On morphisms preserving palindromic richness

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joint work with Edita Pelantová

Day of Short Talks on Combinatorics on Words
One World CoW Seminar
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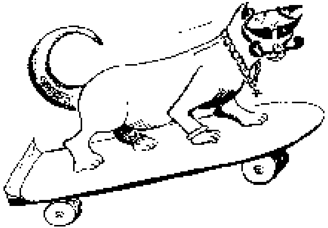
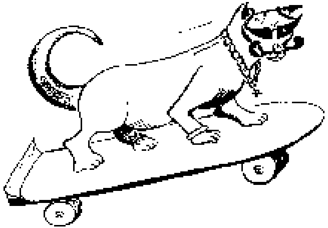


GOFLOWOLFOG

GOFLOWOLFOG, *the spirit who eases traffic blockages so that you can continue your journey.* GOFLOWOLFOG *typically appears in the form of a shades-wearing cat riding a skateboard. He brings with him a wind, and a noise which sounds like "Neeeowww." [..] If nothing else, this act of summoning may take your mind off sources of stress.*

[Phil Hine, *Aspects of Evocation* (1995)]



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Naming the Spirit - several suggestions were made for an appropriate name, and GO FLOW *was chosen. This name was made suitably 'barbaric' by mirroring it, so becoming* GOFLOWOLFOG.

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Palindromes

A *palindrome* is a finite word w such that $w = \widetilde{w}$.

Theorem [Droubay, Justin, Pirillo (2001)]

A word of length n has at most $n + 1$ palindrome factors

A word with maximal number of palindromes is *rich*.

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- $\mathcal{P}\{\texttt{pizza}\} = \{\varepsilon, \texttt{a}, \texttt{i}, \texttt{p}, \texttt{z}, \texttt{zz}\}$

  $\#\mathcal{P}\{\texttt{w}\} = 6 = |\texttt{w}| + 1$ ✓

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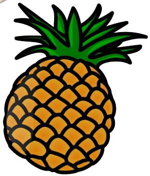
- $\mathcal{P}\{\mathtt{ananas}\} = \{\varepsilon, \mathtt{a}, \mathtt{n}, \mathtt{s}, \mathtt{ana}, \mathtt{nan}, \mathtt{anana}\}$

  $\#\mathcal{P}\{\mathtt{w}\} = 7 = |\mathtt{w}| + 1$ ✓



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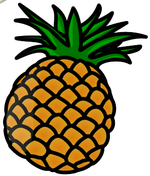

- $\mathcal{P}\{\texttt{hawaiianpizza}\} = \{\varepsilon, \texttt{a}, \texttt{h}, \texttt{i}, \texttt{n}, \texttt{p}, \texttt{w}, \texttt{z}, \texttt{ii}, \texttt{zz}, \texttt{awa}, \texttt{aiia}\}$

  $\#\mathcal{P}\{\texttt{w}\} = 12 < 13 = |\texttt{w}| + 1$ ✗

Rich words

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- Arnoux-Rauzy words

  [Droubay, Justin, Pirillo (2001)]

$$\mathbf{f} = \varphi^{\omega}(\mathtt{a}) = \mathtt{abaababaabaababaababaabaababaabaababaababaab}\cdots$$

$$\text{where} \quad \varphi = \begin{cases} \mathtt{a} \to \mathtt{ab} \\ \mathtt{b} \to \mathtt{a} \end{cases}$$

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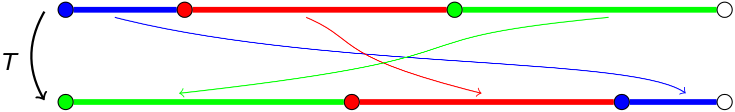

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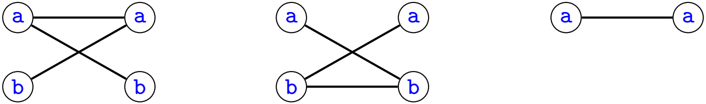
  [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2016)]

