Combinatorics on words for Markoff numbers

One World Combinatorics on Words Seminar

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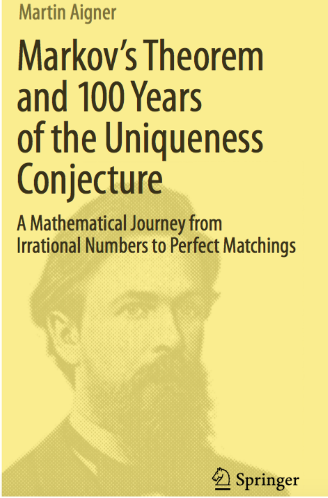
More than 100 years



From CHRISTOFFEL WORDS to MARKOFF NUMBERS

Christophe Reutenauer

OXFORD

Outline

Dirichlet's Theorem

Theorem [Dirichlet, 1855]

Let $\alpha \in \mathbb{R}$ and $N \in \mathbb{N}$. There exists $\frac{p}{q} \in \mathbb{Q}$ with $q \leq N$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qN} \quad \left(\leq \frac{1}{q^2}\right).$$

Corollary

If $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, then there are infinitely many $\frac{p}{q} \in \mathbb{Q}$ with

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^2}.$$

Number Theory

A complex number α is called *algebraic of degree d* if α is the root of a polynomial with integer coefficients of degree d, and d is the smallest degree for which such a polynomial exists.

$\sqrt{2}$ is algebraic of degree 2 because it is solution of $x^2 - 2 = 0$.

Number Theory

A complex number α is called *algebraic of degree d* if α is the root of a polynomial with integer coefficients of degree d, and d is the smallest degree for which such a polynomial exists.

$\sqrt{2}$ is algebraic of degree 2 because it is solution of $x^2 = 2$.

If there is no such polynomial for any degree, then α is called *transcendental*.

For example,

$$\sum_{n=1}^{\infty} \frac{1}{10^{n!}} = 0.110001000000000000000001000000\cdots, e \text{ and } \pi$$

are transcendental.

Liouville's Theorem

Theorem [Liouville,1836]

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be algebraic of degree d. Then there is a constant $C > 0$ such that

$$\frac{C}{q^d} < \left|\alpha - \frac{p}{q}\right| \text{ for all } \frac{p}{q} \in \mathbb{Q}.$$

This implies a nice link between rational approximation and degree of the algebraic number.

Roth's Theorem

Theorem [Liouville,1836]

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be algebraic of degree d. Then there is a constant $C > 0$ such that

$$\frac{C}{q^d} < \left|\alpha - \frac{p}{q}\right| \text{ for all } \frac{p}{q} \in \mathbb{Q}.$$

On the opposite

Theorem [Roth,1955]

Let α be a real number and $\epsilon > 0$. If there are infinitely many $\frac{p}{q} \in \mathbb{Q}$ with

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$$

then α is transcendental.

Lagrange spectrum

In general we cannot improve the exponent 2...

Consider all real numbers $L > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Lq^2}$$

holds for infinitely many $\frac{p}{q} \in \mathbb{Q}$.

Definition

Given $\alpha \in \mathbb{R}$, $L(\alpha) = \sup L$ over all L that satisfy $\left|\alpha - \frac{p}{q}\right| < \frac{1}{Lq^2}$ is called the *Lagrange number* of α.

$\mathcal{L} = \{L(\alpha) : \alpha \in \mathbb{R}\backslash\mathbb{Q}\}$ is the Lagrange spectrum.

The Lagrange spectrum below 3 is
$\mathcal{L}_{<3} = \{L(\alpha) \in \mathcal{L} : L(\alpha) < 3\}$.

First value of the Lagrange spectrum

Theorem [Hurwitz,1891]

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Then there are infinitely many rational numbers $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

And in fact $L\left(\frac{1+\sqrt{5}}{2}\right) = \sqrt{5}$.

Thus $\frac{1+\sqrt{5}}{2}$ is the most badly approximated quadratic number and $\sqrt{5}$ is the first element of the Lagrange spectrum ...

