Open and closed complexity of infinite words

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Table of contents

1. Complexity functions

2. Open and closed complexity functions

3. Words with bounded open complexity are ultimately periodic

4. Words with bounded closed complexity are ultimately periodic

Table of content

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2. Open and closed complexity functions
3. Words with bounded open complexity are ultimately periodic
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Classical complexity function $p_w(n)$

Definition

For an infinite word w, we define its **complexity function** p_w which associates to every non-negative integer n the number of distinct factors of length n in $\mathsf{Fact}(w)$; so we get

$$p_w(n) = \mathsf{Card}(\mathsf{Fact}_n(w)).$$

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Example

For the regular paperfolding word

$$\mathbf{p} = 0010011000110110001001110011011 \cdots$$

we have $p_{\mathbf{p}}(n) = 4n$ for $n \geqslant 7$ (Allouche).

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$$\mathbf{f} = 0100101001001010010100100101001001\cdots$$

the complexity function is $p_{\mathbf{f}}(n) = n + 1$ for every integer n.

Morse and Hedlund's theorem

Theorem (Morse-Hedlund)

An infinite word w is aperiodic if and only if its complexity function is unbounded. If w is aperiodic, we have, for any integer n, $p_w(n) \geqslant n+1$.

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- The proof is quite direct since the complexity function is non-decreasing.
- The Fibonacci word has the minimal complexity for an aperiodic infinite word. Words verifying $p_w(n) \geqslant n+1$ for every integer n are called *Sturmian words* and have many other interesting properties.

Some generalizations

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So Morse and Hedlund's theorem is not always valid.

Example

For the Fibonacci word, the abelian complexity is constant, and for the Thue-Morse word, $a_t(2) = 3 > a_t(3) = 2 > a_t(4) = 1$.

Table of content

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Open and closed words

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The longest border of a closed word is called its **frontier**.

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Complete first returns

Remark

- Closed words are frequently called **complete first returns**, they play a central part in symbolic dynamics.
- In a recurrent word, every factor admits complete first returns to itself.
- In a uniformly recurrent word, every factor admits finitely many complete first returns to itself and for a prefix of the word, those give a new coding of the word.

Open and closed complexity functions

Definition

For a word $x \in \mathbb{A}^{\mathbb{N}}$ we define the **closed complexity function** Cl_x which associates to each $n \in \mathbb{N}$, the number of closed factors of x of length n.

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Example

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Remark

For every infinite word w and integer n we have :

$$\mathrm{Op}_w(n) + \mathrm{Cl}_w(n) = p_w(n).$$

Main result

Theorem (O. Parshina, P. 2020)

Let $x \in \mathbb{A}^{\mathbb{N}}$ be a right-infinite word over a finite alphabet $\mathbb{A}$. The following are equivalent :

1. *x is aperiodic ;*
2. $\liminf_{n \to +\infty} \mathrm{Op}_x(n) = +\infty.$
3. $\limsup_{n \to +\infty} \mathrm{Cl}_x(n) = +\infty$;

Remarks

Using Morse and Hedlund, it is easy to get and

$$\liminf_{n\to+\infty} \mathrm{Op}_x(n) = +\infty \implies x \text{ is aperiodic}$$

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With $\liminf \mathrm{Op}_x(n) = +\infty \Leftrightarrow x$ is aperiodic, we get that

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We can't have $\liminf$ in $\limsup \mathrm{Cl}_x(n) = +\infty \Leftrightarrow x$ is aperiodic : for the paperfolding words, it is known that

$$\liminf_{n\to+\infty} \mathrm{Cl}_x(n) = 0.$$

Table of content

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First part of the theorem : Open complexity function

Theorem

Let x be an infinite word over a finite alphabet $\mathbb{A}$. Let $k \in \mathbb{N}$ be such that $\liminf \mathsf{Op}_x(n) = k$. Then x is ultimately periodic.

Rauzy graphs

For $x \in \mathbb{A}^{\mathbb{N}}$ and $n \in \mathbb{N}$, the **Rauzy graph** of order n of $x \in \mathbb{A}^{\mathbb{N}}$ is the directed graph whose set of vertices (resp. edges) consists of all factors of x of length n (resp. $n+1$). There is a directed edge from u to v labeled w if u is a prefix of w and v a suffix of w.