$\mathcal{E}(\varepsilon)$ $\mathcal{E}(\mathtt{a})$ $\mathcal{E}(\mathtt{b})$



$$\mathcal{L}(\mathbf{f}) = \{\varepsilon, \mathtt{a}, \mathtt{b}, \mathtt{aa}, \mathtt{ab}, \mathtt{ba}, \mathtt{aab}, \mathtt{aba}, \mathtt{baa}, \mathtt{bab}, \ldots\}$$

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- Complementary-symmetric Rote words

  [Blondin-Massé, Brlek, Labbé, Vuillon (2011)]

- Languages closed under reversal with factor complexity $\mathcal{C}(n) = 2n + 1$

  [Balková, Pelantová, Starosta (2009)]

- etc.

How many (finite) rich words?

Theorem [Guo, Shallit, Shur (2016), Rukavicka (2017)]

Let $\mathcal{R}_q(n)$ denote the number of rich words for of length $n \in \mathbb{N}$ over an alphabet of cardinality q.

- $\mathcal{R}_q(n)$ is superpolynomial;
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Can we construct new rich words from known ones?

$$\varphi(\mathtt{aaabbba}) = \mathtt{abababaaaab}$$

where $\varphi : \begin{cases} \mathtt{a} \to \mathtt{ab} \\ \mathtt{b} \to \mathtt{a} \end{cases}$

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Can we construct new rich words from known ones?

Theorem [Vesti (2014)]

Let u be a finite rich word.
There exist an infinite **aperiodic** rich word and an infinite **periodic** rich words such that u is a factor of both of them.

Morphisms

A *morphism* is a map $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ such that $\varphi(uv) = \varphi(u)\varphi(v)$ for all $u, v \in \mathcal{A}^*$.

A *substitution* is a morphism φ such that there exists $a \in \mathcal{A}$ with $\varphi(a) = av$ and $\lim_{n\to\infty} |\varphi^n(a)| = \infty$. The word $\varphi^\omega(a)$ is a *fixed point* of the substitution.

A morphism φ is *primitive* if there exists $k \in \mathbb{N}$ such that b is a factor of $\varphi^k(a)$ for all $a, b \in \mathcal{A}$.

Example (Fibonacci)

$$\varphi : \begin{cases} \mathtt{a} \to \mathtt{ab} \\ \mathtt{b} \to \mathtt{a} \end{cases}, \qquad \varphi^2 : \begin{cases} \mathtt{a} \to \mathtt{aba} \\ \mathtt{b} \to \mathtt{ab} \end{cases}$$

$$\mathbf{f} = \varphi^\omega(\mathtt{a}) = \mathtt{abaababaabaababaababaabaababaabababaab} \cdots$$

Conjugated morphisms

A morphism φ is *right conjugate* to a morphism ψ if there exists a word $x \in \mathcal{A}^*$, called the *conjugate word*, such that $\psi(a)x = x\varphi(a)$ for each $a \in \mathcal{A}$.

The *rightmost conjugate* to φ is (when it exists) a right conjugate to φ that is the only right conjugate to itself. We denote it by φ_R.

Example ($x =$ a)

$$\varphi : \begin{cases} \mathtt{a} \to \mathtt{bba} \\ \mathtt{b} \to \mathtt{a} \end{cases}, \qquad \varphi_R : \begin{cases} \mathtt{a} \to \mathtt{abb} \\ \mathtt{b} \to \mathtt{a} \end{cases}$$

If φ has no rightmost conjugate, then it is called *cyclic* and there exists $z \in \mathcal{A}$ such that $\varphi(a) \in z^*$ for each $a \in \mathcal{A}$. A fixed point of a cyclic morphism if periodic.

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If φ has no rightmost conjugate, then it is called *cyclic* and there exists $z \in \mathcal{A}$ such that $\varphi(a) \in z^*$ for each $a \in \mathcal{A}$. A fixed point of a cyclic morphism if periodic.

If φ and ψ are conjugates and $\mathbf{u}$ is a recurrent infinite word one has $\mathcal{L}(\varphi(\mathbf{u})) = \mathcal{L}(\psi(\mathbf{u}))$.

