The Ellis semigroup of bijective substitution shifts

(Joint work with Johannes Kellendonk)

Given a substitution $\theta : \mathcal{A} \to \mathcal{A}^\ell$ of length ℓ on a finite alphabet $\mathcal{A}$, the substitution shift (X_θ, σ) is the topological dynamical system where X_θ consists of bi-infinite sequences all of whose subwords belong to θ's language, and σ is the left shift map, which defines a $\mathbb{Z}$-action on X_θ. As for any topological dynamical system we can study the *Ellis semigroup* $E(X_\theta, \sigma)$ of (X_θ, σ) (also called the enveloping semigroup). Elements of $E(X_\theta, \sigma)$ are functions on X_θ which are pointwise limits of some net (σ^{n_k}). In other words, $E(X_\theta, \sigma)$ is the compactification of the group action generated by σ in the topology of pointwise convergence on the space $X_\theta^{X_\theta}$. The Ellis semigroup is typically a huge beast, and its computation has been restricted mainly to systems (X, σ) which are metrically group rotations that are *tame*, meaning that the Ellis semigroup has cardinality at most that of the continuum.

In this talk we give an explicit algebraic description of the Ellis semigroup for the family of bijective substitution shifts. These systems are not tame. Along the way we classify the idempotents, left and right ideals, and the kernel of $E(X_\theta, \sigma)$. We also discover a notion of *generalised height* of a substitution, which generalises the classical notion defined by Kamae and Dekking.