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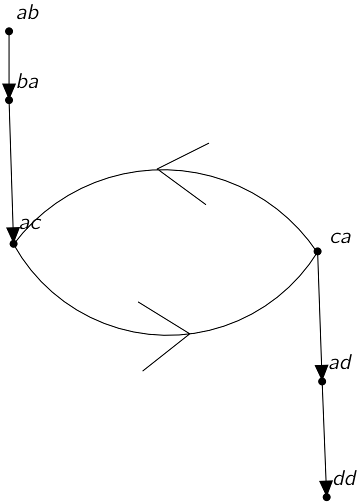
The **distance** between two vertices in a Rauzy graph is the length of the shortest directed path between them. This path does not necessarily correspond to a factor in the word.

Rauzy graph

Rauzy graph of $abacacaddddd\cdots$ of order 2 :



Open complexity function

Proposition

Let $x \in \mathbb{A}^{\mathbb{N}}$ and $N \in \mathbb{N}$. Let w_1 and w_2 be two factors of x, such that there is a path of length i from w_1 to w_2 in the Rauzy graph of order N of x. Suppose w_1 and w_2 are closed with frontiers u_1 and u_2 respectively. Then $||u_1| - |u_2|| < i$.
In particular, if $i = 1$ the frontiers are of the same length : $|u_1| = |u_2|$.

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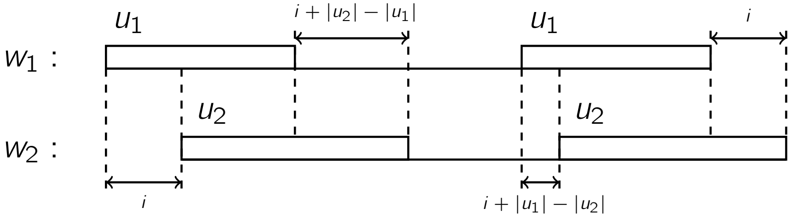

Figure – Factors w_1 and w_2.

Some other lemmas

Lemma

Suppose that x *is aperiodic with* $\liminf \mathrm{Op}_x = k$*. There exists an integer* N *such that* $u^N \notin \mathrm{Fact}(x)$ *for any choice of* u *with* $|u| < 2k$*.*

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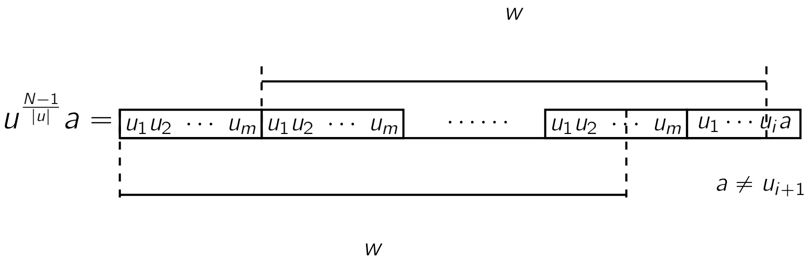

Figure – The frontier should be longer than $w = u^{\frac{N-1}{|u|}-1}$.

Lemma

Suppose that x *is aperiodic with* $\liminf \mathrm{Op}_x = k$*. For every integer* m*, there exists an integer* N *such that the frontier of any closed factor of length* N *is at least* m*.*

Proof - open complexity

$$\begin{tikzcd}
 & \textit{open} & & \textit{closed} \\
 & \boxed{w_2 w_3 \cdots w_N a} \arrow[r] & \cdots \arrow[r] & \boxed{w_{i+1} w_{i+2} \cdots w_N \ a \ y_1 y_2 \cdots y_{i-1}} \\
\boxed{w_1 w_2 w_3 \cdots w_N} \arrow[ur] \arrow[dr] & & & \\
 & \boxed{w_2 w_3 \cdots w_N b} \arrow[r] & \cdots \arrow[r] & \boxed{w_{i+1} w_{i+2} \cdots w_N \ b \ z_1 z_2 \cdots z_{i-1}} \\
 & \textit{closed} & & \textit{closed}
\end{tikzcd}$$

Figure – Open and closed factors in the Rauzy graph of order N.

