Eigenvalues and Constant Arithmetic Progressions for Substitutive Sequences

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$$\sigma : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 00 \end{cases}$$

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➡ **decidability questions**

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➡ **decidability questions**

➡ **dynamical eigenvalues**

Introduction

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Rational eigenvalues

Arithmetic progressions

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Substitutions

Definition.

A **substitution** is an endomorphism of $\mathscr{A}^*$ such that there exist two letters $a, b \in \mathscr{A}$ satisfying :

1 - σ is right-prolongable on a,

2 - σ is left-prolongable on b,

3 - σ is growing.

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Word : $u \in \mathscr{A}^+$

Sequence : $x \in \mathscr{A}^{\mathbb{Z}}$ or $\mathscr{A}^{\mathbb{N}}$

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Word : $u \in \mathscr{A}^+$

Sequence : $x \in \mathscr{A}^{\mathbb{Z}}$ or $\mathscr{A}^{\mathbb{N}}$

Definition.

Purely substitutive sequence :
admissible fixed point of a substitution

$$x = \sigma^\infty(b \,.\, a)$$

Substitutive sequence :
image of a purely substitutive sequence under a letter-to-letter morphism

$$y = \phi(x) = \phi(\sigma^\infty(b \,.\, a))$$

Background

Substitutions

Example.

$$\sigma : \begin{cases} 0 \mapsto 012 \\ 1 \mapsto 001 \\ 2 \mapsto 201 \end{cases} \text{ and } \phi : \begin{cases} 0 \mapsto a \\ 1 \mapsto b \\ 2 \mapsto b \end{cases}$$

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$$x = \sigma^{\infty}(1.0) = \cdots 012012001.012001201 \cdots$$

is a purely substitutive sequence.

$$y = \phi(x) = \phi \circ \sigma^{\infty}(1.0) = \cdots abbabbaab\,.\,abbaabbab \cdots$$

is a substitutive sequence.

Background

Substitutive dynamical systems

Definition.

The **substitutive dynamical system** associated to a substitution σ is the system (X_σ, S) where S is the shift map and

$$X_\sigma = \{x \in \mathscr{A}^{\mathbb{Z}} : \mathscr{L}(x) \subset \mathscr{L}(\sigma)\}.$$

Background

Periodicity of morphic sequences

- **Pansiot, 1986 :** the periodicity of a fixed point of morphism is decidable.
- **Harju and Linna, 1986 :** the ultimate periodicity of a fixed point of morphism is decidable.
- **Honkala, 1986 :** the periodicity of automatic sequences is decidable.
- **Halava, Harju, Kärki and Rigo, 2010 :** the ultimate p-periodicity of a morphic sequence is decidable, for any given period p.
- **Durand, 2012 and 2013 :** the periodicity of morphic (primitive) sequences is decidable.

Background

Constant arithmetic progression

Arithmetic progression of a sequence x : subsequence of the form $(x_{k+np})_{n\in\mathbb{Z}}$

Constant arithmetic progression if $x_{k+np} = x_k$, for all $n \in \mathbb{Z}$.

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Arithmetic progression of a sequence x : subsequence of the form $(x_{k+np})_{n\in\mathbb{Z}}$.

Constant arithmetic progression if $x_{k+np} = x_k$, for all $n \in \mathbb{Z}$.

An integer p is an **essential period** for the constant arithmetic progression $(x_{k+np})_{n\in\mathbb{Z}}$ if, for all divisor q of p, the arithmetic progression $(x_{k+nq})_{n\in\mathbb{Z}}$ is not constant.

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Rational eigenvalues

Arithmetic progressions

Rational eigenvalues

Motivation

Proposition [folklore ; Durand, G].

Let σ be a **primitive** substitution and x a fixed point of σ.

If x has a constant arithmetic progression of essential period p, then $\exp(2i\pi/p)$ is an eigenvalue of the dynamical system (X_σ, S).

Rational eigenvalues

Motivation

Proposition [folklore ; G].

Let σ be a **primitive** substitution and x a fixed point of σ.

If x has a constant arithmetic progression of essential period p, then $\exp(2i\pi/p)$ is an eigenvalue of the dynamical system (X_σ, S).

Idea.

➡ Compute the rational eigenvalues.

➡ Check if there exist some periods among this set.

Rational eigenvalues

Eigenvalues associated to a dynamical system

Definition.

A complex number $\lambda \in \mathbb{C}$ is an **eigenvalue** for the dynamical system $(X, \mathscr{B}, \mu, T)$ if there exists a fonction $f \in L^2(X, \mu)$ such that

$$f \circ T = \lambda f \text{ for } \mu\text{-almost all } x \in X.$$

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If, moreover, $\lambda \in \exp(2i\pi\mathbb{Q})$, we say λ is a **rational eigenvalue**.

Rational eigenvalues

Constant-length case

Theorem [Dekking 1978].

Let σ be a substitution of constant length l. Then, the set of eigenvalues associated to (X_σ, S) is

$$\left\{ \exp\left(\frac{2ik\pi}{hl^m}\right) : k \in \mathbb{Z}, m \in \mathbb{N} \right\},$$

where h is the *height* of σ.

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where h is the *height* of σ.

Definition.

Let σ be a substitution of constant length l and x a fixed point of σ. The **height** of σ is the number h defined by

$$h = \max\{n \geq 1 : (n, l) = 1,\ n \text{ divides } gcd\{i \geq 1 : x_i = x_0\}\}.$$

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$$\sigma : \begin{cases} 0 \mapsto 012 \\ 1 \mapsto 101 \\ 2 \mapsto 210 \end{cases}$$

$x = \cdots 101012101.012101210101012101 \cdots$

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$$x = \cdots 101012101.0121012101010121 01\cdots$$

$$x_{2k} \in \{0,2\} \qquad x_{2k+1} \in \{1\}$$

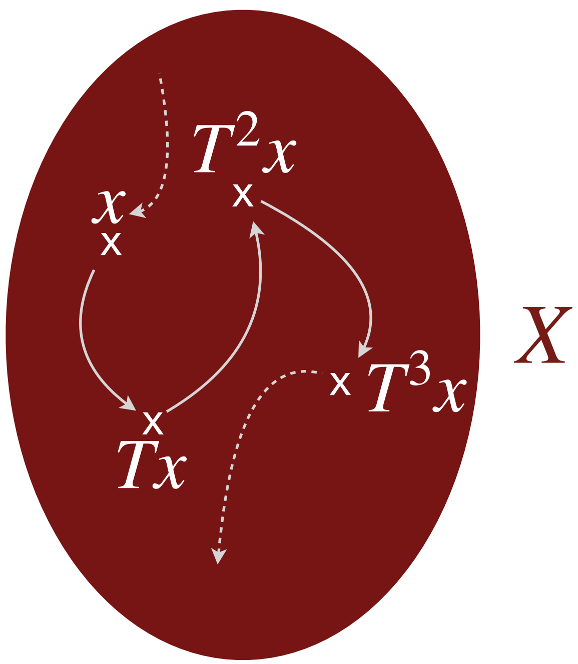
Rational eigenvalues

Characterization of rational eigenvalues

Definition.

We call **periodic spectrum** of a dynamical system (X, T) the set of its essential periods :

$$\mathbb{P}(X,T) = \{p \geq 2 : p \text{ is an essential period for a clopen } U\}.$$



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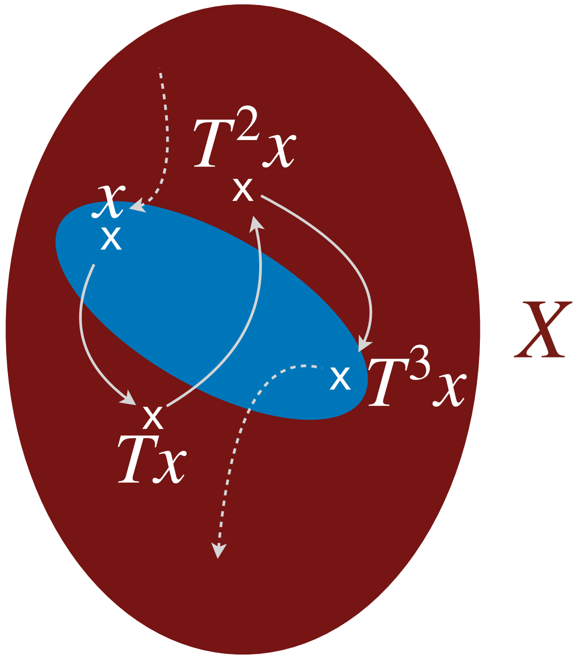
More precisely :

There exists $U \subset X$ and $x \in U$ such that

$$\mathrm{PS}_p(x, U) = \{k \in \mathbb{Z} : T^{k+np}x \in U, \forall n \in \mathbb{Z}\}$$

is non-empty



Rational eigenvalues

Characterization of rational eigenvalues

Proposition [folklore ; Durand, G].

Let (X, T) be a minimal dynamical system and $p \in \mathbb{N}^*$ an integer. The following properties are equivalent.

1. $\lambda = \exp(2i\pi/p)$ is a continuous eigenvalue of (X, T).
2. p belongs to $\mathbb{P}(X, T)$.
3. There exists a minimal closed subset V of X such that $\{V, T^{-1}V, \ldots, T^{-p+1}V\}$ is a partition of X.
4. (X, T) admits a periodic factor with period p.

Rational eigenvalues

Characterization of rational eigenvalues

Exemple. Dynamical system associated to the Thue-Morse sequence.

