On Long Arithmetic Progressions in Binary Morse-Like Words

Ibai Aedo

Joint work with **Uwe Grimm**, **Yasushi Nagai** and **Petra Staynova**.

0	0	1	0	0	1	1	1	0
0	0	1	0	0	1	1	1	0
1	1	0	1	1	0	0	0	1
0	0	1	0	0	1	1	1	0
0	0	1	0	0	1	1	1	0
1	1	0	1	1	0	0	0	1
1	1	0	1	1	0	0	0	1
1	1	0	1	1	0	0	0	1
0	0	1	0	0	1	1	1	0

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Day of Short Talks on Combinatorics on Words.
March 22nd, 2021.

Outline

- Motivation and introduction.
- Arithmetic progressions in the Thue–Morse word.
- Generalised Thue–Morse words.

Motivation

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 ...

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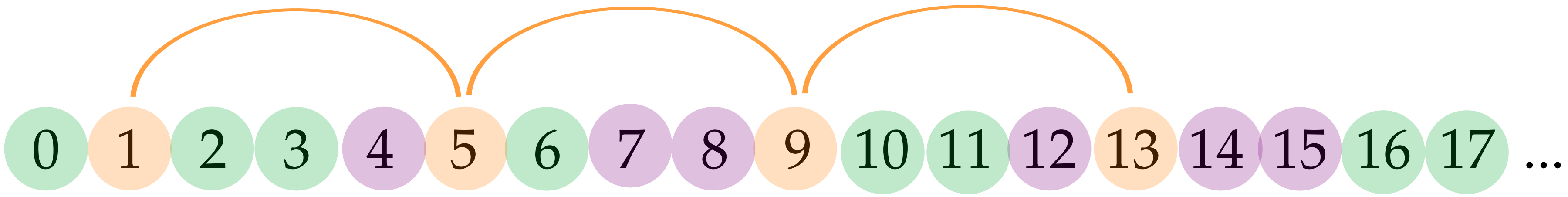
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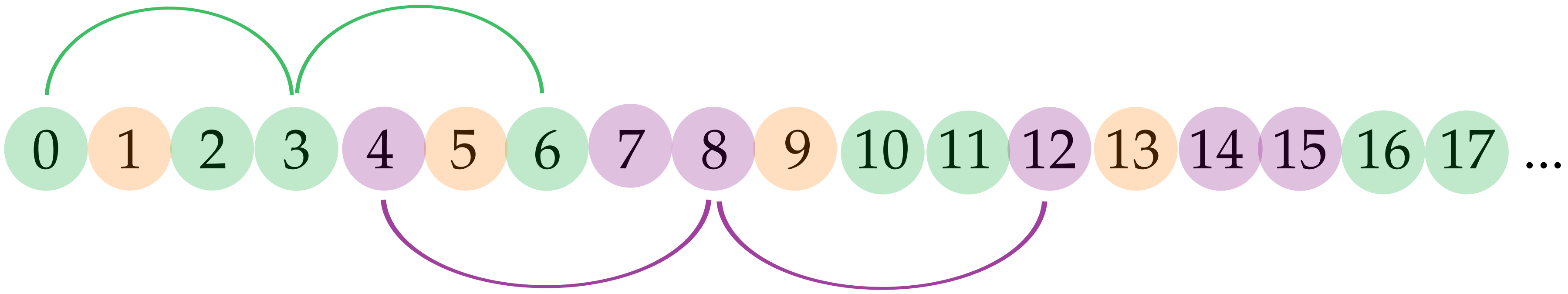
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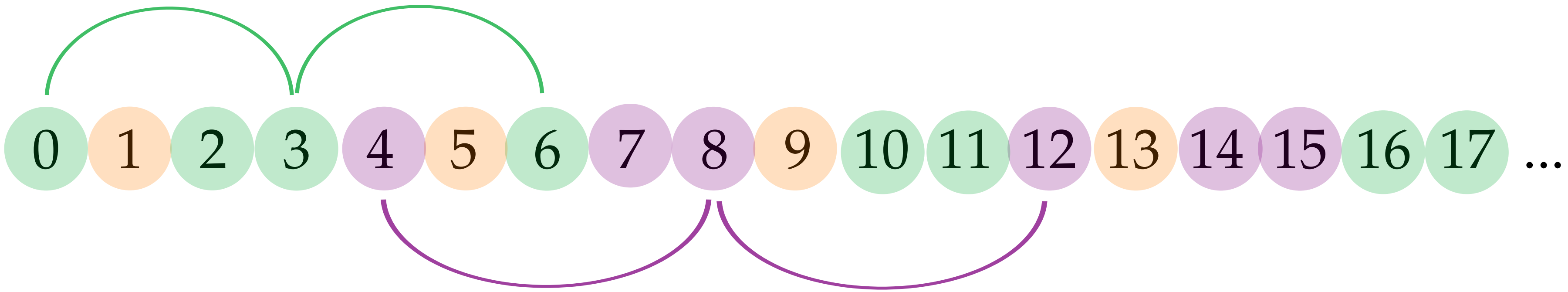