First value of the Lagrange spectrum

In fact $L\left(\frac{1+\sqrt{5}}{2}\right) = \sqrt{5}$.

Thus $\frac{1+\sqrt{5}}{2}$ is the most badly approximated quadratic number and $\sqrt{5}$ is the first element of the Lagrange spectrum ...

In fact all cousins of the golden ratio are badly approximated : two real numbers α and β are cousin if $\beta = \frac{a\alpha+b}{c\alpha+d}$ where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = \pm 1$.
Using homography transformations we find the same Lagrange number : if two real numbers α and β are cousin then $L(\alpha) = L(\beta)$.

Second value of the Lagrange spectrum

If we exclude $\frac{1+\sqrt{5}}{2}$ and all its cousins, then we find the second Lagrange number :

Theorem [Markoff, 1880]

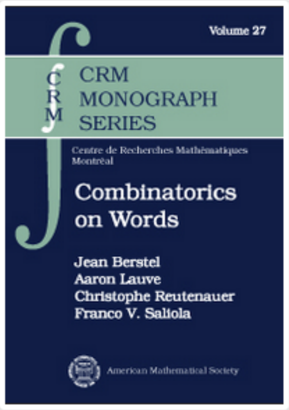
Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ which is neither $\frac{1+\sqrt{5}}{2}$ nor a cousin of $\frac{1+\sqrt{5}}{2}$. Then there are infinitely many rational numbers $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{8}q^2}.$$

Thus $\sqrt{8}$ is the second value of the Lagrange spectrum and in fact $L(1+\sqrt{2}) = \sqrt{8}$ is the second badly approximated quadratic number.

We could find the third value of the Lagrange spectrum which is $\frac{\sqrt{221}}{5}$ and so on ...

Link with continued fractions



Link with continued fractions

Suppose $\alpha \in \mathbb{R}$. The (simple) **continued fraction representation** of α is the sequence of integers $a_0, a_1, a_2, \ldots$ constructed recursively as follows: let

$$\beta_0 = \alpha \quad \text{and} \quad a_0 = \lfloor \beta_0 \rfloor;$$

if $i > 0$ and $a_{i-1} \neq \beta_{i-1}$, then let

$$\beta_i = \frac{1}{\beta_{i-1} - a_{i-1}} \quad \text{and} \quad a_i = \lfloor \beta_i \rfloor;$$

if $i > 0$ and $a_{i-1} = \beta_{i-1}$, then the recursion terminates. The continued fraction representation of α is commonly denoted by

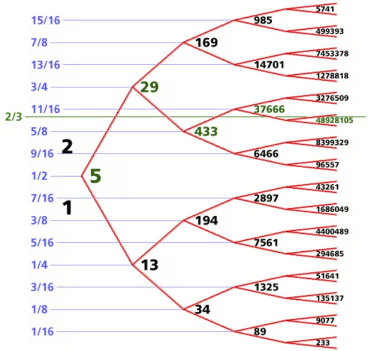
$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

or more compactly by $\alpha = [a_0, a_1, a_2, a_3, \ldots]$.

$\frac{1+\sqrt{5}}{2} = [1, 1, \cdots, 1, \cdots]$; $1 + \sqrt{2} = [2, 2, \cdots, 2, \cdots]$; $[1, 2, \cdots, 1, 2, \cdots]$.

Theorem [Markoff, 1880]

Let $\mathcal{M} =$
$\{1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, \cdots\}$
be the sequence of Markoff numbers. The Lagrange spectrum below 3 is given by $\mathcal{L}_{<3} = \left\{ \frac{\sqrt{9m^2-4}}{m} : m \in \mathcal{M} \right\}$.