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1. Eigenvalues: $\{\exp(2ik\pi/2^m) : m \in \mathbb{N}\}$

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1. Eigenvalues: $\{\exp(2ik\pi/2^m) : m \in \mathbb{N}\}$
2. Periodic spectrum: $\{2^m : m \in \mathbb{N}\}$

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2. Periodic spectrum: $\{2^m : m \in \mathbb{N}\}$
3. For all $m \in \mathbb{N}$, there exists a minimal set V with period 2^m
4. For all $m \in \mathbb{N}$, $(\mathbb{Z}/2^m\mathbb{Z}, +)$ is a factor of (X_σ, S).

Rational eigenvalues

Characterization of rational eigenvalues

Example. Fibonacci dynamical system.

$$\sigma : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases}$$

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Example. Fibonacci dynamical system.

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1. Eigenvalues: $\{\exp\big(n\varphi \pmod 1\big) : n \in \mathbb{N}\}$, rational eigenvalues : $\{1\}$

Rational eigenvalues

Characterization of rational eigenvalues

Example. Fibonacci dynamical system.

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3. There exist no proper periodic subset

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1. Eigenvalues: $\{\exp\big(n\varphi \,(\bmod\ 1)\big) \,:\, n \in \mathbb{N}\}$, rational eigenvalues : $\{1\}$
2. Periodic spectrum : $\{1\}$
3. There exist no proper periodic subset
4. The dynamical system (X_σ, S) has no proper factor

Rational eigenvalues

Computation of rational eigenvalues

Proposition [Durand, G].

The set of rational eigenvalues associated to a substitutive dynamical system (X_σ, S) is computable.

If, moreover, the substitution σ is *proper*, then these eigenvalues only depend on the incidence matrix of σ.

Rational eigenvalues

Computation of rational eigenvalues

Proposition [Durand, G].

The set of rational eigenvalues associated to a substitutive dynamical system (X_σ, S) is computable.

If, moreover, the substitution σ is *proper*, then these eigenvalues only depend on the incidence matrix of σ.

Remark. Case σ is not proper [Durand 2000].

➡ There exists a proper substitution ϕ such that the dynamical system (X_σ, S) is conjugated to (X_ϕ, S)

Rational eigenvalues

Computation of the periodic spectrum - step 1/3

Example.

Computation of rational eigenvalues for the substitution

$$\sigma : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0011 \end{cases}$$

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Lemma [Durand 2000 ; Durand, G].

Let σ be a proper, non-periodic substitution, and M_σ its incidence matrix.

1. An integer p belongs to $\mathbb{P}(X_\sigma, S)$ if and only if there exists $m \in \mathbb{N}$ such that $1M_\sigma^m \in p\mathbb{Z}^d$.

2. A prime number p belongs to $\mathbb{P}(X_\sigma, S)$ if and only if $1M_\sigma^d \in p\mathbb{Z}^d$.

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$M_\sigma = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ thus $M_\sigma^2 = \begin{pmatrix} 3 & 6 \\ 3 & 6 \end{pmatrix}$ and $(1,1)\ M_\sigma^2 = (6,12)$.

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Prime numbers belonging to the spectrum $\mathbb{P}(X_\sigma, S)$: 2 and 3.

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Lemma [Durand 2000].

Let M be a $d \times d$-matrix and p a prime number. The following properties are equivalent.

1. $\forall n \in \mathbb{N}, \exists k \in \mathbb{N} : 1M^k \in p^n \mathbb{Z}^d$
2. p divides $GCD(a_0, \ldots, a_r)$ with $r = \max\{i \in \mathbb{N} : \{1, 1M, \ldots 1M^r\}$ is free$\}$

and $Q(X) = \sum_{i=0}^{r+1} a_i X^i$ is the characteristic polynom of the restriction of M to the subspace generated by $1, 1M, \ldots 1M^r$.

Rational eigenvalues

Computation of the periodic spectrum - step 2/3

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Computation of rational eigenvalues for the substitution

$$\sigma : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0011 \end{cases}$$

$M_\sigma = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ thus $1M_\sigma = (2{,}4)$.

We compute $Q(X) = X^2 - 3X$ and $GCD(0, -3) = 3$.

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Conclusion : 3 has unbounded exponent in $\mathbb{P}(X_\sigma, S)$.

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Computation of the periodic spectrum - step 3/3

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Lemma [Durand, G].

Let p be a prime number and m an integer.

If the maximal power of p that divides $1M^m$ is the same as the one that divides $1M^{m+d}$, then it is equal to the maximal power that divides each $1M^k, k \in \mathbb{N}$.

Moreover, this maximal power divides $1M^{p^d}$.

Rational eigenvalues

Computation of the periodic spectrum - step 3/3

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Computation of rational eigenvalues for the substitution

$$\sigma : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0011 \end{cases}$$

$M_\sigma^4 = \begin{pmatrix} 27 & 54 \\ 27 & 54 \end{pmatrix}$ thus $(1,1)M_\sigma^4 = (54,108) = (2 \times 3^4,\ 2^2 \times 3^4)$.

Lemma [Durand, G].

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Conclusion : the maximal exponent of 2 in $\mathbb{P}(X_\sigma\ , S)$ is 1.

Lemma [Durand, G].

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Moreover, this maximal power divides $1M^{p^d}$.

Rational eigenvalues

Computation of the periodic spectrum - conclusion

Example.

Computation of rational eigenvalues for the substitution

$$\sigma : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0011 \end{cases}$$

Conclusion : $\mathbb{P}(X_\sigma, S) = \{3^m, 2 \times 3^m : m \in \mathbb{N}\}$

Rational eigenvalues

Computation of the periodic spectrum - conclusion

Example.

Computation of rational eigenvalues for the substitution

$$\sigma : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0011 \end{cases}$$

Conclusion : $\mathbb{P}(X_\sigma, S) = \{3^m, 2 \times 3^m : m \in \mathbb{N}\}$

Set of eigenvalues : $\{\exp(2ik\pi/3^m), \ \exp(2ik\pi/(2 \times 3^m)) : m \in \mathbb{N}, k \in \mathbb{Z}\}$

Introduction

Background

Rational eigenvalues

Arithmetic progressions

Arithmetic progressions

General case

Proposition [Durand, G].

Let σ be a primitive substitution and x a fixed point of σ.

If x admits a constant arithmetic progression with essential period p, then p belongs to the periodic spectrum $\mathbb{P}(X_\sigma, S)$.

Arithmetic progressions

General case

Proposition [Durand, G].

Let σ be a primitive substitution and x a fixed point of σ.

If x admits a constant arithmetic progression with essential period p, then p belongs to the periodic spectrum $\mathbb{P}(X_\sigma, S)$.

Idea of the proof.

- Define a p-periodic set defined by the constant arithmetic progression.
- Apply the following proposition.

Arithmetic progressions

General case

Proposition [folklore ; Durand, G].

Let (X, T) be a minimal dynamical system and $p \in \mathbb{N}^*$ an integer. The following properties are equivalent.

1. $\lambda = \exp(2i\pi/p)$ is a continuous eigenvalue of (X, T)
2. p belongs to $\mathbb{P}(X, T)$
3. There exists a closed subset V of X such that $\{V, T^{-1}V, \ldots, T^{-p+1}V\}$ is a partition of X
4. (X, T) admits a periodic factor with period p.

Arithmetic progressions

Proper case

Algorithm. For a given integer p, check if a sequence admits a constant arithmetic progression with period p :

- compute the periodic spectrum $\mathbb{P}(X_\sigma, S)$,

- compute the essential period $\tilde{p} \in \mathbb{P}(X_\sigma, S)$ corresponding to p,

- check if the images of letters under σ^{m_p}, contain a constant arithmetic progression.

These words have lengths multiple of $\tilde{p}$.

Arithmetic progressions

Proper case

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These words have lengths multiple of $\tilde{p}$.

Lemma [Durand 2000 ; Durand, G].

Let σ be a proper, non-periodic substitution, and M_σ its incidence matrix.

1. An integer $\tilde{p}$ belongs to $\mathbb{P}(X_\sigma, S)$ if and only if there exists $m_{\tilde{p}} \in \mathbb{N}$ such that $1M_\sigma^{m_{\tilde{p}}} \in \tilde{p}\mathbb{Z}^d$
2. A prime number $\tilde{p}$ belongs to $\mathbb{P}(X_\sigma, S)$ if and only if $1M_\sigma^d \in \tilde{p}\mathbb{Z}^d$

Arithmetic progressions

Proper case

Example.

For $\sigma : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0011 \end{cases}$ we have $\mathbb{P}(X_\sigma, S) = \{3^m, 2 \times 3^m : m \in \mathbb{N}\}$

➡ Arithmetic progression of period 10 ?

➡ Arithmetic progression of period 12 ?

Arithmetic progressions

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➡ Arithmetic progression of period 10 ? $\tilde{p} = 2$

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Arithmetic progressions

Proper case

Example.

For $\sigma : \begin{cases} 0 \mapsto \boxed{0}1 \\ 1 \mapsto \boxed{0}0\boxed{1}1 \end{cases}$ we have $\mathbb{P}(X_\sigma, S) = \{3^m, 2 \times 3^m : m \in \mathbb{N}\}$

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➡ Arithmetic progression of period 10 ? No.

➡ Arithmetic progression of period 12 ? $\tilde{p} = 6$

Arithmetic progressions

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➡ Arithmetic progression of period 12 ? $\tilde{p} = 6$

$$\sigma^2 : \begin{cases} 0 \mapsto 010011 \\ 1 \mapsto 010100110011 \end{cases}$$

Arithmetic progressions

Proper case

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For $\sigma : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0011 \end{cases}$ we have $\mathbb{P}(X_\sigma, S) = \{3^m, 2 \times 3^m : m \in \mathbb{N}\}$

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Arithmetic progressions

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- ➡ Arithmetic progression of period 12 ? $\tilde{p} = 6$

$$\sigma^2 : \begin{cases} 0 \mapsto \boxed{0}10011 \\ 1 \mapsto \boxed{0}10100\boxed{1}10011 \end{cases}$$

Arithmetic progressions

Proper case

Example.

