposition 4. *Suppose that the image W of $\bar{S}$ is a non-degenerate cycle in $\bar{A}$ and $b_2(\bar{S}) = b_2(\bar{A})$. The map*

$$\vartheta^* : H^2(\bar{A}, \mathbb{Q}_\ell) \to H^2(\bar{S}, \mathbb{Q}_\ell)$$

is an isomorphism of Galois modules. The surface S satisfies the Tate conjecture and $\rho_S = \rho_A$, where $\rho_Z = \dim_{\mathbb{Q}_\ell} A_1(Z)$ for a geometrically smooth irreducible variety $Z_{/k}$.

Proof. Let $f : Y \to X$ be a proper map of smooth complete separated varieties over an algebraically closed field. Let be $a = \dim(X)$, $d = \dim(Y)$ and $c = d - a$. By [8, Remark 11.6(d)], there is a linear map

$$f_* : H^r(Y, \mathbb{Z}_\ell) \to H^{r-2c}(X, \mathbb{Z}_\ell)$$

satisfying the projection formula:

$$f_*\big(y \cdot f^* x\big) = f_* y \cdot x, \quad x \in H^r\big(X, \mathbb{Z}_\ell(d)\big),\ y \in H^s(Y, \mathbb{Z}_\ell).$$

Let W be the image of S in its Albanese variety A. Using the projection formula, we have

$$\eta_{\bar{A}}(x \cdot y \cdot \vartheta_* S) = \eta_{\bar{S}}\big(\vartheta^* x \cdot \vartheta^* y \cdot S\big) = \eta_{\bar{S}}\big(\vartheta^* x \cdot \vartheta^* y\big)$$

for $x, y \in H^2(\bar{A}, \mathbb{Q}_\ell(1))$, and we obtain the following equality:

$$\eta_{\bar{S}}\big(\vartheta^* x \cdot \vartheta^* y\big) = (\deg \vartheta)\eta_{\bar{A}}(x.W.y),$$

where $\deg \vartheta \neq 0$ is the degree of ϑ onto its image W. Since $Q_W(x, y) = \eta_{\bar{A}}(x.W.y)$ and Q_W is a non-degenerate pairing, the map

$$\vartheta^* : H^2\big(\bar{A}, \mathbb{Q}_\ell(1)\big) \to H^2\big(\bar{S}, \mathbb{Q}_\ell(1)\big)$$

is injective. As $b_2(\bar{S}) = b_2(\bar{A})$, the map ϑ^* is then an isomorphism of Galois modules and $H^2(\bar{S}, \mathbb{Q}_\ell(1)) \simeq H^2(\bar{A}, \mathbb{Q}_\ell(1))$. Since the Tate conjecture is satisfied for divisors on Abelian varieties over the field k of finite type over the prime field, we have $\rho_A = \dim H^2(\bar{A}, \mathbb{Q}_\ell(1))^G$. Since ϑ^* is injective, we have $\rho_S \geqslant \rho_A$. On the other hand, for every variety X over k, we have $\dim H^2(\bar{X}, \mathbb{Q}_\ell(1))^G \geqslant \rho_X$ therefore:

$$\dim H^2\big(\bar{S}, \mathbb{Q}_\ell(1)\big)^G \geqslant \rho_S \geqslant \rho_A = \dim H^2\big(\bar{A}, \mathbb{Q}_\ell(1)\big)^G,$$

we thus obtain $\rho_S = \dim H^2(\bar{S}, \mathbb{Q}_\ell(1))^G = \rho_A$ and the Tate conjecture holds for S. □

Let us suppose that the field k has characteristic not 2. Let X be a smooth cubic hypersurface defined over the field k and let S be its Fano surface. The surface S is a smooth geometrically connected variety defined over k [2, Theorem 1.16 i and (1.12)].

Let us suppose that the cubic X contains a k-rational line L_0 such that for every line L' (defined over $\bar{k}$) in X meeting L_0, the plane containing L and L_0 cuts out on $\bar{X}$ three distinct lines. Proposition (1.25) in [10] ensures that such a line L_0 exists on a finite extension of k. Since the Tate conjecture for S over k holds if and only if it holds over any finite extension of k (see [13, Theorem 2.9]), this assumption on the existence of L_0 is not a restriction.

The Albanese variety A of S is defined over k [1, Lemma 3.1] and is 5 dimensional. Let $\vartheta : S \to A$ be the Albanese map such that $\vartheta(s_0) = 0$, where s_0 is the point of the Fano surface corresponding to L_0. Let Θ be the (reduced) image of $S \times S$ by the map $(s_1, s_2) \to \vartheta(s_1) - \vartheta(s_2)$. The variety Θ is a divisor on A defined over k and (A, Θ) is a principally polarized Abelian variety ([3, Proposition 5]; we checked that although [3] deals with an algebraically closed field, the assumption on the existence of L_0 ensures that it remains true for the field k). The Albanese map $\vartheta : S \to A$ is an embedding and the class of $\bar{S}$ in $\bar{A}$ is $\frac{1}{3!}\Theta^3$ [3, Corollaire of §4, and Proposition 7]. Moreover, by [4, p. 11], $b_2(\bar{S}) = b_2(\bar{A}) = 45$.

We thus see that the Fano surface S of X satisfies the hypothesis of Proposition 4 and therefore Theorem 1 holds.

Acknowledgments

The author gratefully thanks Marc-Hubert Nicole for his encouragements to work on this question. He thanks also the Editor and the Referee for their numerous suggestions to improve the exposition of this paper, and Stavros Papadakis, Alessandra Sarti, Matthias Schütt for their comments.

The author has been supported under FCT grant SFRH/BPD/72719/2010 and project Géometria Algébrica PTDC/MAT/099275/2008.

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