

Morphisms Generating Antipalindromic Words

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One World Combinatorics on Words Seminar
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A **palindrome** is a finite word invariant under the **mirror image antimorphism** R :

$$R(a) = a \quad \text{for all } a \in A.$$

Indeed, $R(w_1 \cdots w_n) = w_n \cdots w_1$.

*Czech palindromes:
krk (neck), tahat (pull)*

An **antipalindrome** is a finite binary word invariant under the **exchange map** E antimorphism:

$$E(a) = b \quad \text{and} \quad E(b) = a.$$

*Czech antipalindrome:
Ninini (belonging to Nina)*

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Fibonacci word, $\varphi_F : 0 \mapsto 01, 1 \mapsto 0$

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The diagram shows the Fibonacci word sequence: 0100101001001001010010100100101... Brackets are drawn under the sequence to highlight palindromes of various lengths: a length-2 palindrome '01' at the start, a length-3 palindrome '010' starting at the 3rd position, a length-4 palindrome '1001' starting at the 4th position, a length-5 palindrome '00100' starting at the 5th position, a length-6 palindrome '100100' starting at the 6th position, a length-8 palindrome '01001001' starting at the 8th position, and a length-11 palindrome '01001001001' starting at the 11th position.

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Which known words are palindromic?

- Sturmian
- Arnoux-Rauzy, episturmian
- codings of a symmetric k -interval exchange transformations

Which known words are antipalindromic?

- Thue-Morse
- complementary symmetric Rote sequences

CS Rote sequence \mathbf{v} : $\mathcal{S}(\mathbf{v}) = \mathbf{u}$, \mathbf{u} Sturmian

$$\mathbf{v} = v_0 v_1 v_2 \cdots \quad \mapsto \quad \mathcal{S}(\mathbf{v}) = \mathbf{u} = u_0 u_1 u_2 \cdots$$

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- complementary symmetric Rote sequences

CS Rote sequence \mathbf{v} : $\mathcal{S}(\mathbf{v}) = \mathbf{u}$, \mathbf{u} Sturmian

$$\mathbf{v} = v_0 v_1 v_2 \cdots \quad \mapsto \quad \mathcal{S}(\mathbf{v}) = \mathbf{u} = u_0 u_1 u_2 \cdots$$

$$u_i = v_i + v_{i+1} \pmod{2}$$

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Observation

Let \mathbf{u}, \mathbf{v} be infinite words such that $\mathbf{u} = \mathcal{S}(\mathbf{v})$.

- If \mathbf{u} contains infinitely many palindromes with center 1, then \mathbf{v} contains infinitely many antipalindromes.
- If \mathbf{u} contains infinitely many palindromes with center 0 or ε , then \mathbf{v} contains infinitely many palindromes.

Unrelated remark. If \mathbf{t} is Thue-Morse word then $\mathcal{S}(\mathbf{t})$ is period-doubling sequence ($0 \mapsto 11, 1 \mapsto 10$).

Frid (see talk at OWCW Jan 2021) found the formula for the prefix palindromic length of Thue-Morse and formulated a conjecture concerning the prefix palindromic length of period-doubling word.

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Hof, Knill and Simon (1995) studied spectral properties of

$$(H\phi)(n) = \phi(n+1) + \phi(n-1) + V(n)\phi(n)$$

on $\ell^2(\mathbb{Z})$ with $V : \mathbb{Z} \rightarrow \mathbb{R}$, $\#V(\mathbb{Z})$ finite ... infinite word \mathbf{v}

- “interesting properties” of $H \iff$ purely singular continuous spectrum
- $\sigma(H) = \sigma_{\text{sc}}(H) \iff \mathbf{v}$ aperiodic, palindromic
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Primitive morphism $\varphi : A^* \mapsto A^*$ is in **class \mathcal{P}** ,
if there is a palindrome w such that for each $a \in A$

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Remark (Hof et al.)

We do not know whether all palindromic words generated by primitive morphisms arise from morphisms in class \mathcal{P} .

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Allouche, Baake, Cassaigne, Damanik (2003):

- WLOG we can restrict ourselves to $|w| = 0$ or 1

Theorem (Allouche et al.)

Let u be a periodic sequence that contains arbitrarily long palindromes, then u is a fixed point of a morphism in class \mathcal{P} .

Tan (2007):

- “to be palindromic” is property of $\mathcal{L}(u)$

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Conjecture (version 1)

Let \mathbf{u} be the fixed point of a primitive morphism. Then \mathbf{u} is palindromic if and only if there exists a morphism $\varphi \neq \text{Id}$ such that $\mathbf{u} = \varphi(\mathbf{u})$ and φ has a conjugate in class \mathcal{P} .

Partial proofs:

- Allouche et al. (2003) for **periodic** words
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Class \mathcal{P} conjecture

Labbé (2014) found a counter-example on ternary alphabet.