For $\sigma : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0011 \end{cases}$ we have $\mathbb{P}(X_\sigma, S) = \{3^m, 2 \times 3^m : m \in \mathbb{N}\}$

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Arithmetic progressions

Proper case

Example.

For $\sigma : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0011 \end{cases}$ we have $\mathbb{P}(X_\sigma, S) = \{3^m, 2 \times 3^m : m \in \mathbb{N}\}$

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Arithmetic progressions

Proper case

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For $\sigma : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0011 \end{cases}$ we have $\mathbb{P}(X_\sigma, S) = \{3^m, 2 \times 3^m : m \in \mathbb{N}\}$

➡ Arithmetic progression of period 10 ? No.

➡ Arithmetic progression of period 12 ? Yes.

$$\sigma^2 : \begin{cases} 0 \mapsto 010011 \\ 1 \mapsto 010100110011 \end{cases}$$

$x_{6n+1} = 1$ and $x_{6n+2} = 0$

Arithmetic progressions

Constant length case

Question : Describe all the essential periods of constant arithmetic progressions.

Arithmetic progressions

Constant length case

Question : Describe all the essential periods of constant arithmetic progressions.

Idea.

- Description of the eigenvalues $\Rightarrow$ each essential period divides a hl^m.
- Describe the letters appearing in the arithmetic progressions of the form $(x_{k+nhl^m})_{n\in\mathbb{Z}}$.
- Recursive construction in a graph.

Arithmetic progressions

Constant length case

Definition.

Let σ be a substitution of constant length with height h. We define the **directed labelled graph** $G(\sigma)$ by the following process :

- the first h vertices are the alphabet $\mathscr{A}_i$, $0 \leq i \leq h-1$ containing the letters with the same label when computing the height,

- $(\mathscr{C}, \mathscr{D})$ is an edge of the graph with label i if

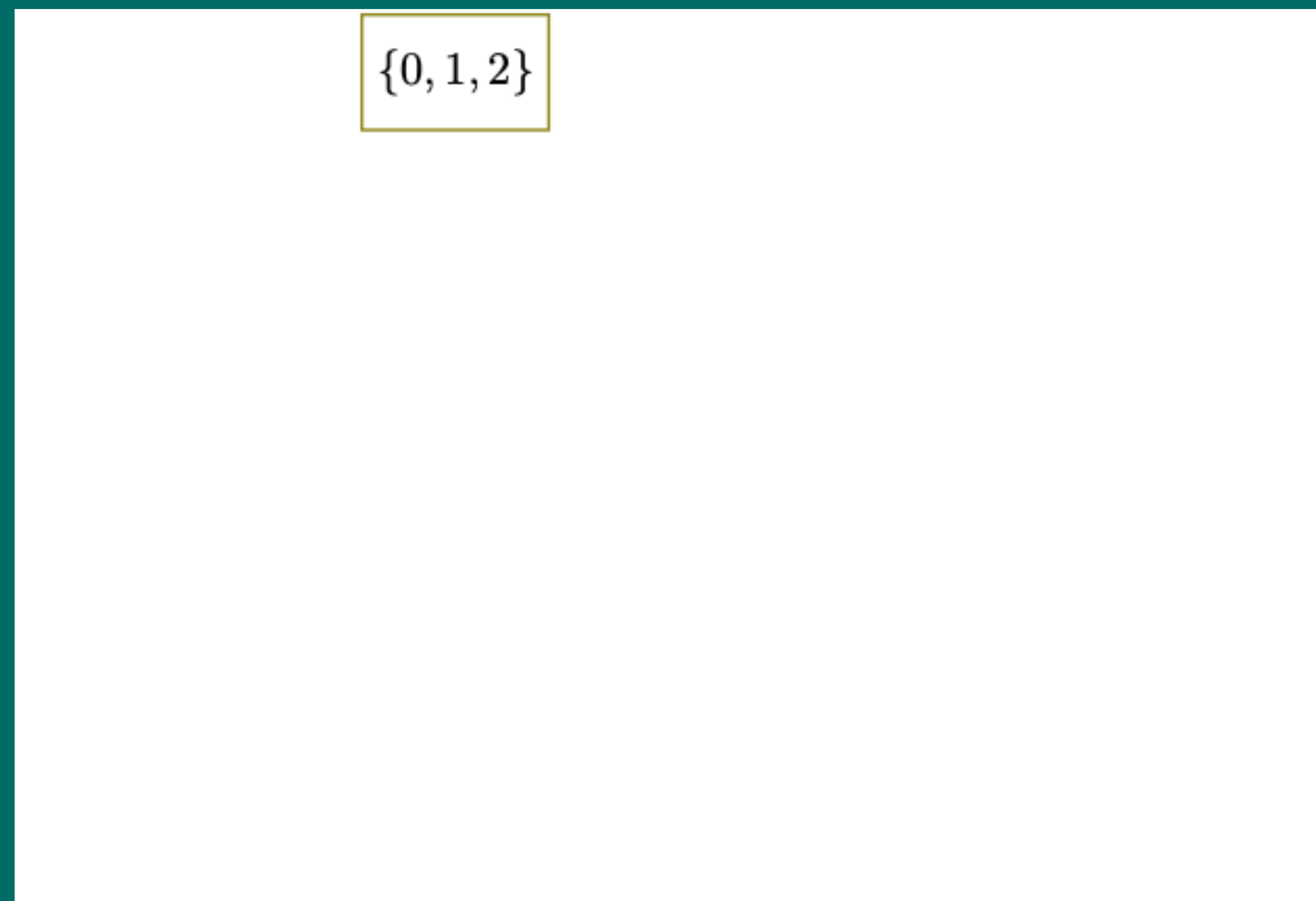
$$\mathscr{D} = \{\sigma(b)_i : b \in \mathscr{C}\}.$$

Arithmetic progressions

Constant length case

Example : purely substitutive sequence.

$$\sigma : \begin{cases} 0 \mapsto 012 \\ 1 \mapsto 001 \\ 2 \mapsto 201 \end{cases}$$



Arithmetic progressions

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Example : purely substitutive sequence.

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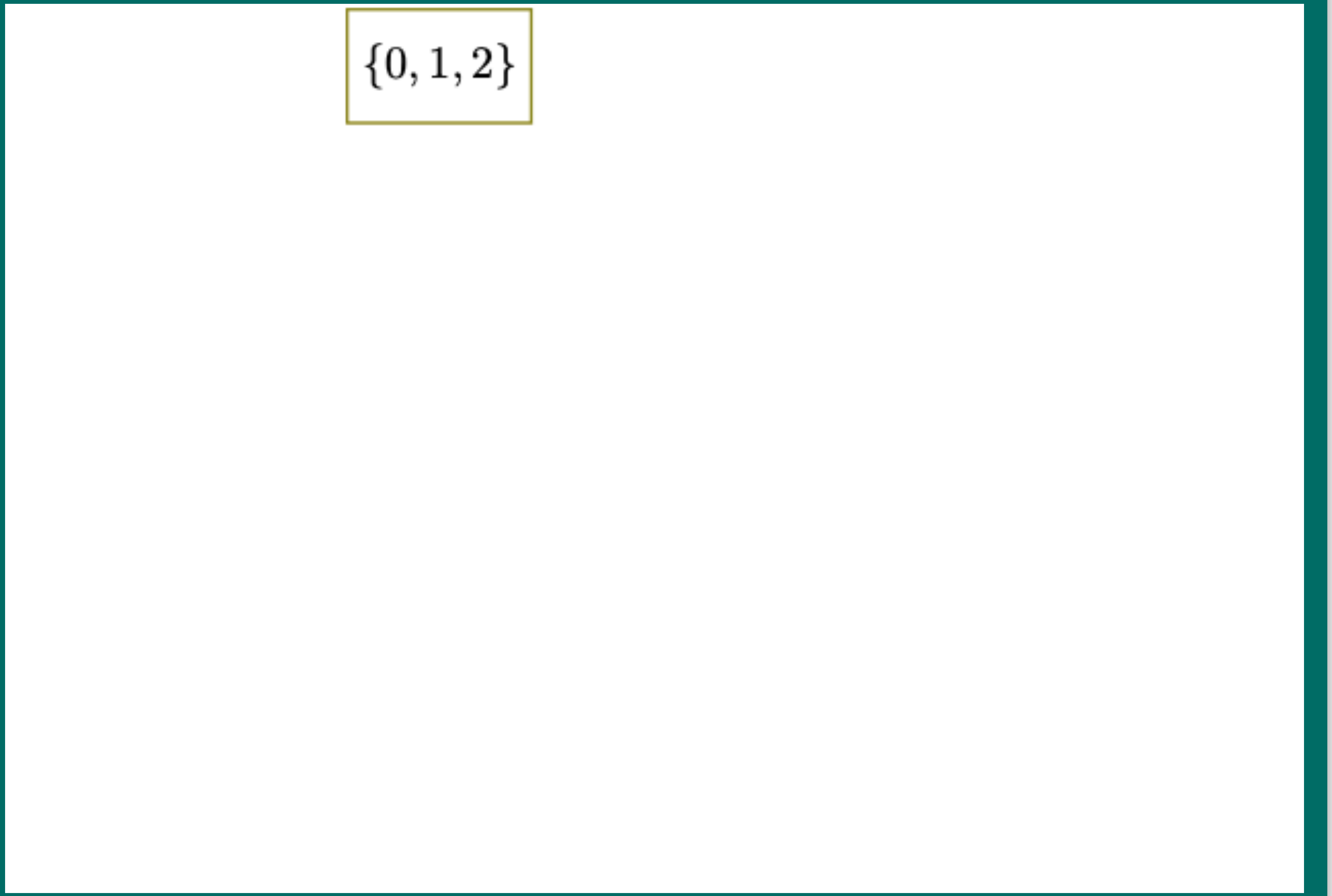

