Hidden automatic sequences

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(joint work with Michel Dekking and Martine Queffélec)

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1. Quickly recalling some notions

Recall the following definitions: an *alphabet* $\mathcal{A}$ is a finite set, whose elements are called *letters*. The set of all *words* (i.e., finite sequences) on $\mathcal{A}$, denoted $\mathcal{A}^*$, and equipped with *concatenation* of words has a structure of (free) monoid. The *length* of a word is the total number of its letters (the *empty word* has length 0).

A *morphism* from monoid $\mathcal{A}^*$ to monoid $\mathcal{B}^*$ is a map that preserves concatenation. It is determined by its values on $\mathcal{A}$. It is called *uniform* or *of constant length (ℓ)* if all the images of letters have the same length (ℓ).

If $\mathcal{A} = \mathcal{B}$, let φ be a morphism from $\mathcal{A}^*$ to itself. If there exists a letter $a \in \mathcal{A}$ such that $\varphi(a)$ begins with a and the length of the iterates of φ on a, $\varphi^k(a)$, tends to infinity with k, the sequence $\varphi^k(a)$ tends to an infinite sequence that is its limit for a natural topology (the topology of simple convergence) and is an *(iterative) fixed point* of φ for the extension (by continuity) of φ to infinite sequences on $\mathcal{A}$.

This iterative fixed point is called a *purely morphic* sequence.

Example 1:

Thue-Morse sequence:

$\mathcal{A} = \{0, 1\}$.

Morphism $0 \to 01$, $1 \to 10$. Can be iterated on, say, 0:

0
0 1
0 1 1 0
0 1 1 0 1 0 0 1
...

Example 2:

(Binary) Fibonacci sequence:

$\mathcal{A} = \{0, 1\}$.

Morphism $0 \to 01$, $1 \to 0$. Can be iterated on 0:

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Morphic and automatic sequences

A sequence $(a_n)_{n \geq 0}$ with values in some alphabet $\mathcal{A}$ is called *morphic* if there exists an alphabet $\mathcal{B}$, a morphism $\varphi : \mathcal{B}^* \to \mathcal{B}^*$ admitting an iterative fixed point say the sequence $(b_n)_{n \geq 0}$ on $\mathcal{B}$, and a map $f : \mathcal{B} \to \mathcal{A}$ such that for all $n \geq 0$, one has $a_n = f(b_n)$.

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If, furthermore, the morphism φ has constant length (ℓ), then the sequence $(a_n)_{n \geq 0}$ is called *(ℓ-)automatic*.

Example 3

The Golay-Shapiro sequence (or the Rudin-Shapiro sequence*) is the 2-automatic sequence defined as follows.

Alphabet $\{0, 1\}$. "Auxiliary" alphabet $\mathcal{B} := \{a, b, c, d\}$.

Morphism on $\mathcal{B}$, $\varphi : a \to ab,\ b \to ac,\ c \to db,\ d \to dc$,

Fixed point of φ: $\lim \varphi^n(a) = a\ b\ a\ c\ a\ b\ d\ b \ldots$.

f is the map: $a \to 0,\ b \to 0,\ c \to 1,\ d \to 1$.

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$(*)$ GS versus RS...

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2. Beginning(s) of the story

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[2] The Lysënok morphism provides a presentation by generators and (infinitely many) defining relations of the first Grigorchuk group.

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Other examples were given by Dekking in 1978 (see below).

2. A first idea: the *Anagram theorem*

We played with some examples of morphic sequences given in (or inspired by) the OEIS. For example let ψ be the morphism

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Defining a new alphabet $\{w'_1, w'_2\}$ and a new morphism ψ' by: $\psi'(w'_1) = w'_1 w'_2 w'_2 w'_1$, $\psi'(w'_2) = w'_1 w'_1 w'_2 w'_2$, it is clear that the fixed point of ψ is the image of the fixed point of ψ' beginning with w'_1 by the map $w'_1 \to 01$, $w'_2 \to 10$. Hence the fixed point of ψ is 4-automatic (hence 2-automatic).

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Anagram Theorem. *Let W be a finite set of anagrams on the alphabet $\mathcal{A}$. Let ψ a morphism on $\mathcal{A}$, admitting an infinite iterative fixed point, and such that the image by ψ of any letter in $\mathcal{A}$ is a concatenation of words in W. Then the fixed point is d-automatic, where $d = \text{length}(\psi(w))/\text{length}(w)$ for any/all $w \in W$.*

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3. A theorem of Dekking

It happens that the anagram theorem is superseded by an "old" theorem of Dekking, 1978 (Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete / Probability Theory and Related Fields).

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Theorem (Dekking). *Let φ be a morphism on the alphabet $[0, r-1]$, non-erasing, and admitting an iterative fixed point $(a_n)_{n \geq 0}$. For all $i \in [0, r-1]$ let ℓ_i be the length of the word $\varphi(i)$. If the vector $(\ell_0, \ell_1, \ldots \ell_{r-1})$ is a left eigenvector of the incidence matrix (transition matrix) M of φ, then the sequence $(a_n)_{n \geq 0}$ is q-automatic, where q is the spectral radius of M.*

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Exercise. Prove that the anagram theorem is implied by the theorem of Dekking above.

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(Prove that the fixed point of $0 \rightarrow 10,\ 1 \rightarrow 1100$ is 3-automatic.)

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For example, the Grigorchuk group can be defined as a group generated by an automaton with five states over an alphabet of two letters. This group is a particularly simple example of an infinite finitely generated torsion group and is the first example of a group whose growth is intermediate between polynomial and exponential.

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(v) Generalize.

Thank you! / Merci !

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– Last but not least: generalization of Cobham by F. Durand.

Proving that a sequence is not automatic (in preparation, JPA-J. Shallit)

– Infinite q-kernels and many variations: Ex.: $a \to aab, \ b \to b$ (cf. $v_n := u_{2^k n + 2^k - k}$).

– Irrationality of frequencies. Ex.: $0 \to 01, \ 1 \to 0$.

– Block complexity. Ex.: Pascal triangle modulo d, with $d \neq p^{\alpha}$.

– Growth, gaps, etc. Ex.: Characteristic function of primes.

– Dirichlet series.

– Orbit properties.

– Last but not least: generalization of Cobham by F. Durand.

– ...

Thank you! / Merci !