Theorem (van der Waerden. 1927)**.** *For each pair of positive integers L and c, there exist a positive integer N such that any c-colouring of the segment $\{1, 2, \ldots, N\}$ contains a monochromatic arithmetic progression of length L.*

Notation

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ $\qquad$ $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$

Alphabet: $\mathcal{A} = \{0, 1\}$.

Finite word: $u = u_0 u_1 \cdots u_{n-1}$, with $u_i \in \mathcal{A}$. Write $u \in \mathcal{A}^*$.

Infinite word: $u = u_0 u_1 u_2 \cdots$, with $u_i \in \mathcal{A}$. Write $u \in \mathcal{A}^{\mathbb{N}}$.

Definition. Let $v = v_0 v_1 v_2 \cdots$ with $v_i \in \mathcal{A}$. We say that v contains an arithmetic progression of difference $d \in \mathbb{N}^+$ and length $L \in \mathbb{N}^+$, if there exists $n \in \mathbb{N}$ such that $v_n = v_{n+d} = v_{n+2d} = \cdots = v_{n+(L-1)d}$.

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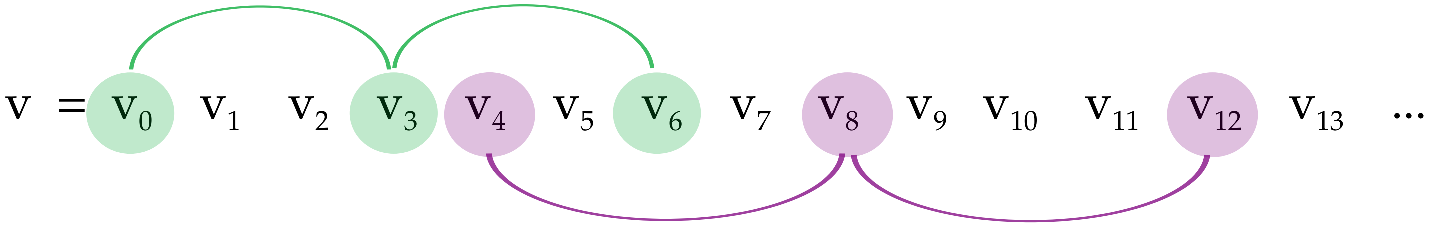

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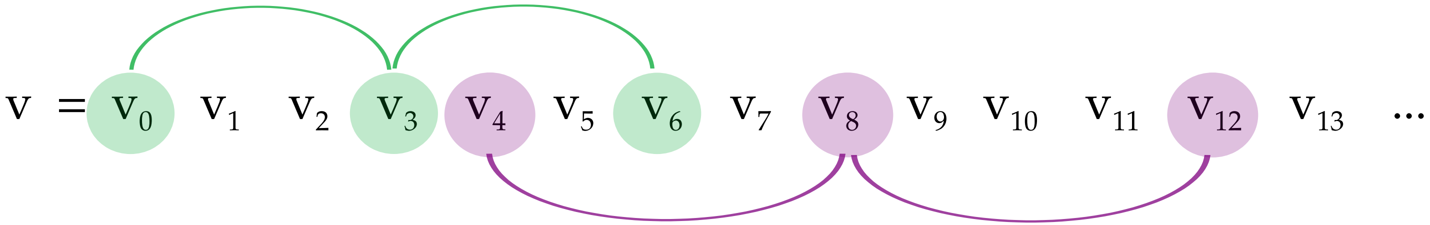