Arithmetic progressions

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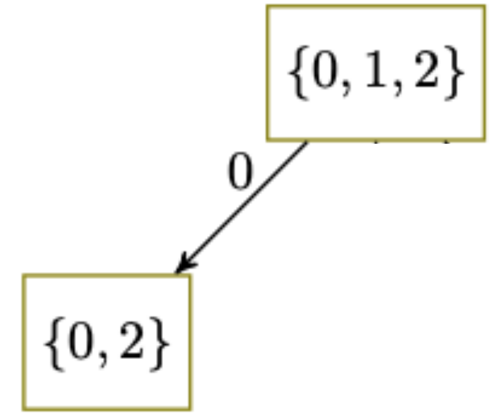
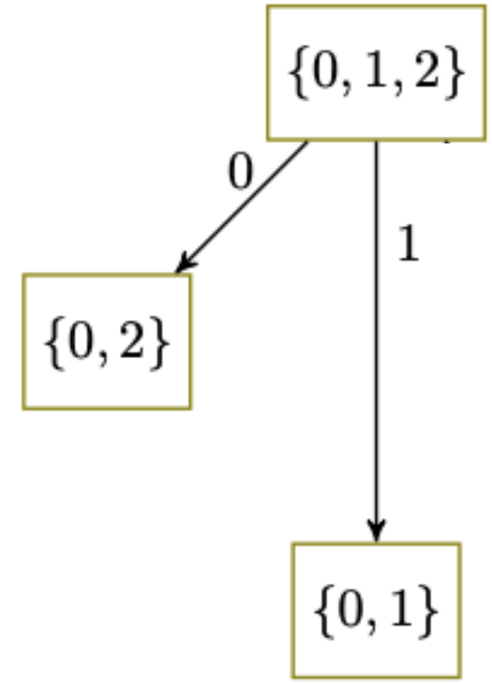


Arithmetic progressions

Constant length case

Example : purely substitutive sequence.

$$\sigma : \begin{cases} 0 \mapsto 0\boxed{1}2 \\ 1 \mapsto 0\boxed{0}1 \\ 2 \mapsto 2\boxed{0}1 \end{cases}$$

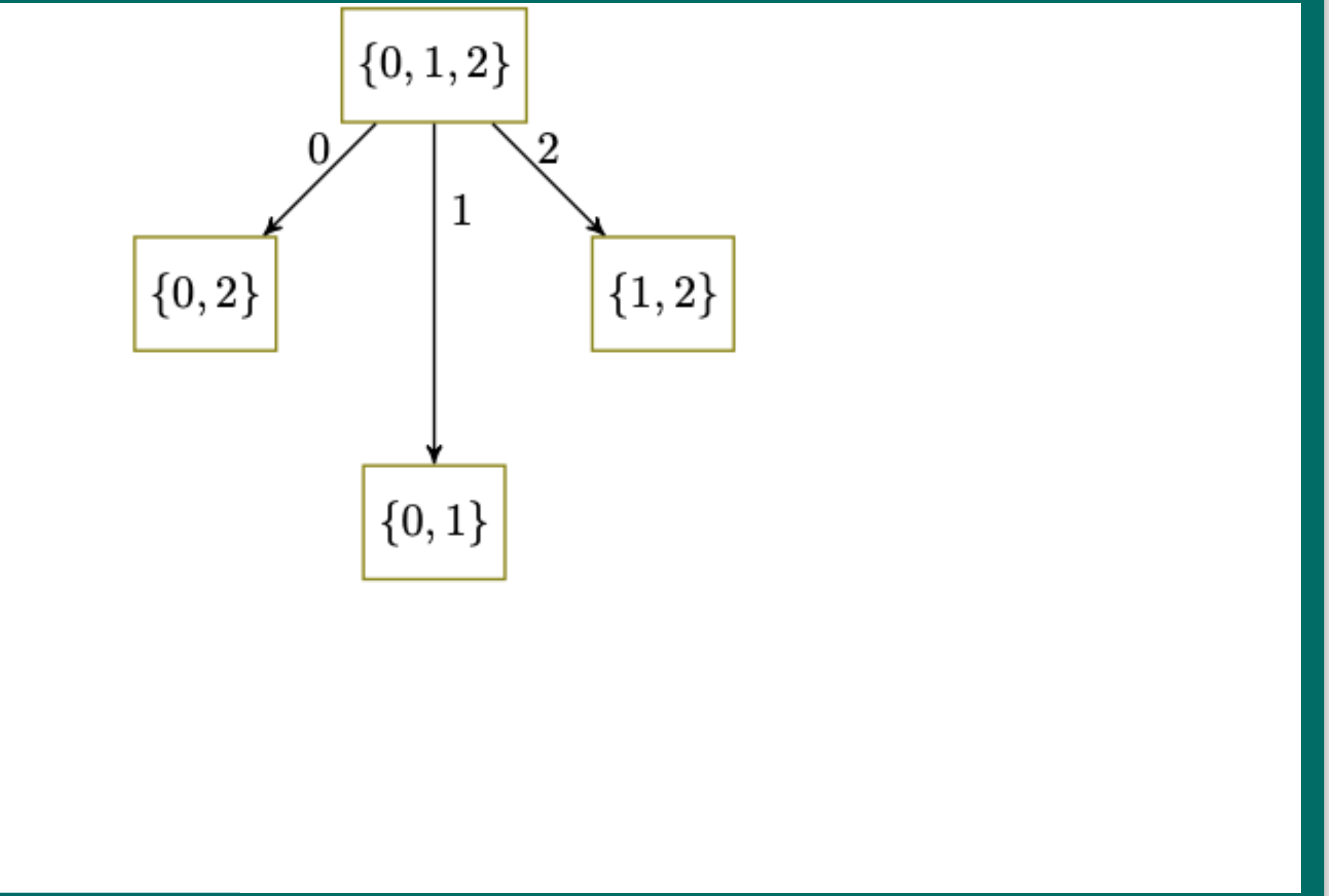


Arithmetic progressions

Constant length case

Example : purely substitutive sequence.

$$\sigma : \begin{cases} 0 \mapsto 01\boxed{2} \\ 1 \mapsto 00\boxed{1} \\ 2 \mapsto 20\boxed{1} \end{cases}$$



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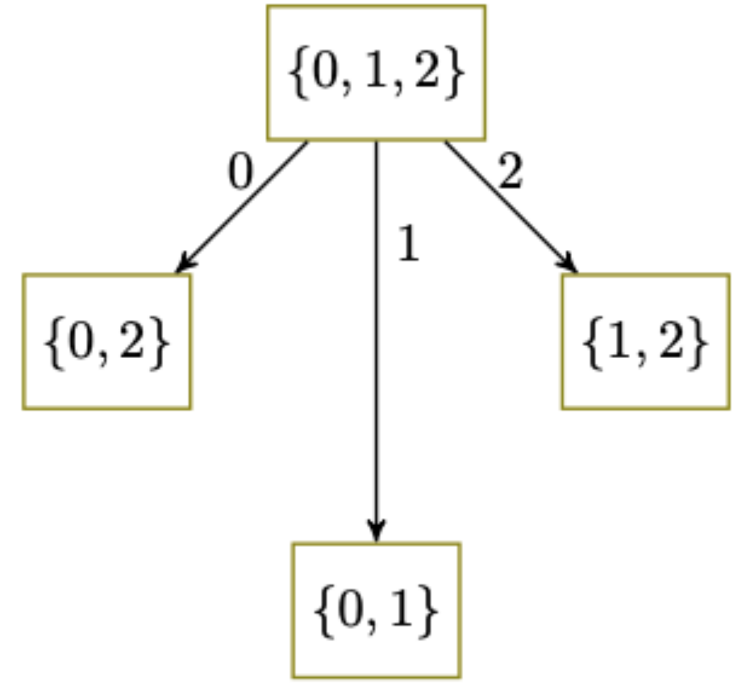
$$\sigma : \begin{cases} 0 \mapsto \boxed{0}12 \\ 1 \mapsto 001 \\ 2 \mapsto \boxed{2}01 \end{cases}$$

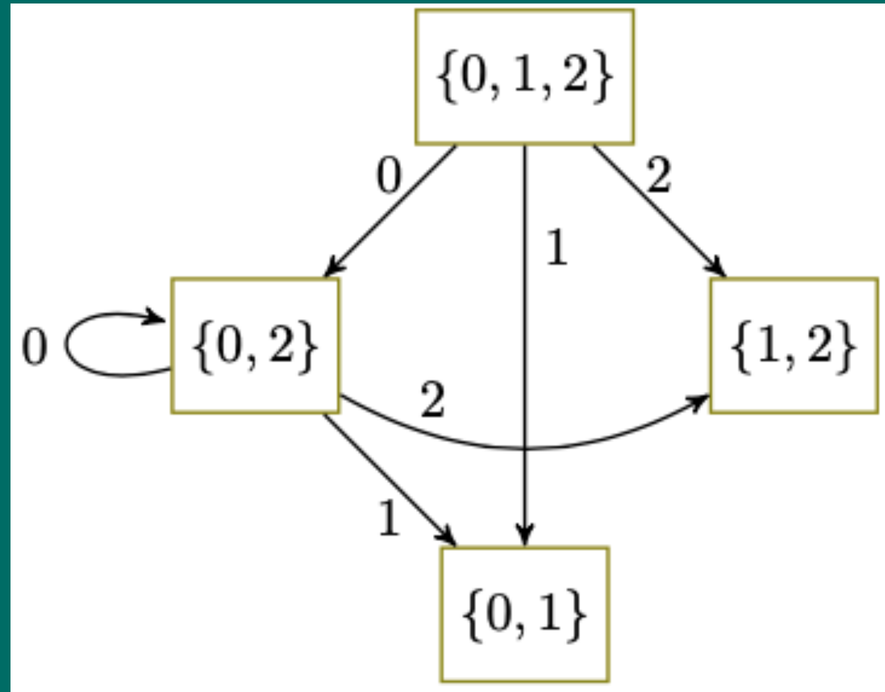


Arithmetic progressions

Constant length case

Example : purely substitutive sequence.