Markoff's Theorem

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be the sequence of Markoff numbers. The Lagrange spectrum below 3 is given by

$$\mathcal{L}_{<3} = \left\{ \frac{\sqrt{9m^2 - 4}}{m} : m \in \mathcal{M} \right\}.$$

And to define the sequence of Markoff numbers, we consider the solutions of the following Diophantine equation :

$$x^2 + y^2 + z^2 = 3xyz \text{ with } x, y, z \in \mathbb{N}^*.$$

First triples

The triplets (1,1,1) and (1,1,2) are solutions of the equation

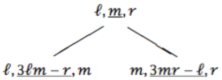
$$x^2 + y^2 + z^2 = 3xyz \text{ with } x, y, z \in \mathbb{N}^*.$$

The triple (1,2,5) is a solution with non repeated values.
In fact, if we note the maximal value in the middle of the triples, we find a recursive rule in order to generate new solutions :

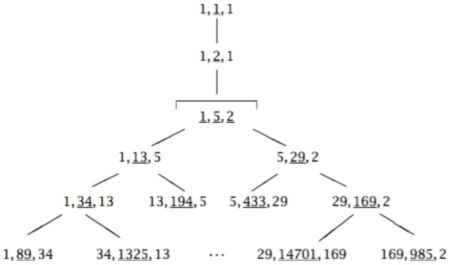
$$\ell, \underline{m}, r$$

$$\ell, \underline{3\ell m - r}, m \qquad m, \underline{3mr - \ell}, r$$

Markoff tree

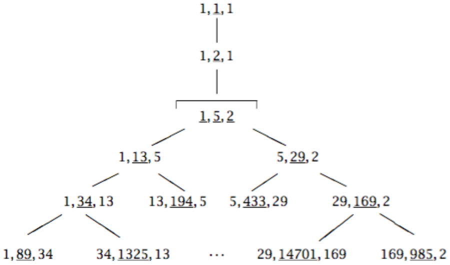


The recursive rule gives birth to an infinite tree



Frobenius' conjecture

All the Markov numbers appear in the Markoff tree.



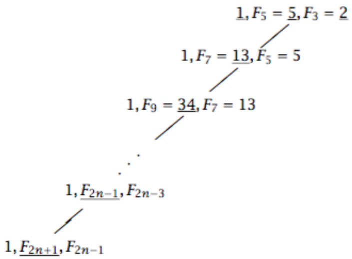
The **Frobenius' conjecture** asserts that :

Conjecture [Frobenius,1913]

Each Markoff number is the **maximum of a unique Markoff triple** in the Markoff tree.

Left branch of the Markoff tree

Consider the Fibonacci numbers F_n, with $F_0 = 0, F_1 = 1$, and the recurrence $F_{n+1} = F_n + F_{n-1}\ \ (n \geq 1)$.
The first values are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$ and F_{2n+1} with $n \geq 1$ are in the left branch of the Markoff tree :



Right branch of the Markoff tree

Consider the Pell numbers P_n, with $P_0 = 0, P_1 = 1$, and the recurrence $P_{n+1} = 2P_n + P_{n-1} \ (n \geq 1)$.
The first values are $0, 1, 2, 5, 12, 29, 70, 169, 408, 985 \cdots$ and P_{2n+1} with $n \geq 1$ are in the right branch of the Markoff tree :

$$P_1 = \underline{1}, P_3 = \underline{5}, \underline{2}$$

$$\searrow$$

$$P_3 = 5, P_5 = \underline{29}, 2$$

$$\ddots$$

$$\searrow$$

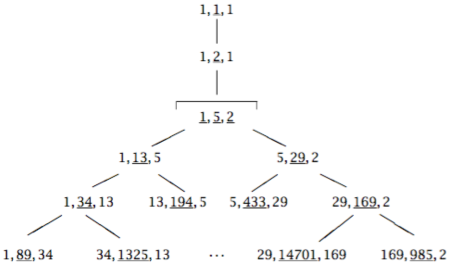
$$P_{2n-3}, \underline{P_{2n-1}}, 2$$

$$\searrow$$

$$P_{2n-1}, \underline{P_{2n+1}}, 2$$

Fibonacci and Pell numbers

The **Frobenius' conjecture** asserts that each Markoff number is the **maximum of a unique Markoff triple** in the Markoff tree !