Question. *Does v contain arbitrarily long arithmetic progressions of a fixed difference d?*

Question. *Given $d \in \mathbb{N}^+$, is there $n \in \mathbb{N}^+$ such that $v_n = v_{n+md}$ for each $m \in \mathbb{N}^+$?*

Arithmetic progressions in the Thue-Morse word

Thue-Morse substitution and Thue-Morse word

$$\theta\colon \begin{array}{ccc} 0 & \mapsto & 0\,1 \\ 1 & \mapsto & 1\,0 \end{array} \qquad\qquad v = \lim_{n\to\infty} \theta^n(0) = 0110100110010110 1001\cdots \in \{0,1\}^{\mathbb{N}}$$

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$\bar{a} = 1 - a$
for $a \in \{0,1\}$

$\overline{\theta^n w} = \theta^n \overline{w}$
for all $n \in \mathbb{N}$ and $w \in \mathcal{A}^*$

$\theta^{n+1}(a) = \theta^n(a)\,\overline{\theta^n(a)} = \theta^n(a)\,\theta^n(\overline{a})$
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$$\begin{array}{llr}
\theta^0(0) = & & 0 \\
\theta^1(0) = & \theta^0(0)\,\overline{\theta^0(0)} = & 01 \\
\theta^2(0) = & \theta^1(0)\,\overline{\theta^1(0)} = & 0110 \\
\theta^3(0) = & \theta^2(0)\,\overline{\theta^2(0)} = & 01101001 \\
\theta^4(0) = & \theta^3(0)\,\overline{\theta^3(0)} = & 0110100110010110 \\
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v is overlap-free : For any $u \in \{0,1\}^+$, v does not contain uuu_0 as a subword.

Question. *Does the Thue-Morse word contain arbitrarily long arithmetic progressions of a fixed difference d?*

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Proposition. *The Thue–Morse word does not contain arbitrarily long monochromatic arithmetic progressions for any fixed difference* d.

The proof is based on a result by S. Akiyama, J-Y. Lee and Y. Nagai (2020) about the existence of infinite arithmetic progressions in non-periodic tilings.

Definition. For a positive integer d, let $A(d)$ be the maximum length of a monochromatic arithmetic progression of difference d within the Thue–Morse word.

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$A(2^n d) = A(d)$ for any $n \in \mathbb{N}$. In particular, $A(2^n) = A(1) = 2$ for all $n \in \mathbb{N}$.

Lemma. *$A(d) = 2$ if, and only if, $d = 2^n$ for some $n \in \mathbb{N}$.*

Theorem (Parshina. 2015). *For all $n \in \mathbb{N}^+$, we have that*

$$\max_{d<2^n} A(d) \;=\; A(2^n-1) \;=\; \begin{cases} 2^n+4, & \text{if } 2|n, \\ 2^n & \text{otherwise.} \end{cases}$$

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Proposition. *For all $n > 1$, we have that $A(2^n + 1) = 2^n + 2$.*

Let Θ be the substitution of binary blocks given by

$$\Theta\colon \quad 0 \longmapsto \begin{matrix} 0\,1 \\ 1\,0, \end{matrix} \quad 1 \longmapsto \begin{matrix} 1\,0 \\ 0\,1. \end{matrix}$$

Iterating Θ on a single letter produces square blocks of size $2^n \times 2^n$, for instance

$$\begin{matrix} \mathbf{0} \end{matrix} \longmapsto \begin{matrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{matrix} \longmapsto \begin{matrix} \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \end{matrix} \longmapsto \begin{matrix} \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \end{matrix} \longmapsto \cdots$$

Lemma. *For* $a \in \{0,1\}$ *and* $n \in \mathbb{N}^+$, *the block* $\Theta^n(a)$, *read row-wise from top to bottom, is the word* $\theta^{2n}(a)$ *with* θ *the Thue–Morse substitution on* $\{0,1\}$.