Let \mathbf{x} be the fixed point of

$$a \mapsto aca, \quad b \mapsto cab, \quad c \mapsto b.$$

Then \mathbf{x} is palindromic but no morphism φ such that $\varphi(\mathbf{x}) = \mathbf{x}$ has a conjugate in class \mathcal{P} .

Conjecture (version 2)

Let \mathbf{u} be the fixed point of a primitive morphism φ . If \mathbf{u} is palindromic then there exists a morphism ψ in class \mathcal{P} such that the languages of both morphisms coincide.

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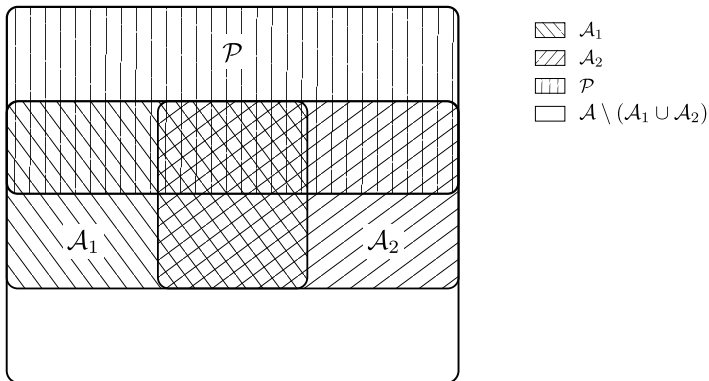
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Study a modification of class \mathcal{P} conjecture for antipalindromes.

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Class \mathcal{A}_1 – uniform morphisms

A morphism $\varphi : \{0, 1\}^* \rightarrow \{0, 1\}^*$ belongs to **class \mathcal{A}_1** if there exist words $\mathfrak{p}, \mathfrak{s} \in \{0, 1\}^*$ such that $\mathfrak{p} \neq \varepsilon$, \mathfrak{s} is an antipalindrome and

$$\varphi(0) = \mathfrak{p}\mathfrak{s}, \quad \varphi(1) = E(\mathfrak{p})\mathfrak{s}.$$

$$\begin{array}{l} 0 \mapsto \boxed{11} \boxed{101} \\ 1 \mapsto \boxed{00} \boxed{1} \\ \quad \quad \quad E(\mathfrak{p}) \mathfrak{s} \end{array}$$

Remarks.

- All morphisms in class \mathcal{A}_1 are uniform.
- All morphisms in class \mathcal{A}_1 are primitive, except the trivial case $\varphi(0) = 0^k$, $\varphi(1) = 1^k$.
- Class \mathcal{A}_1 has already been considered by Labbé (2008).

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- Class \mathcal{A}_1 has already been considered by Labbé (2008).

Class \mathcal{A}_1 – uniform morphisms

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Let φ be a primitive morphism in class \mathcal{A}_1 , u its fixed point. Then $\mathcal{L}(u)$ contains infinitely many antipalindromes.

Lemma. Let $\varphi \in \mathcal{A}_1$. Then $E(s\varphi(w)) = s\varphi(E(w)) \forall w \in \{0, 1\}^*$.

Proof.

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Class \mathcal{A}_2 – non-uniform morphisms

A morphism $\psi : \{0, 1\}^* \rightarrow \{0, 1\}^*$ belongs to **class \mathcal{A}_2** if there exist a non-empty word $\mathfrak{w} \in \{0, 1\}^*$ and $k, h \in \mathbb{N}$ such that

$$\psi(0) = \Theta(\mathfrak{w}(R(\mathfrak{w})\mathfrak{w})^k), \quad \psi(1) = \Theta((R(\mathfrak{w})\mathfrak{w})^h R(\mathfrak{w})).$$

If $\mathfrak{w} = 01$, $k = h = 0$
then $\psi = \Theta^2$.

Remarks.

- In general, morphisms in class \mathcal{A}_2 are non-uniform.
- Morphisms in class \mathcal{A}_2 are primitive.

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Main results

From now on:

Let \mathbf{u} be a fixed point of a primitive morphism, \mathbf{u} antipalindromic.

Conjecture

There is a primitive morphism $\psi \in \mathcal{A}_1 \cup \mathcal{A}_2$ such that languages of \mathbf{u} and of a fixed point of ψ coincide.

Supporting fact:

- \mathbf{u} is eventually periodic $\Rightarrow \mathbf{u} = (w_1 w_2)^\omega$,
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Then \mathbf{u} is fixed by $\psi(0) = \psi(1) = w_1 w_2$, and $\psi \in \mathcal{A}_1$.