It is not easy to prove that the sets $\{F_{2n+1} : n \geq 1\}$ and $\{P_{2n+1} : n \geq 1\}$ are distinct except for the value 5 (see the paper in 2009 of Y. Bugeaud, C. Reutenauer, S. Siksek).

Palindrome

Definition

A *word* w is a finite sequence of letters $w_1 w_2 \cdots w_n$ on a finite alphabet Σ.
The set of all finite words on the alphabet Σ is denoted by Σ^*.

$w = aabb$ is a word on the alphabet $\Sigma = \{a, b\}$.

Definitions

A *prefix* of a word $w = w_1 w_2 \cdots w_n$ with $w_i \in \Sigma$ is a word $p = w_1 w_2 \cdots w_j$ where $j \leq n$.
The *reversal* of a word $w = w_1 w_2 \cdots w_n$ with $w_i \in \Sigma$ is the word $\widetilde{w} = w_n w_{n-1} \cdots w_1$.
A word w is a *palindrome* if it is equal to its reversal (that is $w = \widetilde{w}$).

$w = aba$ is a palindrome on the alphabet $\Sigma = \{a, b\}$.
$p = ab$ is a prefix of w.

Palindromic closure

Definition [de Luca, 1997]

The palindromic closure of a word x on the alphabet $\Sigma = \{a, b\}$ is the shortest palindrome having x as a prefix, it exists and is unique, it is denoted by $x^{(+)}$.

For example, if $x = a$, then $x^{(+)} = a$ because x is a palindrome. If $x = ab$, then $x^{(+)} = aba$.

It is known that $x^{(+)} = x'y\widetilde{x'}$ where $x = x'y$ with y the longest palindrome suffix of x. A *suffix* of a word $w = w_1 w_2 \cdots w_n$ with $w_i \in \Sigma$ is a word $s = w_j w_{j+1} \cdots w_n$ where $j \geq 1$.

Iterated palindromic closure

Definition [de Luca, 1997]

We consider the iterated palindromic closure, denoted by $\mathrm{Pal}(d)$, is defined recursively by $\mathrm{Pal}(d_1 d_2 \cdots d_n) = (\mathrm{Pal}(d_1 d_2 \cdots d_{n-1}) d_n)^{(+)}$, $d_i \in \Sigma$, with the initial condition $\mathrm{Pal}(\epsilon) = \epsilon$, where ϵ denotes the empty word. The word d is called the *directive word* of $\mathrm{Pal}(d)$.

For example $\mathrm{Pal}(aba) = abaaba$; indeed
$\mathrm{Pal}(a) = a$ and
$\mathrm{Pal}(ab) = (\mathrm{Pal}(a)b)^{(+)} = (ab)^{(+)} = aba$ and then
$\mathrm{Pal}(aba) = (\mathrm{Pal}(ab)a)^{(+)} = (abaa)^{(+)} = abaaba = \underline{a}\ \underline{b}a\ \underline{a}ba$.

Thue-Morse substitution and the main theorem

We also use the *Thue-Morse substitution*, denoted by $\theta = (ab, ba)$ that maps the letter a to ab and the letter b to ba.

Theorem[C. Reutenauer, LV, 2017]

For each word $v \in \{a, b\}^*$, the number $|\mathrm{Pal} \circ \theta \circ \mathrm{Pal}(av)| + 2$ is a Markoff number $\neq 1, 2$.
The mapping defined in this way from $\{a, b\}^*$ into the set of Markoff numbers different from $1, 2$ is surjective. Injectivity of this mapping is equivalent to the Frobenius' conjecture.

The Markoff number $m = 5$ is given by $v = \epsilon$; indeed $\mathrm{Pal}(a) = a$, thus $\theta \circ \mathrm{Pal}(a) = ab$ and then $\mathrm{Pal} \circ \theta \circ \mathrm{Pal}(a) = \mathrm{Pal}(ab) = aba$, which is of length 3.

Thue-Morse substitution and the main theorem

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Theorem[C. Reutenauer, LV, 2017]

For each word $v \in \{a, b\}^*$, the number $|\mathrm{Pal} \circ \theta \circ \mathrm{Pal}(av)| + 2$ is a Markoff number $\neq 1, 2$.