$$\sigma : \begin{cases} 0 \mapsto 012 \\ 1 \mapsto 001 \\ 2 \mapsto 201 \end{cases}$$



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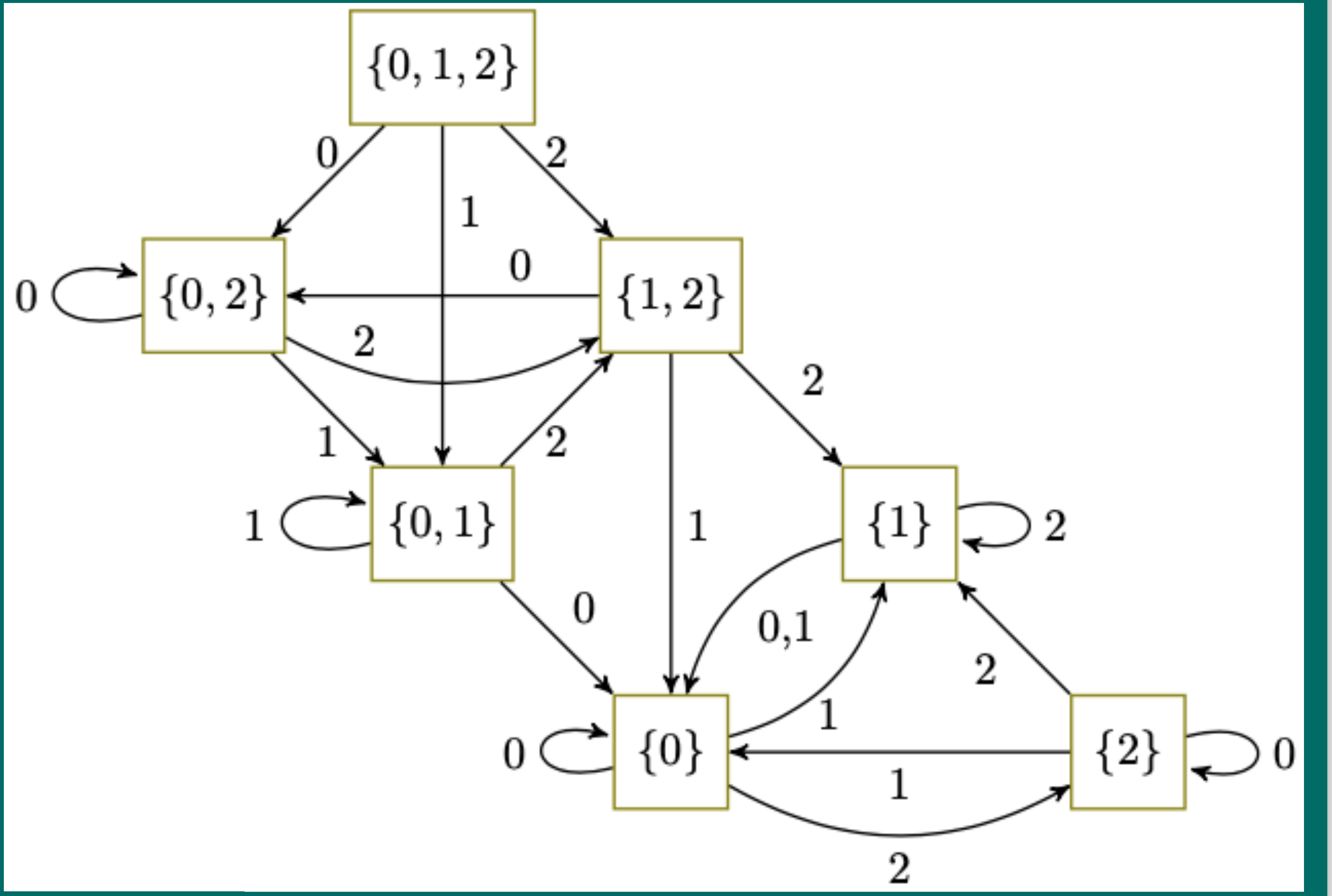

Arithmetic progressions

Constant length case

Proposition [Durand, G].

Let $x = \sigma^\infty(a)$ be a purely substitutive sequence and $y = \phi(x)$ a substitutive one, defined by a proper substitution σ and a letter-to-letter morphism ϕ.

1. The set $\mathscr{A}((x_{k+nhl^m})_{n\in\mathbb{Z}})$ is the final vertex of the admissible walk of $G(\sigma)$ starting from vertex $\mathscr{A}_{k_m}$ and labelled $(k_{m-1},\ldots,k_1,k_0)$, where $k = k_m l^m + k_{m-1} l^{m-1} + \cdots k_1 l + k_0$.
2. The set $\mathscr{A}((y_{k+nhl^m})_{n\in\mathbb{Z}})$ is the image under ϕ of the vertex described above.

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2. The set $\mathscr{A}((y_{k+nhl^m})_{n\in\mathbb{Z}})$ is the image under ϕ of the vertex described above.

Proposition [Durand, G].

The sequence x (resp. y) admits a constant arithmetic progression if, and only if, there exists a vertex of $G(\sigma)$ (resp. of the image of $G(\sigma)$ under ϕ) that is a singleton.

Arithmetic progressions

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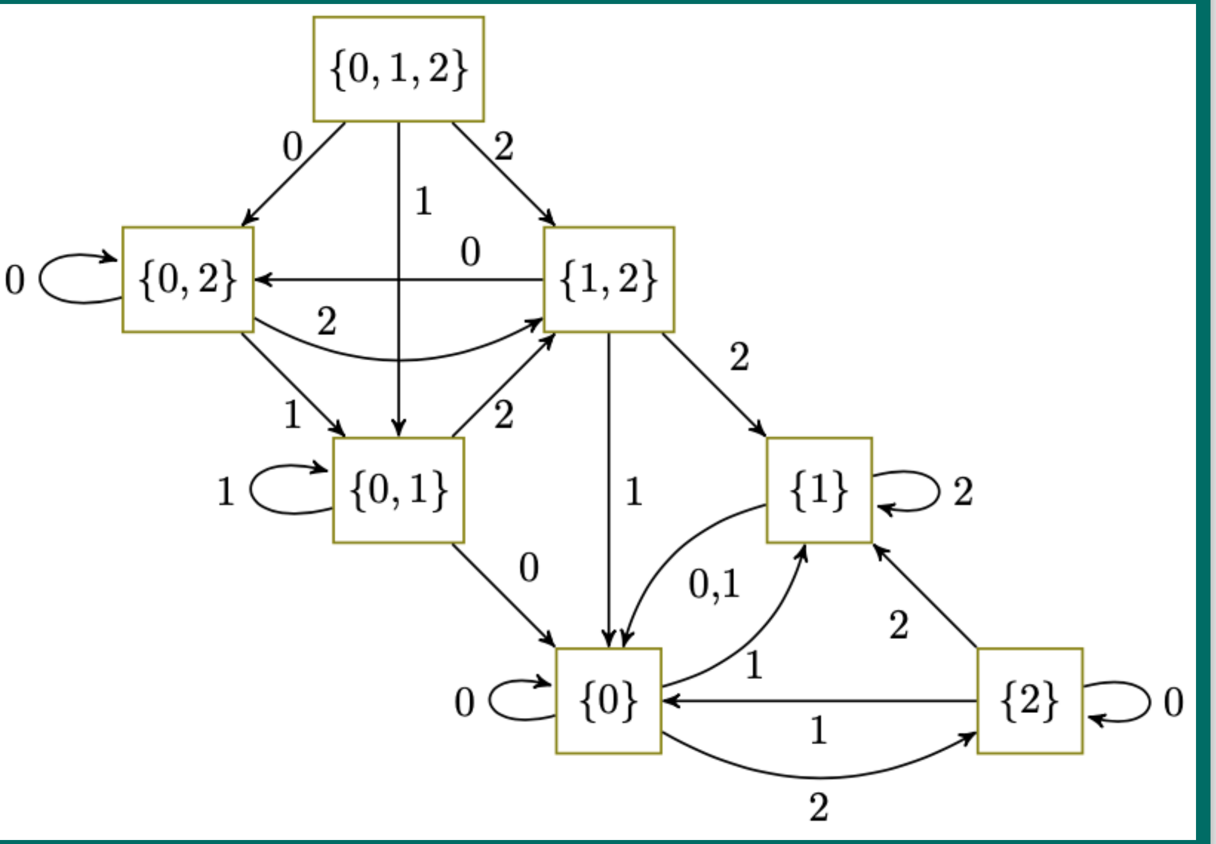