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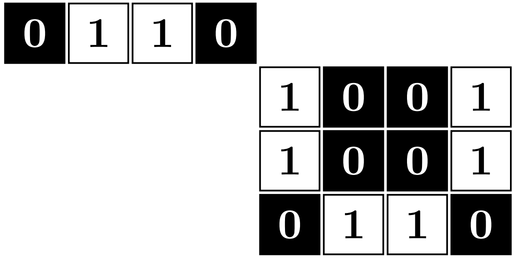
$\Theta^2(0) =$ 

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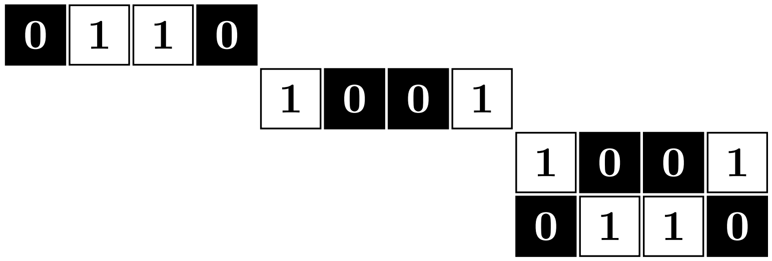

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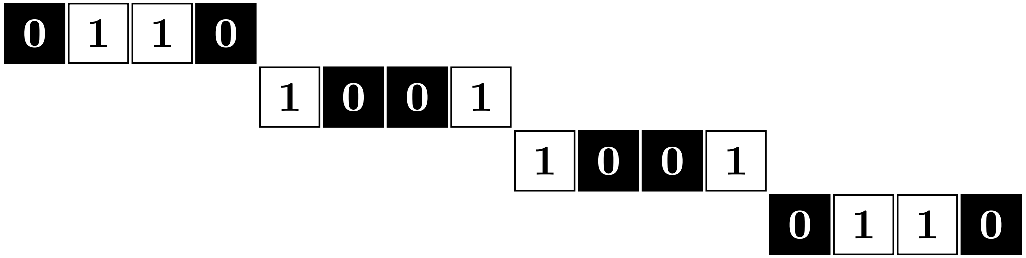

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Lemma. *For* $a \in \{0, 1\}$ *and* $n \in \mathbb{N}^+$, *the blocks* $\Theta^n(a)$ *consists of only two types of row and column words, and are symmetric under reflection in either diagonals. All entries on the main diagonal are* a, *while entries of the other diagonal are* a *for even* n *and* $\overline{a}$ *otherwise.*

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$$d = 2^n + 1$$

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For all $n \in \mathbb{N}^+$, we have that $A(2^n + 1) \geqslant 2^n$.

$$d = 2^n - 1$$

$$\Theta^2(0) = \begin{array}{cccc} \boxed{0} & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}$$

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For all $n \in \mathbb{N}^+$, we have $A(2^n - 1) \geqslant 2^n$.
For even n, we further have $A(2^n - 1) \geqslant 2^n + 2$.

Lemma. *For all $n \in \mathbb{N}$, $n > 1$, we have that*

$$A(2^n - 1) \;=\; \begin{cases} 2^n + 4, & \textit{if } 2|n, \\ 2^n, & \textit{otherwise.} \end{cases}$$

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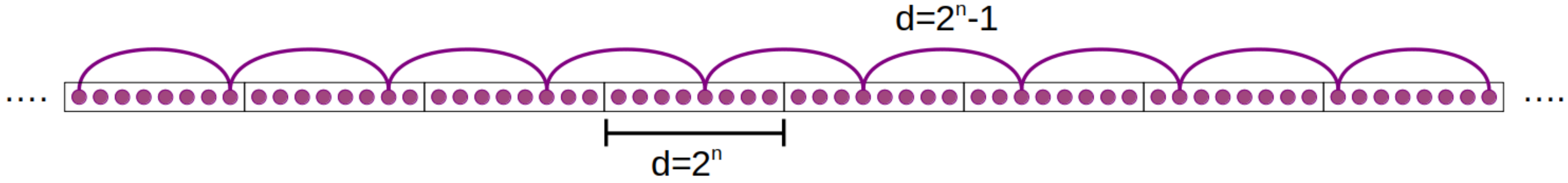

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Assume this happens. If the progression continued to the left and to the right, we are forced to have $\theta^n(\bar{a})\theta^n(\bar{a})\theta^n(a)\theta^n(\bar{a})\theta^n(\bar{a})$.