The Markoff number $m = 13$ is given by $v = a$; indeed $\mathrm{Pal}(aa) = aa$, thus $\theta \circ \mathrm{Pal}(aa) = abab$ and then $\mathrm{Pal} \circ \theta \circ \mathrm{Pal}(aa) = \mathrm{Pal}(abab) = \underline{a}\,\underline{b}a\,\underline{a}ba\,\underline{b}aaba$, which is of length 11.

Thue-Morse substitution and the main theorem

We also use the *Thue-Morse substitution*, denoted by $\theta = (ab, ba)$ that maps the letter a to ab and the letter b to ba.

Theorem[C. Reutenauer, LV, 2017]

For each word $v \in \{a, b\}^*$, the number $|\mathrm{Pal} \circ \theta \circ \mathrm{Pal}(av)| + 2$ is a Markoff number $\neq 1, 2$.

The Markoff number $m = 29$ is given by $v = b$ indeed $\mathrm{Pal}(ab) = aba$, thus $\theta \circ \mathrm{Pal}(aba) = abbaab$ and then $\mathrm{Pal} \circ \theta \circ \mathrm{Pal}(ab) = \mathrm{Pal}(abbaab) =$
$\underline{a}\ \underline{b}a\ \underline{b}a\ \underline{a}baba\ \underline{a}baba\ \underline{b}aababaababa$, which is of length 27.

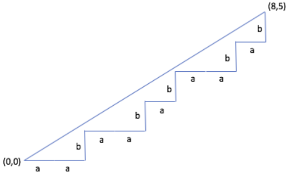
Christoffel words

The Markoff number $m = 13$ is given by $v = a$ and
$\mathrm{Pal} \circ \theta \circ \mathrm{Pal}(aa) = \underline{a}\ \underline{b}a\ \underline{a}ba\ \underline{b}aaba$.
The word $a\,\mathrm{Pal} \circ \theta \circ \mathrm{Pal}(av)b$ is a Christoffel word :

Definition

A Christoffel word coding of a discrete segment from $(0, 0)$ to (i, j) where i and j are co-prime (i.e. $\gcd(i, j) = 1$).

For $v = a$ we compute $C = a\ abaababaaba\ b$



More on Christoffel words

Definition 1.2. Suppose $a \perp b$ and $(a,b) \neq (0,1)$. The **label** of a point (i,j) on the (lower) Christoffel path of slope $\frac{b}{a}$ is the number $\frac{ib-ja}{a}$. That is, the label of (i,j) is the vertical distance from the point (i,j) to the line segment from $(0,0)$ to (a,b).

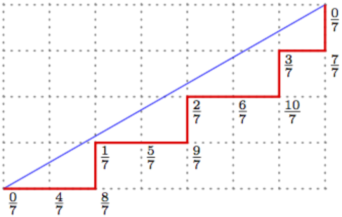


FIGURE 1.3: The labels of the points on the Christoffel path of slope $\frac{4}{7}$.

Definition 1.4. Suppose $a \perp b$. Consider the Cayley graph of $\mathbb{Z}/(a+b)\mathbb{Z}$ with generator b. It is a cycle, with vertices $0, b, 2b, 3b, \ldots, a, 0 \bmod (a+b)$. Starting from zero and proceeding in the order listed above,

(i) label those edges (s,t) satisfying $s < t$ by x;

(ii) label those edges (s,t) satisfying $s > t$ by y;

(iii) read edge-labels in the prescribed order, i.e., $0 \xrightarrow{x} b \xrightarrow{*} \cdots \xrightarrow{*} a \xrightarrow{y} 0$.

The **lower Christoffel word** of slope $\frac{b}{a}$ is the word $x \cdots y$ formed above.