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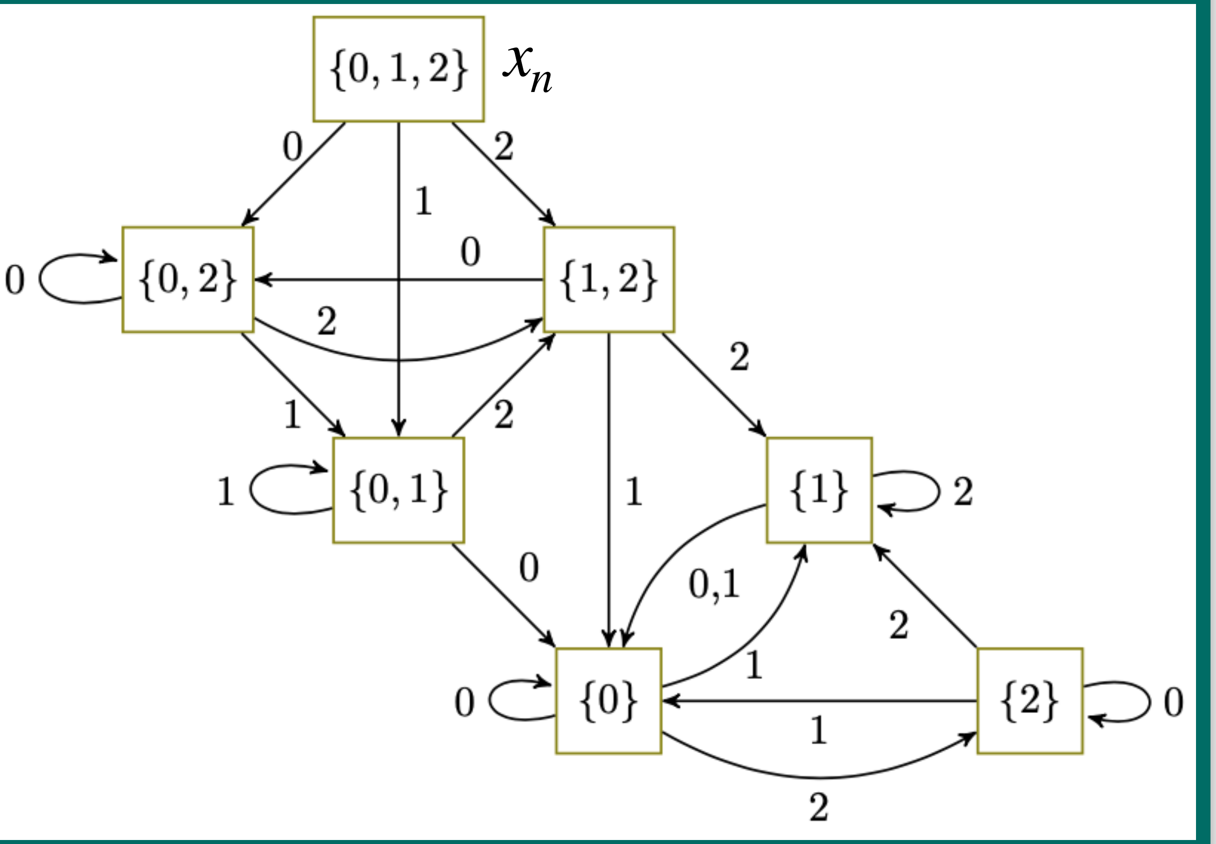

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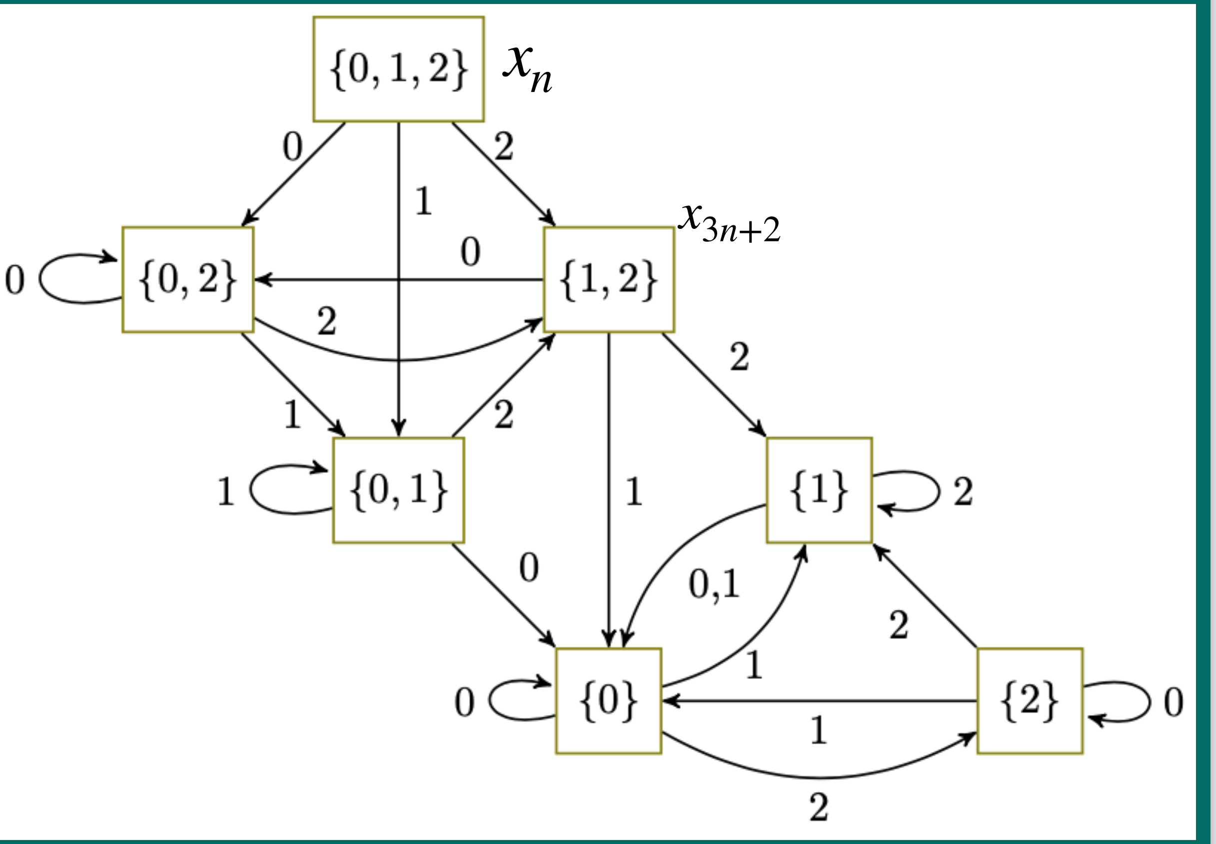

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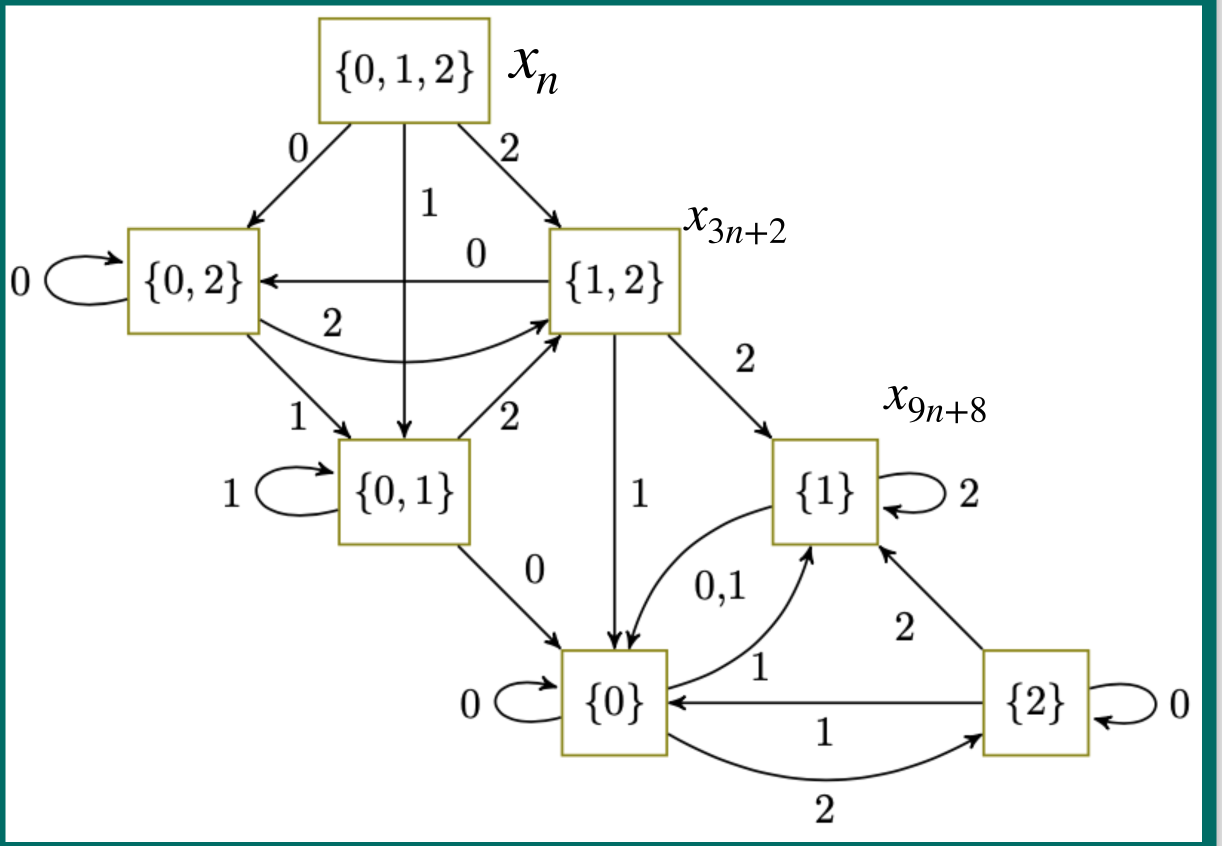