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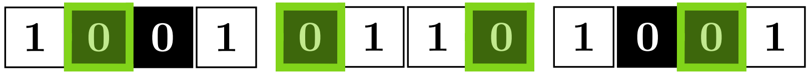

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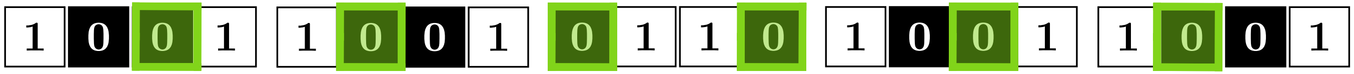

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$$1\,0\,0\,1 \quad 1\,0\,0\,1 \quad 0\,1\,1\,0 \quad 1\,0\,0\,1 \quad 1\,0\,0\,1 = \theta^2(11011)$$

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The word $\bar{a}\bar{a}a\bar{a}\bar{a}$ does not belong to the Thue–Morse language.
Therefore, $A(2^n-1) \leqslant 2^n + 4$.

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The bound $2^n + 4$ is attained because $\theta^{2n}(a)$, which contains a progression of length $2^n + 2$, can be extended in both directions with $\theta^{2n}(\bar{a})$, giving $\theta^{2n}(\bar{a}a\bar{a})$ and $\bar{a}a\bar{a}$ is in the Thue–Morse language.

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$$\theta^4(0) \approx \Theta^2(0) = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ \hline \end{array}$$

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0	1	1	0
1	0	0	1
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Assume this happens. If the progression continued to the left and to the right, we are forced to have $\theta^n(\bar{a})\theta^n(\bar{a})\theta^n(a)\theta^n(\bar{a})\theta^n(\bar{a})$.

The word $\bar{a}\bar{a}a\bar{a}\bar{a}$ does not belong to the Thue–Morse language. Therefore, $A(2^n-1) \leqslant 2^n + 4$.

The bound $2^n + 4$ is attained because $\theta^{2n}(a)$, which contains a progression of length $2^n + 2$, can be extended in both directions with $\theta^{2n}(\bar{a})$, giving $\theta^{2n}(\bar{a}a\bar{a})$ and $\bar{a}a\bar{a}$ is in the Thue–Morse language.

0	1	1	0
1	0	0	1
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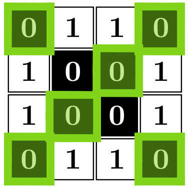

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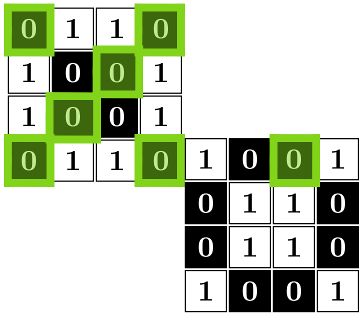

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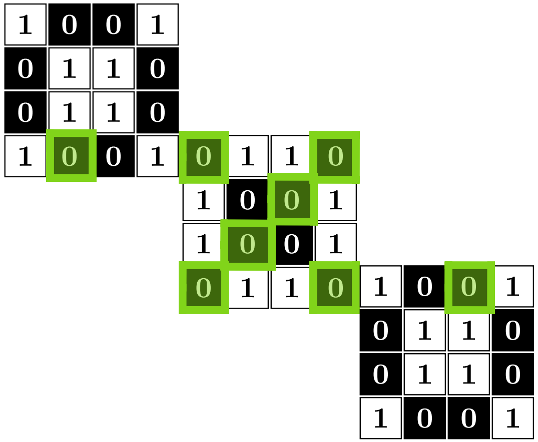

