Morphisms Generating Antipalindromic Words

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One World Combinatorics on Words Seminar
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A palindrome is a finite word invariant under the mirror image antimorphism R:

$$\mathrm{R}(a) = a \quad \text{for all } a \in A.$$

Indeed, $R(w_1 \cdots w_n) = w_n \cdots w_1$.

Czech palindromes:
krk (neck), tahat (pull)

An antipalindrome is a finite binary word invariant under the exchange map E antimorphism:

$$\mathrm{E}(a) = b \quad \text{and} \quad \mathrm{E}(b) = a.$$

Czech antipalindrome:
Ninini (belonging to Nina)

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An infinite word is called palindromic if it contains arbitrarily long palindromes / infinitely many palindromes.

Fibonacci word, $\varphi_F : 0 \mapsto 01, 1 \mapsto 0$

$$0100101001001010010100100101\cdots$$

Thue-Morse word, $\Theta : 0 \mapsto 01, 1 \mapsto 10$

$$0110100110010110100101100110\cdots$$

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$010010100100101001010010010 1\cdots$

2 4 8

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3 4 6 8

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Which known words are palindromic?

- Sturmian
- Arnoux-Rauzy, episturmian
- codings of a symmetric k-interval exchange transformations

Which known words are antipalindromic?

- Thue-Morse
- complementary symmetric Rote sequences

CS Rote sequence $\boldsymbol{v}$: $\mathcal{S}(\boldsymbol{v}) = \boldsymbol{u}$, $\boldsymbol{u}$ Sturmian

$$\boldsymbol{v} = v_0 v_1 v_2 \cdots \quad \mapsto \quad \mathcal{S}(\boldsymbol{v}) = \boldsymbol{u} = u_0 u_1 u_2 \cdots$$

$$u_i = v_i + v_{i+1} \mod 2$$

Fibonacci: $\boldsymbol{f} = \mathcal{S}(\boldsymbol{v}) = 010010100100101\cdots$

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Observation

Let $\boldsymbol{u}, \boldsymbol{v}$ be infinite words such that $\boldsymbol{u} = \mathcal{S}(\boldsymbol{v})$.

- If $\boldsymbol{u}$ contains infinitely many palindromes with center 1, then $\boldsymbol{v}$ contains inifinitely many antipalindromes.
- If $\boldsymbol{u}$ contains infinitely many palindromes with center 0 or ε, then $\boldsymbol{v}$ contains inifinitely many palindromes.

Unrelated remark. If $\boldsymbol{t}$ is Thue-Morse word then $\mathcal{S}(\boldsymbol{t})$ is period-dubling sequence $(0 \mapsto 11, 1 \mapsto 10)$.

Frid (see talk at OWCW Jan 2021) found the formula for the prefix palindromic length of Thue-Morse and formulated a conjecture concerning the prefix palindromic length of period-doubling word.

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Motivation

Hof, Knill and Simon (1995) studied spectral properties of

$$(H\phi)(n) = \phi(n+1) + \phi(n-1) + V(n)\phi(n)$$

on $\ell^2(\mathbb{Z})$ with $V : \mathbb{Z} \to \mathbb{R}$, $\#V(\mathbb{Z})$ finite . . . infinite word $\boldsymbol{v}$

- "interesting properties" of $H \iff$ purely singular continuous spectrum
- $\sigma(H) = \sigma_{sc}(H) \iff \boldsymbol{v}$ aperiodic, palindromic
- class $\mathcal{P}$ of morphisms that generate palindromic words

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Class $\mathcal{P}$

Primitive morphism $\varphi : A^* \mapsto A^*$ is in class $\mathcal{P}$,
if there is a palindrome w such that for each $a \in A$

$$\varphi(a) = wq_a, \text{ where } q_a \text{ is a palindrome.}$$

$$\varphi_F = \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases}$$

$$\Theta = \begin{cases} a \mapsto ab \\ b \mapsto ba \end{cases} \qquad \Theta^2 = \begin{cases} a \mapsto abba \\ b \mapsto baab \end{cases}$$

Remark (Hof et al.)

We do not know whether all palindromic words generated by primitive morphisms arise from morphisms in class $\mathcal{P}$.

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- WLOG we can restrict ourselves to $|w| = 0$ or 1

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Let u be a periodic sequence that contains arbitrarily long palindromes, then u is a fixed point of a morphism in class $\mathcal{P}$.

Tan (2007):

- "to be palindromic" is property of $\mathcal{L}(\boldsymbol{u})$

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Conjecture (version 1)

Let $\boldsymbol{u}$ be the fixed point of a primitive morphism. Then $\boldsymbol{u}$ is palindromic if and only if there exists a morphism $\varphi \neq \mathrm{Id}$ such that $\boldsymbol{u} = \varphi(\boldsymbol{u})$ and φ has a conjugate in class $\mathcal{P}$.

Partial proofs:

- Allouche et al. (2003) for **periodic** words
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Labbé (2014) found a counter-example on ternary alphabet.

Let $\boldsymbol{x}$ be the fixed point of

$$a \mapsto aca, \quad b \mapsto cab, \quad c \mapsto b.$$

Then $\boldsymbol{x}$ is palindromic but no morphism φ such that $\varphi(\boldsymbol{x}) = \boldsymbol{x}$ has a conjugate in class $\mathcal{P}$.

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Let $\boldsymbol{u}$ be the fixed point of a primitive morphism φ. If $\boldsymbol{u}$ is palindromic then there exists a morphism ψ in class $\mathcal{P}$ such that the languages of both morphisms coincide.

