(Trying to do a)
Counting of distinct repetitions in words[1]

One World Combinatorics

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[1]Szilárd Zsolt Fazekas and RM

1. Preliminaries
2. Basics
3. Results
4. Final Remarks

Basics

[2] https://cs.uwaterloo.ca/∼shallit/repetitions.html

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kth power: $x^k = x \cdot x \cdot \dots \cdot x$, where x is some word

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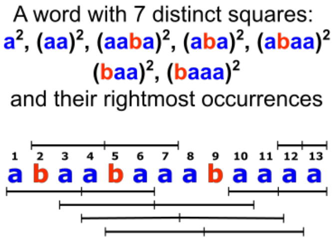

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lower bound: $\frac{2k-1}{2k+2}n$, k=number of b's

Upper bounds

Lemma (Three square prefixes, [Crochemore and Rytter, 1995])

If u^2 prefix of v^2, v^2 prefix of w^2, and u primitive then $|u| + |v| \leq |w|$.

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Circular squares [Amit and Gawrychowski, 2017]: $>1.25n$ and $<3.14n$

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Strategy: Lyndon roots (of sorts)

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Strategy: (primitive) roots, or very techical

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Lemma (Synchronization)

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Lemma (Fine and Wilf)

If p and q are periods of a word with length at least $p + q - \gcd(p, q)$, then $\gcd(p, q)$ is also a period.

Suffix Array

order the suffixes lexicographically

a	a	b	a	b	a
1	2	3	4	5	6

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useful for indexing text

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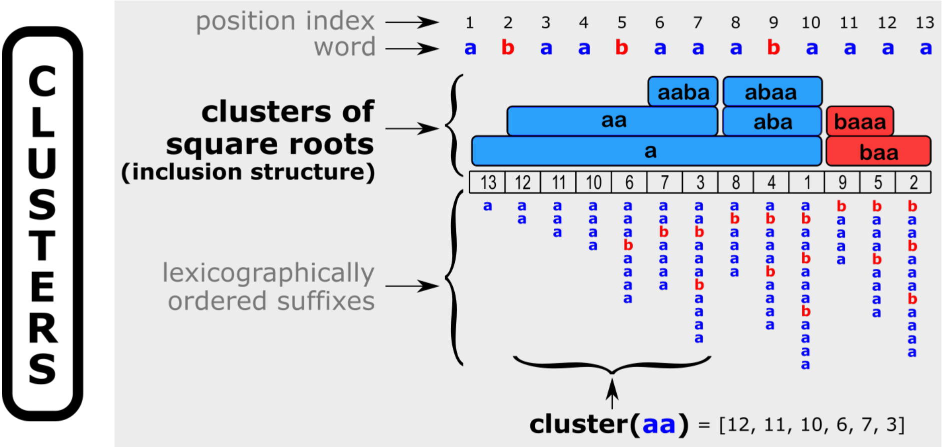
Mountain example



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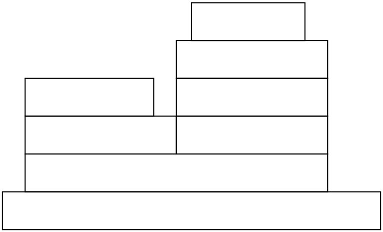

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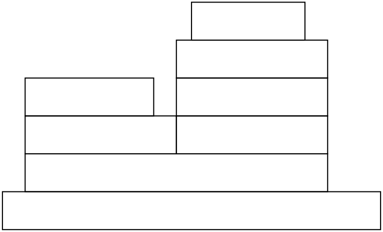

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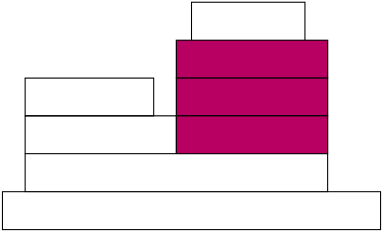

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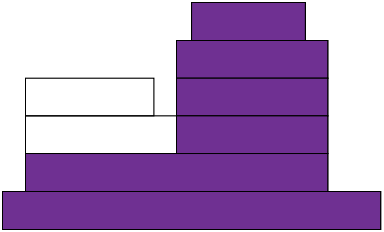

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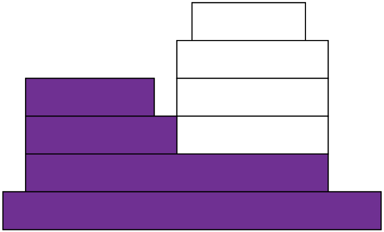

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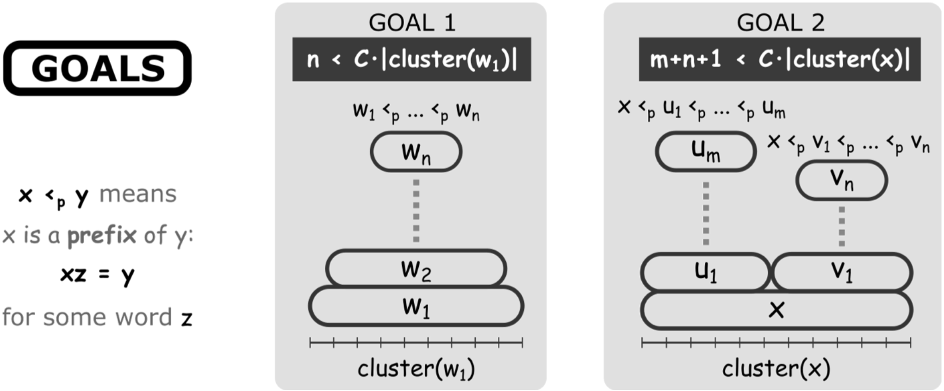