Example. Pick $a = 7$ and $b = 4$. Figure 1.4 shows the Cayley graph of $\mathbb{Z}/11\mathbb{Z}$ with generator 4 and edges $u \to v$ labelled x or y according to whether or not $u < v$. Reading the edges clockwise from 0 yields the word $xxyxxyxxyxy$,

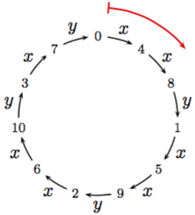


FIGURE 1.4: The Cayley graph of $\mathbb{Z}/(7{+}4)\mathbb{Z}$ with generator 4 and the associated Christoffel word.

Example. From the previous example, the continued fraction representation of $\frac{10}{23}$ is given by the sequence $[0, 2, 3, 3]$. Therefore, the continuants of $\frac{10}{23}$ are

$$0 + \frac{1}{2} = \mathbf{\frac{1}{2}}, \quad 0 + \cfrac{1}{2 + \cfrac{1}{3}} = \mathbf{\frac{3}{7}}, \quad 0 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{3}}} = \mathbf{\frac{10}{23}}. \tag{7.1}$$

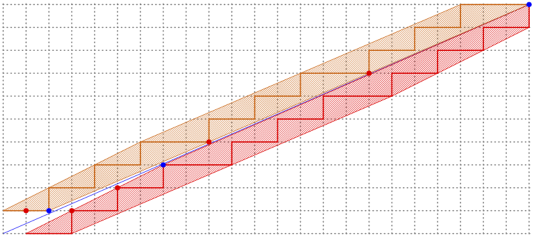


FIGURE 7.1: The convex hulls of the lower Christoffel path from $(0, 1)$ to $(23, 10)$ and the upper Christoffel path from $(1, 0)$ to $(23, 10)$.

Computation of Markoff numbers

Theorem[C. Reutenauer, LV, 2017]

Consider $d = \theta \circ \mathrm{Pal}(av)$ with $v \in \{a,b\}^*$. We write $d = d_1 d_2 \cdots d_{|d|}$ with $d_i \in \{a,b\}$. We let $L_0 = L_1 = 1$ and $L_2 = L_1 + L_0 = 2$. For $j \geq 3$ we define recursively the L_j :

$$L_j = \begin{cases} L_{j-1} & \text{if } d_j = d_{j-1}, \\ L_{j-1} + L_{j-2} & \text{if } d_j \neq d_{j-1} \neq d_{j-2}, \\ L_{j-1} + L_{j-2} + L_{j-3} & \text{if } d_j \neq d_{j-1} = d_{j-2}. \end{cases}$$

Then the Markoff number m_v is given by

$$m_v = 1 + \sum_{j=0}^{|d|} L_j.$$

Computation of Markoff numbers

Consider $d_v = \theta \circ \mathrm{Pal}(av)$ with $v \in \{a, b\}^*$.

$$L_j = \begin{cases} L_{j-1} & \text{if } d_j = d_{j-1}, \\ L_{j-1} + L_{j-2} & \text{if } d_j \neq d_{j-1} \neq d_{j-2}, \\ L_{j-1} + L_{j-2} + L_{j-3} & \text{if } d_j \neq d_{j-1} = d_{j-2}. \end{cases}$$

$d_\epsilon = \theta(a) = ab$ and then $\begin{array}{ccc} d_\epsilon = & a & b \\ 1 & 1 & 2 \end{array}$ and thus

$m_\epsilon = 1 + (1 + 1 + 2) = 5.$

Computation of Markoff numbers

Consider $d_v = \theta \circ \mathrm{Pal}(av)$ with $v \in \{a, b\}^*$.

$$L_j = \begin{cases} L_{j-1} & \text{if } d_j = d_{j-1}, \\ L_{j-1} + L_{j-2} & \text{if } d_j \neq d_{j-1} \neq d_{j-2}, \\ L_{j-1} + L_{j-2} + L_{j-3} & \text{if } d_j \neq d_{j-1} = d_{j-2}. \end{cases}$$

$d_\epsilon = \theta(a) = ab$ and then $\begin{array}{ccc} d_\epsilon = & a & b \\ 1 & 1 & 2 \end{array}$ and thus

$m_\epsilon = 1 + (1 + 1 + 2) = 5$.