Since the palindromic richness can be seen as a property of a language (and not of an infinite word itself) it is enough to examine richness for one of these languages.

Arnoux-Rauzy morphisms

The *Arnoux-Rauzy* monoid is generated by *elementary Arnoux-Rauxy* morphisms:

- permutations over $\mathcal{A}$ and
- for each $a \in \mathcal{A}$

$$\psi_a : \begin{cases} a \to a \\ b \to ab \end{cases} \text{if } b \neq a \qquad \text{and} \qquad \widetilde{\psi}_a : \begin{cases} a \to a \\ b \to ba \end{cases} \text{if } b \neq a$$

Example (Fibonacci and Tribonacci)

$$\varphi = \psi_{\mathtt{a}} \circ \pi_{(\mathtt{ab})} : \begin{cases} \mathtt{a} \to \mathtt{ab} \\ \mathtt{b} \to \mathtt{a} \end{cases} , \qquad \tau = \psi_{\mathtt{a}} \circ \pi_{(\mathtt{abc})} : \begin{cases} \mathtt{a} \to \mathtt{ab} \\ \mathtt{b} \to \mathtt{ac} \\ \mathtt{c} \to \mathtt{a} \end{cases}$$

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A morphism over the binary alphabet $\{\mathtt{a}, \mathtt{b}\}$ is called *standard Sturmian* if it belongs to the monoid generated by $\pi_{(\mathtt{ab})}$ and φ.

Arnoux-Rauzy morphisms

Theorem [Glen, Justin, Widmer, Zamboni (2009)]

Let $\psi : \mathcal{A}^* \to \mathcal{A}^*$ be an Arnoux-Rauzy morphism and $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ s.t. $\mathcal{L}(\mathbf{u})$ is closed under reversal. Then

$$\mathbf{u} \text{ is rich} \iff \psi(\mathbf{u}) \text{ is rich.}$$

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$$\mathbf{u} \text{ is rich} \iff \psi(\mathbf{u}) \text{ is rich.}$$

Example (Fibonacci after Tribonacci)

The infinite word

$$\tau(\mathbf{f}) = \mathtt{abacababacabacababacababacabacababacabacababacababacabac}\cdots$$

is rich.

Class P_{ret}

A morphism $\psi : \mathcal{A}^* \to \mathcal{A}^*$ belongs to *Class P_{ret}*, if there exists a palindrome w, called *marker*, such that:

- $\psi(a)w$ is a palindromic complete return word to w for each $a \in \mathcal{A}$,

  (i.e., $\psi(a)w = w\widetilde{\psi(a)}$ and $|\psi(a)w|_w = 2$)
- $\psi(a) \neq \psi(b)$ for each $a, b \in \mathcal{A}$, $a \neq b$.

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Example ($\ell, p, q \in \mathbb{N}$, $\ell > 0$, $p \neq q$)

$$\psi_1 : \begin{cases} \mathtt{a} \to \mathtt{aba} \\ \mathtt{b} \to \mathtt{abaab} \end{cases}, \qquad \psi_2 : \begin{cases} \mathtt{a} \to \mathtt{bba} \\ \mathtt{b} \to \mathtt{b} \end{cases}, \qquad \psi_3 : \begin{cases} \mathtt{a} \to \mathtt{a}^\ell\mathtt{b}^p \\ \mathtt{b} \to \mathtt{a}^\ell\mathtt{b}^q \end{cases}$$

$$w_1 = \mathtt{abaaba} \qquad w_2 = \mathtt{bb} \qquad w_3 = \mathtt{a}^\ell$$

$$\psi_1(\mathtt{a})w_1 = \mathtt{abaabaaba} \quad , \quad \psi_1(\mathtt{b})w_1 = \mathtt{abaababaaba}$$

$\mathcal{C}lass\ P_{ret}$

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◇ Every permutation on $\mathcal{A}$ is in Class P_{ret} with marker ε,