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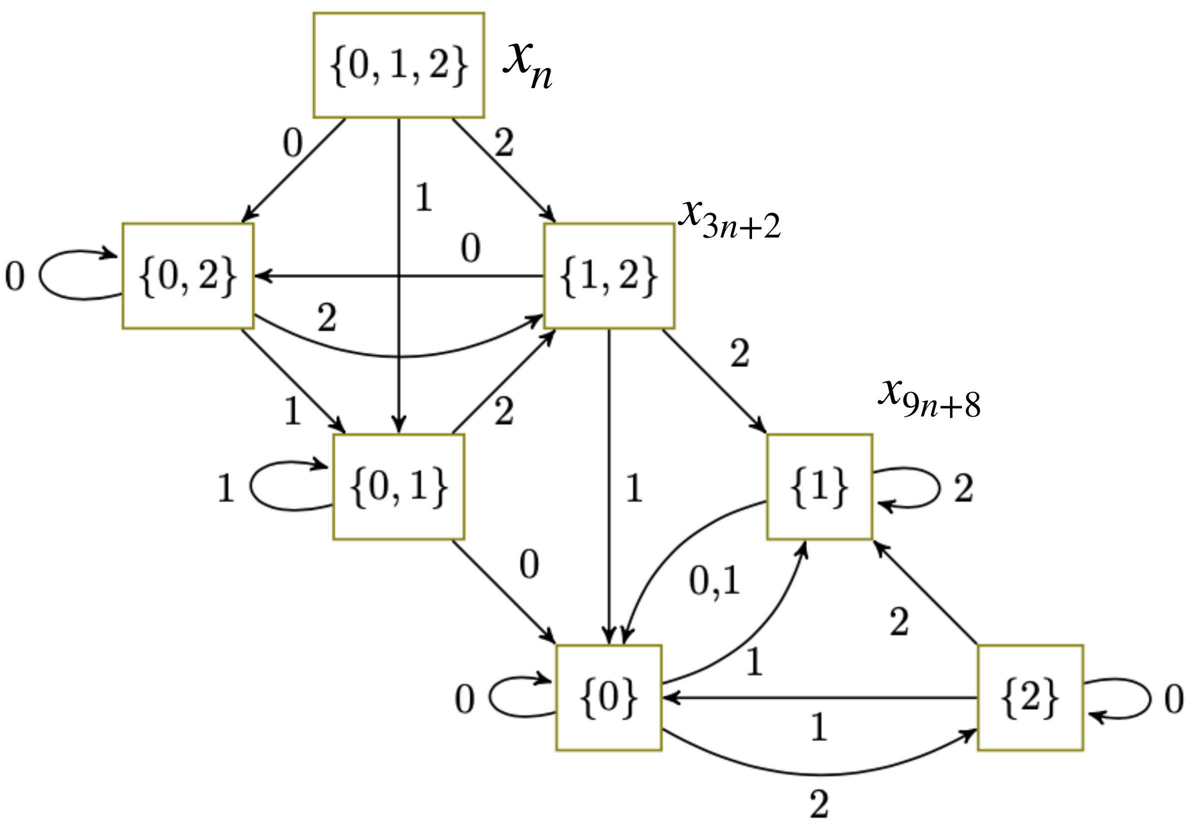


We have, for example

$$x_{9n+8} = 1 \ , \ x_{9n+3} = 0 \ , \ x_{9n+7} = 0 \ ,$$

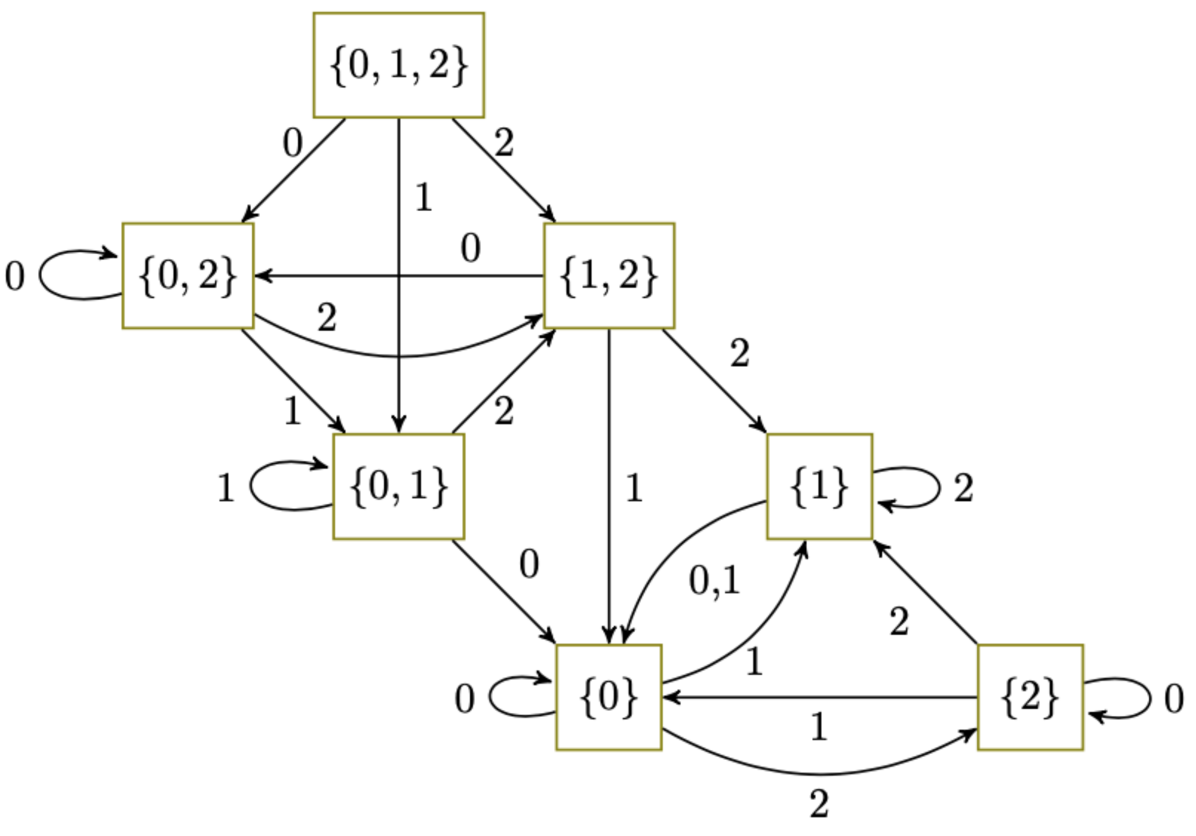
$\forall n \in \mathbb{Z}$

Arithmetic progressions

Constant length case

Example : substitutive sequence.

$$\sigma : \begin{cases} 0 \mapsto 012 \\ 1 \mapsto 001 \\ 2 \mapsto 201 \end{cases} \text{ and } \phi : \begin{cases} 0 \mapsto a \\ 1 \mapsto b \\ 2 \mapsto b \end{cases}$$



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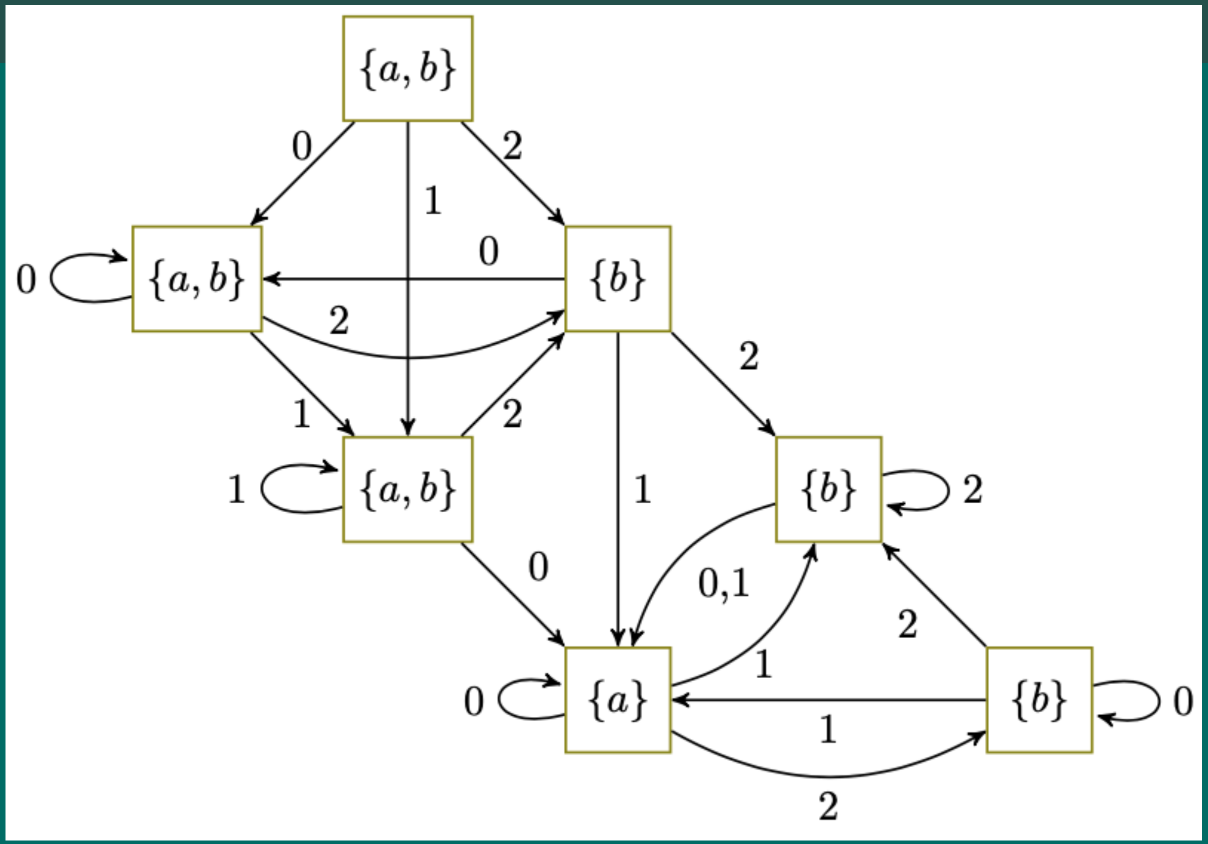