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Closed complexity function

Theorem

Let $x \in \mathbb{A}^{\mathbb{N}}$ *be such that* $\limsup_{n \in \mathbb{N}} \mathrm{Cl}_x(n) < k$. *Then* x *is ultimately periodic.*

The branching lemma

Lemma

Let x be aperiodic with closed complexity bounded by k. At least one of the two following assertions holds :

1. $\forall u \in \mathrm{RecFact}(x), \exists (p \leqslant k, a_1, \ldots, a_p, a, b) \in \mathbb{N} \times \mathbb{A}^{p+2}, a \neq b$, *such that*

$$aa_1 \cdots a_p u \in \mathrm{RecFact}(x) \quad \textit{and} \quad ba_1 \cdots a_p u \in \mathrm{RecFact}(x);$$

2. $\forall u \in \mathrm{RecFact}(x), \exists (p \leqslant k, a_1, \cdots, a_p, a, b) \in \mathbb{N} \times \mathbb{A}^{p+2}, a \neq b$, *such that*

$$ua_1 \cdots a_p a \in \mathrm{RecFact}(x) \quad \textit{and} \quad ua_1 \cdots a_p b \in \mathrm{RecFact}(x).$$

Sketch of proof

Let u be such that u is always preceded by $a_1 a_2 \cdots a_k$.

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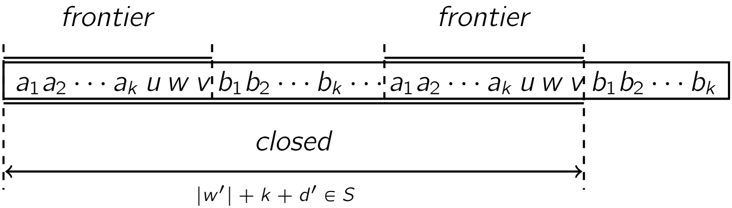

Figure – k closed factors of x.

Corollary

Let $x \in \mathbb{A}^{\mathbb{N}}$ with bounded closed complexity. If x is uniformly recurrent, then it is periodic.

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$$\begin{tikzcd} & a\, a_2 a_3 \cdots \\ v_1 v_2\, v_3 \cdots\cdots v_j \arrow[ur] \arrow[dr] & \\ & b\, b_2\, b_3 \cdots \end{tikzcd}$$

Figure – It is possible to avoid seeing u.

Lemma

Let $x \in \mathbb{A}^{\mathbb{N}}$ with $\limsup \mathrm{Cl}_x(n) < k$. Let u be a primitive word in $\mathrm{Fact}(x)$ such that, for every $n \in \mathbb{N}$, $u^n \in \mathrm{Fact}(x)$. Then $|u| < k$.

Lemma

Let $x \in \mathbb{A}^{\mathbb{N}}$ *with* $\limsup \mathsf{Cl}_x(n) < k$. *Let* u *be a primitive word in* $\mathsf{Fact}(x)$ *such that, for every* $n \in \mathbb{N}$, $u^n \in \mathsf{Fact}(x)$. *Then* $|u| < k$.

The word uuu contains the square $r(u)r(u)$ for every rotation of u.

An operator often used in symbolic dynamics is the **shift operator** T. That operator has many uses in combinatorics on words. It is defined on infinite words as follows :

$$\forall x \in \mathbb{A}^{\mathbb{N}}, x = x_0 x_1 x_2 \cdots , T(x) = x_1 x_2 x_3 \cdots$$

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For $x \in \mathbb{A}^{\mathbb{N}}$, we let $\Omega(x)$ denote the **shift orbit closure** of x, i.e. the closure in $\mathbb{A}^{\mathbb{N}}$ for the usual topology of the set

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Theorem (Furstenberg)

For every infinite word x over a finite alphabet, $\Omega(x)$ contains at least one uniformly recurrent element.

Sketch of proof

- Starting from x, we construct a word y that does not contain powers of great order.

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- Starting from x, we construct a word y that does not contain powers of great order.
- We consider a uniformly recurrent element in $\Omega(y)$ and derive a contradiction.

Closed complexity function

Definition

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$$S \cap [n, n+d] \neq \emptyset.$$

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$$S \cap [n, n+d] \neq \varnothing.$$

Theorem

Let $x \in \mathbb{A}^{\mathbb{N}}$ be such that there exist a positive integer d and a syndetic subset $S \subseteq \mathbb{N}$ with gaps smaller than d on which the closed complexity of x is bounded, i.e. there exists $k \in \mathbb{N}$ such that $\mathrm{Cl}_x(n) < k$ for every $n \in S$. Then x is ultimately periodic.

Conclusion

Theorem (O. Parshina, P. 2020)

Let $x \in \mathbb{A}^{\mathbb{N}}$ be a right-infinite word over a finite alphabet $\mathbb{A}$. The following are equivalent :

1. *x is aperiodic ;*
2. $\liminf\limits_{n \to +\infty} \mathrm{Op}_x(n) = +\infty$.
3. *There exists a syndetic set S such that $\mathrm{Cl}_x(n)$ is bounded on S.*

Thank you for your attention !