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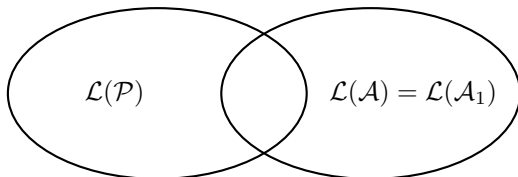
Let \mathbf{u} be an aperiodic fixed point of a primitive binary **uniform** morphism φ such that $\mathcal{L}(\mathbf{u})$ contains infinitely many antipalindromes. Then φ or φ^2 is conjugated to a morphism in class \mathcal{A}_1 .

Remark. If $\varphi \in \mathcal{A}_1$, \mathbf{u} its aperiodic fixed point.
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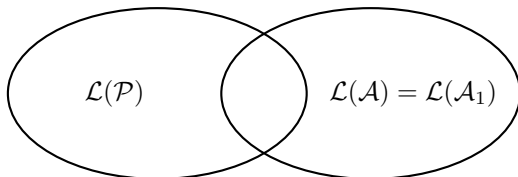


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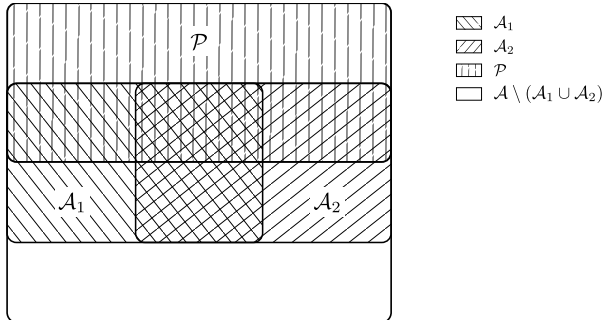
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Let \mathbf{u} be an aperiodic fixed point of a primitive binary non-uniform morphism φ such that $\mathcal{L}(\mathbf{u})$ contains infinite number of **palindromes as well as antipalindromes**. Then either φ or φ^2 is a morphism in class \mathcal{A}_2 (with \mathfrak{w} being an antipalindrome).

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Comments

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- 2 Similar problem studied by Labbé:
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$$\varphi(a) = pp_a \quad a = 0, 1 \quad p, p_0, p_1 \text{ antipalindromes.}$$
 - He conjectures that always the latter is true.
We proved this conjecture.
- 3 Initial intuition: the problem for an antipalindromic word u over $\{0, 1\}$ should not be difficult:
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- Palindromic and antipalindromic words can be constructed using the so-called palindromic and pseudopalindromic closure.
- Introduced by de Luca and De Luca (2006):

$$\Delta = (d_1, \psi_1), (d_2, \psi_2), \dots \quad d_i \in \{0, 1\}, \psi_i \in \{R, E\}.$$

$$u_0 = \varepsilon$$

$$u_{i+1} = \text{shortest } \psi_i\text{-palindrome with prefix } u_i d_i$$

Then u_i are prefixes of Thue-Morse word.

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- Introduced by de Luca and De Luca (2006):

$$\Delta = (d_1, \psi_1), (d_2, \psi_2), \dots \quad d_i \in \{0, 1\}, \psi_i \in \{R, E\}.$$

$$u_0 = \varepsilon$$

$$u_{i+1} = \text{shortest } \psi_i\text{-palindrome with prefix } u_i d_i$$

Then u_i are prefixes of Thue-Morse word.

- CS Rote words can be generated by (pseudo)palindromic closure. (Blondin Massé et al. 2013).

Related problems

1 Palindromic/pseudopalindromic closure:

- Dvořáková, Velká (2018): Which words generated by pseudopalindromic closure are fixed points of morphisms?

Conjecture: only morphisms $\varphi : \{0, 1\}^* \rightarrow \{0, 1\}^*$ of the form

$$\varphi(0) = 0(110)^k, \quad \varphi(1) = 1(001)^k, \quad k \in \mathbb{N}, k \geq 1,$$

generate such fixed points.

- Above morphisms belong to $\mathcal{P} \cap \mathcal{A}_1$.
- Do other morphisms in \mathcal{A}_1 or \mathcal{A}_2 have fixed points arising by pseudopalindromic closure?

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2 Richness:

- Not all palindromic infinite words are rich in palindromes.
- The question on which morphisms in class \mathcal{P} have rich fixed point is not solved even for the binary case.
Partial results by Glen et al. (2009)
- Which are morphisms of classes $\mathcal{A}_1 \cap \mathcal{P}$, $\mathcal{A}_2 \cap \mathcal{P}$ such that their fixed points are H -rich, where H is the group of morphisms and antimorphisms generated by E and R?

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- ③ Generalization to multiliteral alphabets A :
- Consider a group G generated by antimorphisms over the monoid A^* .
 - Ask when an infinite word contains infinitely many f -palindromes for each antimorphism $f \in G$.

Here f -palindrome is a finite word $v \in A^*$ such that $f(v) = v$.

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Thank you for your attention.