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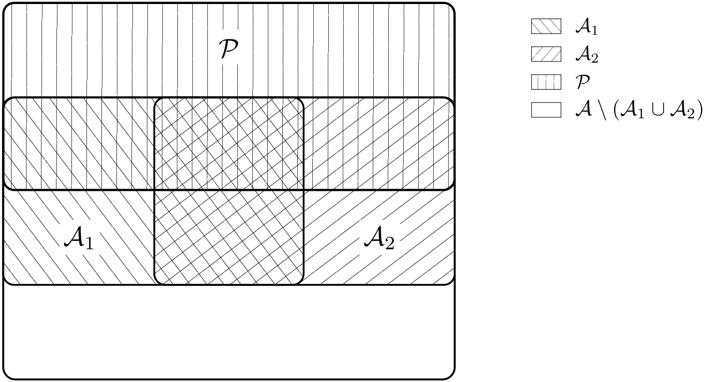

Class $\mathcal{A}_1$ – uniform morphisms

A morphism $\varphi : \{0,1\}^* \to \{0,1\}^*$ belongs to class $\mathcal{A}_1$ if there exist words $\mathfrak{p}, \mathfrak{s} \in \{0,1\}^*$ such that $\mathfrak{p} \neq \varepsilon$, $\mathfrak{s}$ is an antipalindrome and

$$\varphi(0) = \mathfrak{p}\mathfrak{s}, \qquad \varphi(1) = \mathsf{E}(\mathfrak{p})\mathfrak{s}.$$

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Remarks.

- All morphisms in class $\mathcal{A}_1$ are uniform.
- All morphisms in class $\mathcal{A}_1$ are primitive, except the trivial case $\varphi(0) = 0^k$, $\varphi(1) = 1^k$.
- Class $\mathcal{A}_1$ has already been considered by Labbé (2008).

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Let φ be a primitive morphism in class $\mathcal{A}_1$, $\boldsymbol{u}$ its fixed point. Then $\mathcal{L}(u)$ contains infinitely many antipalindromes.

Lemma. Let $\varphi \in \mathcal{A}_1$. Then $\mathrm{E}(\mathfrak{s}\varphi(w)) = \mathfrak{s}\varphi(\mathrm{E}(w))\ \forall w \in \{0,1\}^*$.

Proof.

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A morphism $\psi : \{0,1\}^* \to \{0,1\}^*$ belongs to class $\mathcal{A}_2$ if there exist a non-empty word $\mathfrak{w} \in \{0,1\}^*$ and $k, h \in \mathbb{N}$ such that

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Remarks.

- In general, morphisms in class $\mathcal{A}_2$ are non-uniform.
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From now on:
Let $\boldsymbol{u}$ be a fixed point of a primitive morphism, $\boldsymbol{u}$ antipalindromic.

Conjecture

There is a primitive morphism $\psi \in \mathcal{A}_1 \cup \mathcal{A}_2$ such that languages of $\boldsymbol{u}$ and of a fixed point of ψ conicide.

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- $\boldsymbol{u}$ is eventually periodic $\Rightarrow \boldsymbol{u} = (w_1 w_2)^\omega$, w_1, w_2 antipalindromes (by result of Labbé (2008)).
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Let $\boldsymbol{u}$ be an aperiodic fixed point of a primitive binary uniform morphism φ such that $\mathcal{L}(\boldsymbol{u})$ contains infinitely many antipalindromes. Then φ or φ^2 is conjugated to a morphism in class $\mathcal{A}_1$.

Remark. If $\varphi \in \mathcal{A}_1$, $\boldsymbol{u}$ its aperiodic fixed point.
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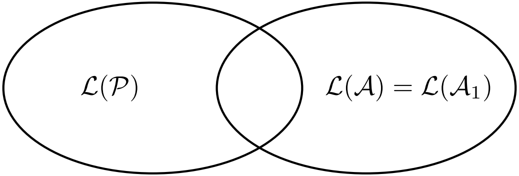

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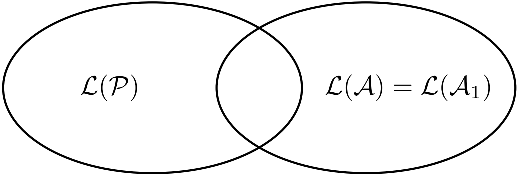

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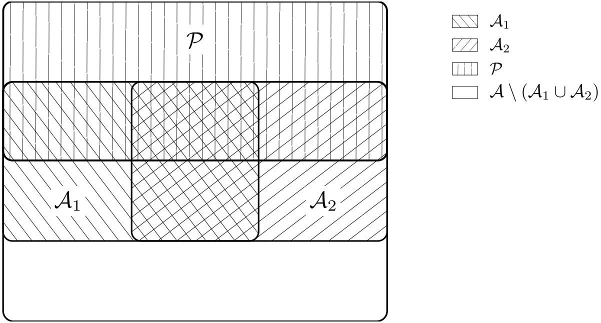

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2. Similar problem studied by Labbé:
   - If a uniform morphism φ has an antipalidromic fixed point, then φ or $\varphi\Theta$ is conjugated to a morphism in $\mathcal{E}$-$\mathcal{P}$:
   $$\varphi(a) = pp_a \qquad a = 0, 1 \qquad p, p_0, p_1 \text{ antipalindromes.}$$
   - He conjectures that always the latter is true. We proved this conjecture.

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1. Palindromic/pseudopalindromic closure:

   - Palindromic and antipalindromic words can be constructed using the so-called palindromic and pseudopalindromic closure.
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    - Consider a group G generated by antimorphisms over the monoid A^*.
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3. Generalization to multiliteral alphabets A:

   - Consider a group G generated by antimorphisms over the monoid A^*.
   - Ask when an infinite word contains infinitely many f-palindromes for each antimorphism $f \in G$.

Here f-palindrome is a finite word $v \in A^*$ such that $f(v) = v$.

Thank you for your attention.