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$$\pi_{(\mathtt{abc})} : \begin{cases} \mathtt{a} \to \mathtt{b} \\ \mathtt{b} \to \mathtt{c} \\ \mathtt{c} \to \mathtt{a} \end{cases} , \qquad \psi_{\mathtt{a}} : \begin{cases} \mathtt{a} \to \mathtt{a} \\ \mathtt{b} \to \mathtt{ab} \\ \mathtt{c} \to \mathtt{ac} \end{cases} , \qquad \widetilde{\psi}_{\mathtt{a}} : \begin{cases} \mathtt{a} \to \mathtt{a} \\ \mathtt{b} \to \mathtt{ba} \\ \mathtt{c} \to \mathtt{ca} \end{cases}$$

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Theorem [D., Pelantová (2021)]

Every Arnoux-Rauzy morphism is conjugate to a morphism in Class P_{ret}.

Class P_{ret}

Theorem [Balková, Pelantová, Starosta (2011)]

Let ψ_1, ψ_2 be in Class P_{ret} with marker w_1, w_2 respectively.
Then $\psi_2 \circ \psi_1$ is in Class P_{ret} with marker $\psi_2(w_1)w_2$.

Example

$$\psi_1 : \begin{cases} \mathtt{a} \to \mathtt{a} \\ \mathtt{b} \to \mathtt{ab} \end{cases}, \qquad \psi_2 : \begin{cases} \mathtt{a} \to \mathtt{bba} \\ \mathtt{b} \to \mathtt{b} \end{cases}, \qquad \psi_2 \circ \psi_1 : \begin{cases} \mathtt{a} \to \mathtt{bba} \\ \mathtt{b} \to \mathtt{bbab} \end{cases}$$

$$w_1 = \mathtt{a} \qquad w_2 = \mathtt{bb} \qquad \psi_2(w_1)w_2 = \mathtt{bba\,bb}$$

$$(\psi_2 \circ \psi_1)(\mathtt{a})\mathtt{bbabb} = \mathtt{bbabbabb} \quad , \qquad (\psi_2 \circ \psi_1)(\mathtt{b})\mathtt{bbabb} = \mathtt{bbabbbabb}$$

Class P_{ret} and Class P

A morphism $\psi : \mathcal{A}^* \to \mathcal{A}^*$ belongs to *Class P* if there exists a palindrome $p \in \mathcal{A}^*$ such that $\psi(a) = pq_a$ for each $a \in \mathcal{A}$, where q_a is a palindrome.

Any fixed point of a substitution from Class P contains infinitely many palindromes.

Proposition

Any morphism from Class P_{ret} is conjugate to an acyclic morphism from Class P.

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Example (The converse is not true)

$$\psi : \begin{cases} \mathtt{a} \to \mathtt{ababab} \\ \mathtt{b} \to \mathtt{ababaab} \end{cases}, \qquad \psi_R : \begin{cases} \mathtt{a} \to \mathtt{ababab} \\ \mathtt{b} \to \mathtt{abababa} \end{cases}$$

in Class P, acyclic $\qquad w_R = \mathtt{abababa}$

$$|\psi_R(\mathtt{a})w_R|_{w_R} = |\mathtt{abababababa}|_{w_R} = 4$$

Marked morphisms

An acyclic morphism ψ is

- *right marked* if the mapping $a \to \mathrm{Lst}(\psi_R(a))$ is injective on $\mathcal{A}$.
- *left marked* if the mapping $a \to \mathrm{Fst}(\psi_L(a))$ is injective on $\mathcal{A}$.

A morphism is *marked* if it is both right marked and left marked.

A marked morphism is *well-marked* if the mappings above are the identity on $\mathcal{A}$.