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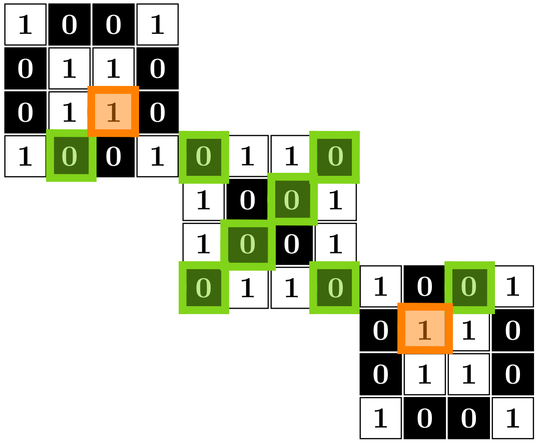

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Hence the progression can be continued by one additional step to either direction, and the bound is attained.

$\square$

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$\square$

A bit more of work shows that $\max_{d<2^n} A(d) = A(2^n - 1)$.

Arithmetic progressions in generalised Thue-Morse word

Generalised Thue-Morse substitutions

$$\theta_{p,q}\colon \begin{array}{l} 0 \mapsto 0^p 1^q \\ 1 \mapsto 1^p 0^q \end{array}, \quad \text{for } p, q \in \mathbb{N}^+.$$

$$v_{p,q} = \lim_{n\to\infty} \theta_{p,q}^n(0)$$

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Proposition. *The generalised Thue–Morse words do not contain arbitrarily long monochromatic arithmetic progressions for any fixed difference d.*

Definition. For $d \in \mathbb{N}$, $A_{p,q}(d)$ is the maximum length of a monochromatic arithmetic progression of difference d within the generalised Thue–Morse word.

Let $\Theta_{p,q}$ be the block substitution on the alphabet $\{0,1\}$ defined by

$$\Theta_{p,q}\colon \quad 0 \longmapsto \begin{array}{l} \left.\begin{array}{l} 0^p1^q \\ 0^p1^q \\ \vdots \\ 0^p1^q \end{array}\right\} p \\ \left.\begin{array}{l} 1^p0^q \\ 1^p0^q \\ \vdots \\ 1^p0^q \end{array}\right\} q \end{array}, \qquad 1 \longmapsto \begin{array}{l} \left.\begin{array}{l} 1^p0^q \\ 1^p0^q \\ \vdots \\ 1^p0^q \end{array}\right\} p \\ \left.\begin{array}{l} 0^p1^q \\ 0^p1^q \\ \vdots \\ 0^p1^q \end{array}\right\} q \end{array}.$$

Iterating $\Theta_{p,q}$ on a single letter produces square blocks of size $Q^n \times Q^n$, where $Q = p + q$.