$d_a = \theta(aa) = abab$ and then $\begin{array}{ccccc} d_a = & a & b & a & b \\ 1 & 1 & 2 & 3 & 5 \end{array}$

and thus $m_a = 1 + (1 + 1 + 2 + 3 + 5) = 13$.

Computation of Markoff numbers

$$L_j = \begin{cases} L_{j-1} & \text{if } d_j = d_{j-1}, \\ L_{j-1} + L_{j-2} & \text{if } d_j \neq d_{j-1} \neq d_{j-2}, \\ L_{j-1} + L_{j-2} + L_{j-3} & \text{if } d_j \neq d_{j-1} = d_{j-2}. \end{cases}$$

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$d_a = \theta(aa) = abab$ and then $\begin{array}{ccccc} d_a = & a & b & a & b \\ 1 & 1 & 2 & 3 & 5 \end{array}$

and thus $m_a = 1 + (1 + 1 + 2 + 3 + 5) = 13.$

$d_{aa} = \theta(aaa) = ababab$ and then

$\begin{array}{ccccccc} d_{aa} = & a & b & a & b & a & b \\ 1 & 1 & 2 & 3 & 5 & 8 & 13 \end{array}$ and thus

$m_{aa} = 1 + (1 + 1 + 2 + 3 + 5 + 8 + 13) = 34.$

The directive words $d_{aa\cdots a}$ compute the Markoff numbers that are Fibonacci numbers...

Computation of Markoff numbers

Consider $d_v = \theta \circ \mathrm{Pal}(av)$ with $v \in \{a, b\}^*$.

$$L_j = \begin{cases} L_{j-1} & \text{if } d_j = d_{j-1}, \\ L_{j-1} + L_{j-2} & \text{if } d_j \neq d_{j-1} \neq d_{j-2}, \\ L_{j-1} + L_{j-2} + L_{j-3} & \text{if } d_j \neq d_{j-1} = d_{j-2}. \end{cases}$$

$d_b = \theta(aba) = abbaab$ and then

$$\begin{array}{cccccccc} d_b = & & a & b & b & a & a & b \\ & 1 & 1 & 2 & 2 & 5 & 5 & 12 \end{array}$$

and thus

$m_{aa} = 1 + (1 + 1 + 2 + 2 + 5 + 5 + 12) = 29.$

The directive words d_{b^k} compute the Markoff numbers that are Pell numbers...

Computation of Markoff numbers

Consider $d_v = \theta \circ \mathrm{Pal}(av)$ with $v \in \{a, b\}^*$.

$$L_j = \begin{cases} L_{j-1} & \text{if } d_j = d_{j-1}, \\ L_{j-1} + L_{j-2} & \text{if } d_j \neq d_{j-1} \neq d_{j-2}, \\ L_{j-1} + L_{j-2} + L_{j-3} & \text{if } d_j \neq d_{j-1} = d_{j-2}. \end{cases}$$

$d_b = \theta(aba) = abbaab$ and then

$$\begin{array}{cccccccc} d_b = & & a & b & b & a & a & b \\ & 1 & 1 & 2 & 2 & 5 & 5 & 12 \end{array}$$

and thus

$m_{aa} = 1 + (1 + 1 + 2 + 2 + 5 + 5 + 12) = 29.$

The directive words d_{b^k} compute the Markoff numbers that are Pell numbers...

$d_{ab} = \theta(aabaa) = ababbaabab$ and then

$$\begin{array}{ccccccccccccc} d_{ab} = & & a & b & a & b & b & a & a & b & a & b \\ & 1 & 1 & 2 & 3 & 5 & 5 & 13 & 13 & 31 & 44 & 75 \end{array}$$

$m_{ab} = 194.$

Conclusion

We are able to compute all the Markoff numbers by Iterated palindromic closures.

How to prove that the words $\mathrm{Pal} \circ \theta \circ \mathrm{Pal}(av)$ have two by two distinct lengths ?

What are the properties of the L_i's ?

How to use discrete geometry and Christoffel words in order to prove the Frobenius' conjecture ?