Example (Tribonacci)

$$\tau = \tau_R : \begin{cases} \mathtt{a} \to \mathtt{a}\underline{\mathtt{b}} \\ \mathtt{b} \to \mathtt{a}\underline{\mathtt{c}} \\ \mathtt{c} \to \underline{\mathtt{a}} \end{cases} , \qquad \tau_L : \begin{cases} \mathtt{a} \to \underline{\mathtt{b}}\mathtt{a} \\ \mathtt{b} \to \underline{\mathtt{c}}\mathtt{a} \\ \mathtt{c} \to \underline{\mathtt{a}} \end{cases}$$

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Proposition [D., Pelantová (2021)]

Let ψ be in Class P_{ret} and right marked. Then ψ is left marked too. Moreover there exists $k \geq 1$ such that ψ^k is well-marked.

Example (Tribonacci)

$$\tau^3 = \tau_R^3 : \begin{cases} \mathtt{a} \to \mathtt{abacab\underline{a}} \\ \mathtt{b} \to \mathtt{abaca\underline{b}} \\ \mathtt{c} \to \mathtt{aba\underline{c}} \end{cases}, \qquad \tau_L^3 : \begin{cases} \mathtt{a} \to \mathtt{\underline{a}bacaba} \\ \mathtt{b} \to \mathtt{\underline{b}acaba} \\ \mathtt{c} \to \mathtt{\underline{c}aba} \end{cases}$$

Marked morphisms

Theorem [D., Pelantová (2021)]

Let ψ be a marked morphism in Class P_{ret} and $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ s.t. $\mathcal{L}(\mathbf{u})$ is closed under reversal. If $\psi(\mathbf{u})$ is rich, then $\mathbf{u}$ is rich.

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Let $\psi : \{\mathtt{a}, \mathtt{b}\}^* \to \{\mathtt{a}, \mathtt{b}\}^*$ be a morphism conjugated to a morphism in Class P_{ret}, and let w be the marker associated to ψ_R. Assume that $\psi_R(\mathtt{ab})w$ is rich. Then

- If $\mathbf{u} \in \{\mathtt{a}, \mathtt{b}\}^{\mathbb{N}}$ is recurrent and rich, then $\psi(\mathbf{u})$ is rich.
- If $\mathbf{u} \in \{\mathtt{a}, \mathtt{b}\}^{\mathbb{N}}$ is a fixed point of ψ, and ψ is primitive, then $\psi(\mathbf{u}) = \mathbf{u}$ is rich.

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Corollary

Let $\psi : \{\mathtt{a}, \mathtt{b}\}^* \to \{\mathtt{a}, \mathtt{b}\}^*$ be a morphism from Class P_{ret} and $\mathbf{u} \in \{\mathtt{a}, \mathtt{b}\}^{\mathbb{N}}$ a non-unary recurrent word. If $\psi(\mathbf{u})$ is rich, then $\psi(\mathbf{v})$ is rich for every recurrent rich word $\mathbf{v} \in \{\mathtt{a}, \mathtt{b}\}^{\mathbb{N}}$.

To sum up

We can construct new rich words from known ones.

- Applying an arbitrary Arnoux-Rauzy morphism to a symmetric regular IET word gives a new rich word which is neither Arnoux-Rauzy nor a IET word.

  (see Fibonacci after Tribonacci).

- We can apply the results both to finite and infinite words.

  ([Vesti (2014)])

- Improve lower bound of rich words over a binary alphabet.

  (Each word of the form $\mathtt{a}^{m_1}\mathtt{b}^{n_1}\mathtt{a}^{m_2}\mathtt{b}^{n_2}\cdots\mathtt{a}^{m_k}\mathtt{b}^{n_k}$, with $m_1 \leq m_2 \leq \cdots \leq m_k$ and $n_1 \leq n_2 \leq \cdots \leq n_k$ is rich [Guo, Shallit, Shur (2016)])

Open questions

- Which tame morphisms preserve richness?
- How characterize dendric languages closed under reversal?
- How many finite rich words of given length are there over a given alphabet?
- Can we determine an optimal lower bound for the critical exponent?
  (Lower bounds on alphabets of cardinality $k = 2, 3, 4, 5$. [Baranwal, Shallit (2019)]
  The bound is the best possible for $k = 2$. [Currie, Mol, Rampersad (2020)]
  What about $k \geq 3$?)

THANK YOUOY KNAHT