$p = q$

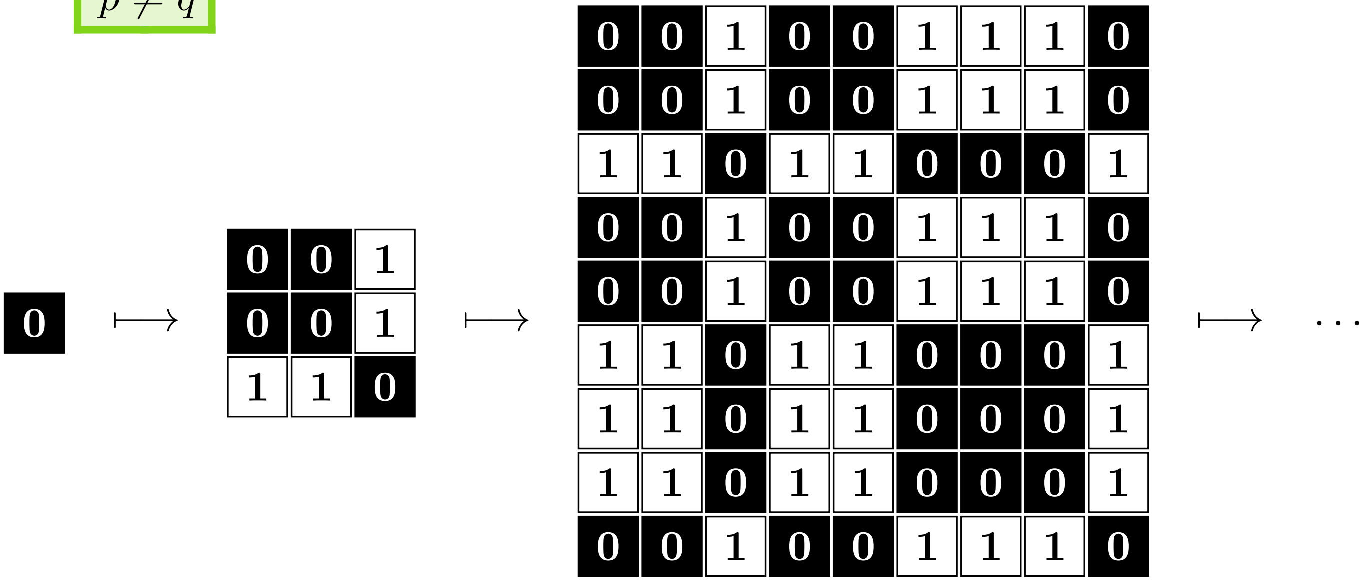
$0 \longmapsto$

0	0	1	1
0	0	1	1
1	1	0	0
1	1	0	0

$\longmapsto$

0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0
0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0
1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1
1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1
0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0
0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0
1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1
1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1
1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1
1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1
0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0
0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0
1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1
1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1
0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0
0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0

$\boxed{p \neq q}$



0
0 0 1
0 0 1
1 1 0
0 0 1 0 0 1 1 1 0
0 0 1 0 0 1 1 1 0
1 1 0 1 1 0 0 0 1
0 0 1 0 0 1 1 1 0
0 0 1 0 0 1 1 1 0
1 1 0 1 1 0 0 0 1
1 1 0 1 1 0 0 0 1
1 1 0 1 1 0 0 0 1
0 0 1 0 0 1 1 1 0
...

Proposition. *For all* $n, p, q \in \mathbb{N}^+$, $Q = p + q$, $n > 1$, *we have that*

$$A_{p,q}(Q^n+1) = \begin{cases} Q^n + Q - 2, & \text{if } p > 1 \text{ and } q > 1, \\ Q^n + Q - 1, & \text{if } q > p = 1 \text{ or } p > q = 1, \\ Q^n + Q, & \text{if } p = q = 1. \end{cases}$$

$(p = q)$ **Proposition.** *For all* $n, p \in \mathbb{N}^+$, $Q = 2p$, $n > 1$, *we have that*

$$A_{p,p}(Q^n-1) = \begin{cases} Q^n, & \text{if } n \text{ is odd} \\ Q^n + Q, & \text{if } n \text{ is even and } p > 1 \\ Q^n + Q + 2, & \text{if } n \text{ is even and } p = 1. \end{cases}$$

We have results for differences $d = Q^n + 1$, as well as for differences $d = Q^n - 1$ in the case that $p = q$. But we have not been able to show that these are again the longest arithmetic progressions up to the given difference.

Conjecture. *For all $n, p, q \in \mathbb{N}^+$, $Q = p + q$, $n > 2$, we have that*

$$\max_{d \leqslant Q^n+1} A_{p,q}(d) = \begin{cases} A_{p,q}(Q^n{-}1) = Q^n + Q + 2, & \text{if } p = q = 1 \text{ and } n \text{ even}, \\ A_{p,q}(Q^n{-}1) = Q^n + Q, & \text{if } p = q > 1 \text{ and } n \text{ even}, \\ A_{p,q}(Q^n{+}1) = Q^n + Q, & \text{if } p = q = 1 \text{ and } n \text{ odd}, \\ A_{p,q}(Q^n{+}1) = Q^n + Q - 1, & \text{if } q > p = 1 \text{ or } p > q = 1, \\ A_{p,q}(Q^n{+}1) = Q^n + Q - 2, & \text{if } p, q > 1 \text{ and } p \neq q \text{ or } n \text{ odd}. \end{cases}$$

Thank you, merci :-)