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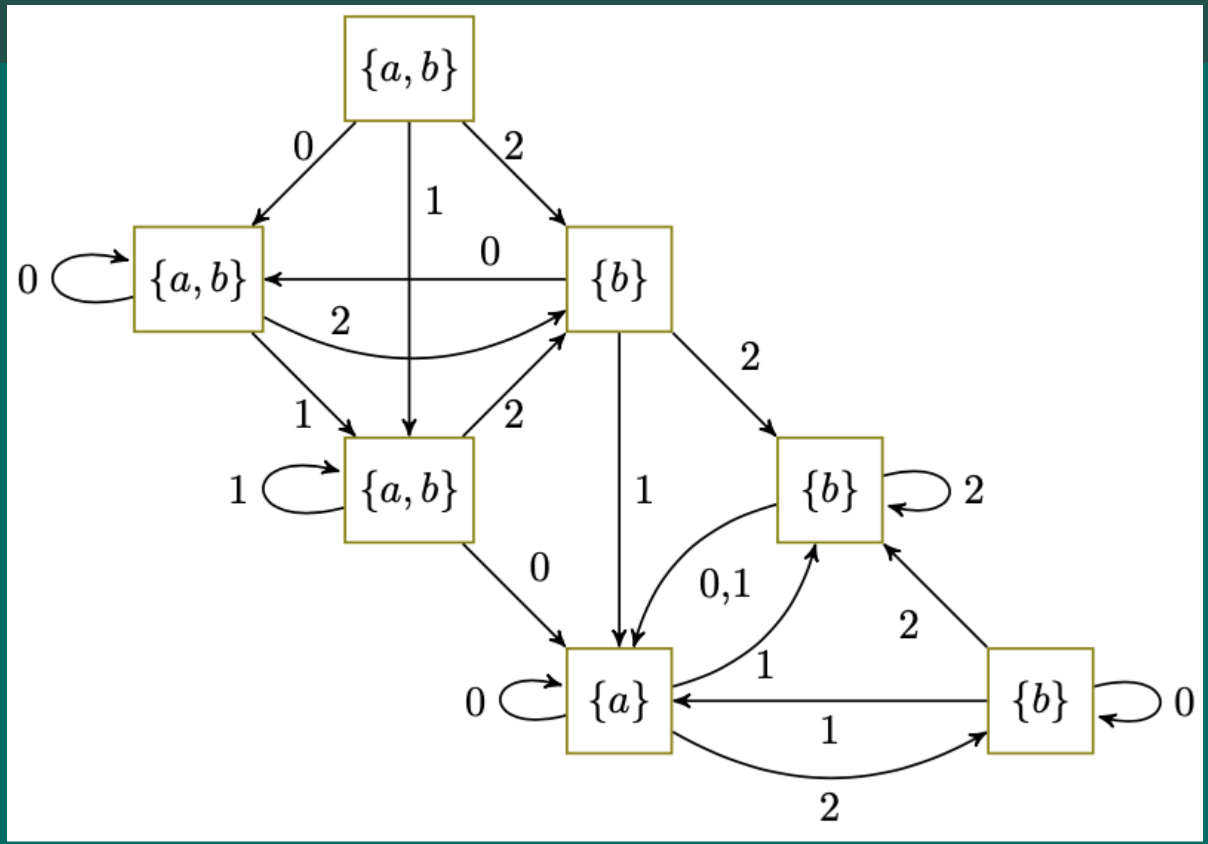

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Arithmetic progressions

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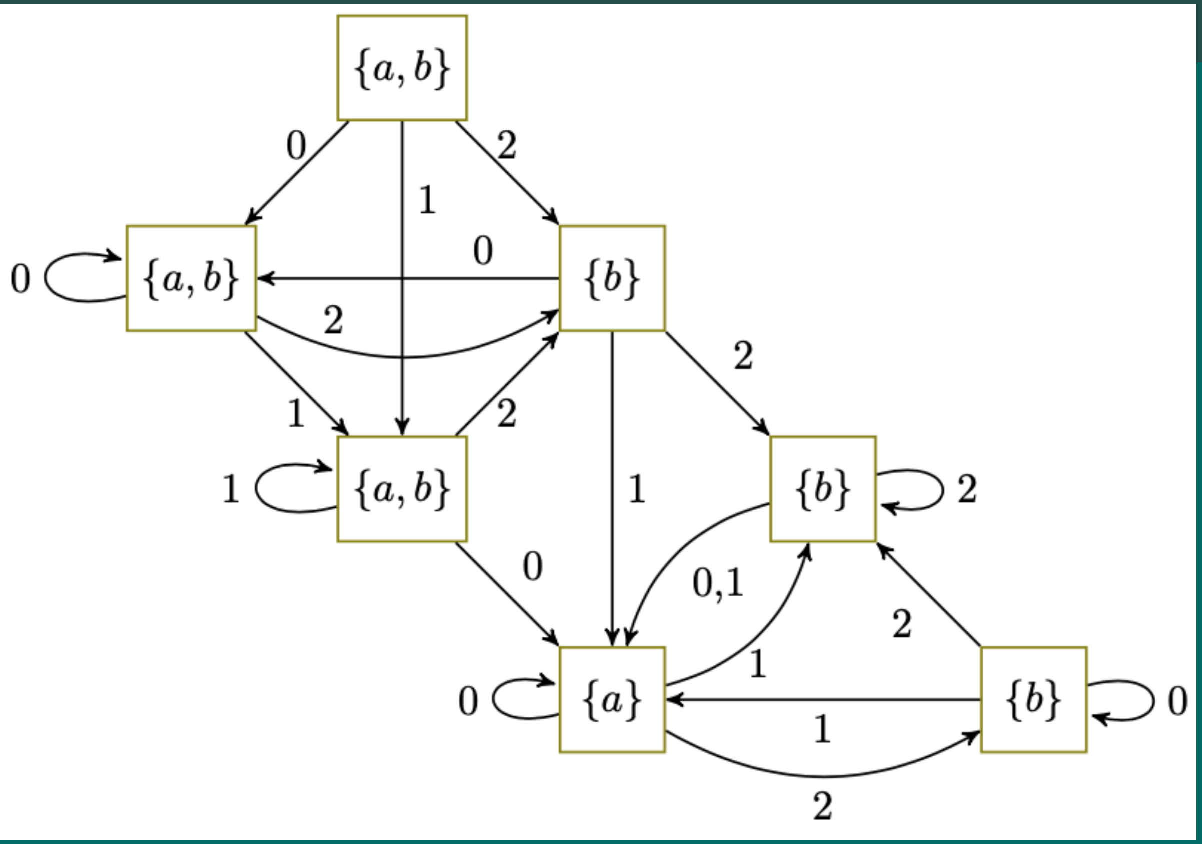
We have, for example

$y_{9n+8} = b$, $y_{9n+3} = a$, $y_{9n+7} = a$,

$y_{3n+2} = b$,

$\forall n \in \mathbb{Z}$



Arithmetic progressions

Constant length case

Proposition [Durand, G].

The graph $G(\sigma)$ satisfies exactly one of the following properties :

1. It has no singleton.
2. Every long enough path ends in a singleton.
3. There exists a cycle among vertices with cardinal greater or equal to 2, with a least one singleton in their descendants.

Arithmetic progressions

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Arithmetic progressions

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Proposition [Durand, G].

The graph $G(\sigma)$ satisfies exactly one of the following properties :

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The graph $G(\sigma)$ satisfies exactly one of the following properties :

1. It has no singleton. $\Longleftrightarrow$ x has no arithmetic progression.
2. Every long enough path ends in a singleton. $\Longleftrightarrow$ x is periodic.
3. There exists a cycle among vertices with cardinal greater or equal to 2, with a least one singleton in their descendants. $\Longleftrightarrow$ The essential periods of letters in x are unbounded.

Arithmetic progressions

Conclusion

➡ General case : the set of rational eigenvalues is computable.

Arithmetic progressions

Conclusion

- ➡ General case : the set of rational eigenvalues is computable.
- ➡ Constant-length case : all the periods can be described by an automaton.

Arithmetic progressions

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➡ General case : the set of rational eigenvalues is computable.

➡ Constant-length case : all the periods can be described by an automaton.

➡ General case : given an integer p, we can check if there exists a constant arithmetic progression of period p.

Arithmetic progressions

Conclusion

➡ General case : the set of rational eigenvalues is computable.

➡ Constant-length case : all the periods can be described by an automaton.

➡ General case : given an integer p, we can check if there exists a constant arithmetic progression of period p.

➡ Open question : describe the set of periods for constant arithmetic progressions in the general case.

Thank you.

F. Durand; V. Goyheneche
Decidability, arithmetic subsequences and eigenvalues of morphic subshifts.
Bull. Belg. Math. Soc. Simon Stevin 26 (2019), no. 4, 591–618.
https://arxiv.org/abs/1811.03942