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- if $\mathbf{clust}(u) \cap \mathbf{clust}(v) \neq \emptyset$, then either u is prefix of v or v is prefix of u

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Lemma

Let $u^k \neq v^k$ *with* $k > 1$ *be factors of* w, *where* $\mathbf{clust}(u) = \mathbf{clust}(v)$ *and* $u \leq_p v$. *If their corresponding rightmost occurrences start at positions* u_s *and* v_s, *respectively, then* $|u_s - v_s| \geq |u|$.

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Corollary

Let $u_1^k, \ldots, u_n^k$ be squares in w such that $\mathbf{clust}_w(u_1) = \cdots = \mathbf{clust}_w(u_n)$. Then, $|\mathbf{clust}_w(u_1)| > (k-1)n$.

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Idea was also mentioned/used in [Jonoska et al., 2014, Lemma 2]
and [Bannai et al., 2014].)

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a-rep of u^2 is $abaaba = u^2$,

Representatives (of squares)

For $x \leq_p u$, the *x-representative* (x-rep) of u^2 is the longest prefix of u^2 which ends in x.

x-rep is of length at least $|u| + |x|$

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Anchors

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The ab-rep of u^2 is $abaab$, first occurring at 1,

$$\Psi(u^2, ab) = 1 + 3 = 4.$$

Collision Lemma

Lemma

Let w be an arbitrary word with two square factors u^2, v^2 such that $u <_p v$, and let $x \leq_p u$ be a common prefix of theirs.

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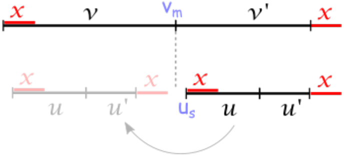
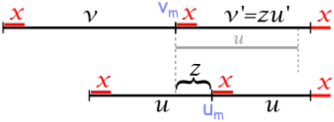
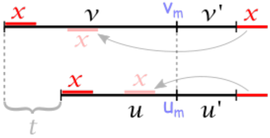
(1)



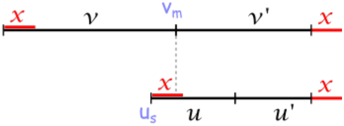
(3a)



(2)



(3b)



Chain chain chain

Corollary

Let $u_1^2, \ldots, u_n^2$ and $v_1^2, \ldots, v_n^2$ be squares in a word w with their roots from the same chain, and x a common prefix of theirs.

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If the x-anchor of some u_i^2 and v_i^2 coincide, there is some primitive t_i with $|t_i| < |x|$ such that $u_i = t_i^{k_i}$ with $k_i \geq 2$ and $t_i x$ is a prefix of v_i.

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Since $|t_i x| > |t_i| + |t_j| > |t_i| + |t_j| - \gcd(|t_i|, |t_j|)$, by [Fine and Wilf] we have that t_i and t_j have a common primitive root t, so $t_i = t_j = t$. □

Main Result

Theorem

For all words w and squares $u_1^2, \ldots, u_n^2$ in w with $u_1 <_p \cdots <_p u_n$:

$$|\mathbf{clust}_w(u_1)| \geq n + 1.$$

Proof idea

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Then $\mathbf{\Psi}_w((yzzyz)^2, x_i) = |w| - |x_i| + 1 = \mathbf{\Psi}_w((zyz)^2, x_i)$ in:

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A refinement of the anchor positions and assignment strategy might work.

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Using the lower bound construction in [Jonoska et al., 2014], we easily illustrate the extremal cases

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- $|\textbf{clust}_w(u_1)| = |\textbf{clust}_w(u_n)| = n + 1$ is realised by the roots $u_i = a^{n-1}ba^{i-1}$ and $w = u_1 u_2 \cdots u_n u_n$.

1. Preliminaries
2. Basics
3. Results
4. Final Remarks

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- (conjecture) the idea from [Jonoska et al., 2014] can be modified to produce any combination of cluster sizes
- take words $u_i = a^D ba^{\ell_i}$, adjusting ℓ_i so u_i occurs exactly at positions $p_j = \sum_{k=1}^{j} |u_k|$ for $j \geq i - d_i$, which are the unique starting positions of squares u_j^2 in the word $w = u_1 \ldots u_n u_n$.

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We expect that investigating the shortest words which realise a combination of cluster sizes could lead to improvements in both lower and upper bounds on distinct repetitions (the above are NOT).

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An upper bound on the number of run ending squares is implicitly an upper bound on the number of runs.

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- argument does not extend easily to overlapping chains of run ending squares (might cover cases when two chains overlap, but does not seem to work when we have 3 peeks)

Closing remarks

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4. the big challenge: overlapping chains
   (find a definition for anchor of u^2 which depends on anchors of all squares having u as a prefix)

Cheers

QUESTIONS