Singular Words

Joint with Alessandro De Luca & Marcia Edson

One World Combinatorics on Words Seminar
April 26, 2021

Warm up : Lyndon words

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3. $x \in \mathbb{A}^+$ is alt-Lyndon if $x \leq_{alt} x'$ for each suffix x' of x. Eg. ab, $abaabb, \ldots$.

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Outline

1. Extremal problems in the theory of finite continued fractions
2. Singular words
3. A non-commutative variant of the Euclidean algorithm

Regular continuant

For $x = x_1 x_2 \cdots x_n \ (x_i \in \mathbb{N})$

$$K(x) = K_n(x_1, x_2, \ldots, x_n)$$

$K_0() = 1, \ K_1(x_1) = x_1$

$K_n(x_1, x_2, \ldots, x_n) = x_n K_{n-1}(x_1, x_2, \ldots, x_{n-1}) + K_{n-2}(x_1, x_2, \ldots, x_{n-2})$

$$\boxed{K(x^*) = K(x)} \qquad x^* = x_n x_{n-1} \cdots x_1$$

- $K(x)$ is the denominator of the terminating regular continued fraction $[0; x_1, x_2, \ldots, x_n]$.

Semi-regular continuant

$$\dot{K}(x) = \dot{K}_n(x_1, x_2, \ldots, x_n)$$

$\dot{K}_0() = 1,\ \dot{K}_1(x_1) = x_1$

$$\dot{K}_n(x_1, x_2, \ldots, x_n) = x_n \dot{K}_{n-1}(x_1, x_2, \ldots, x_{n-1}) - \dot{K}_{n-2}(x_1, x_2, \ldots, x_{n-2})$$

$$\boxed{\dot{K}(x) = \dot{K}(x^*)}$$

If $\boxed{x_i \geq 2,}$ then $\dot{K}(x)$ is the denominator of the semi-regular c.f.

$$[x]^\bullet = \cfrac{1}{x_1 - \cfrac{1}{x_2 - \cfrac{1}{x_3 - \cdots}}}$$

Matrix comparison

$$X = \begin{pmatrix} x_1 & 1 & 0 & \cdots & 0 \\ 1 & x_2 & 1 & \ddots & 0 \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & x_{n-1} & 1 \\ 0 & \cdots & 0 & 1 & x_n \end{pmatrix}$$

$$\boxed{K(x_1 x_2 \cdots x_n) = \mathrm{perm}(X).}$$

$$\boxed{\dot{K}(x_1 x_2 \cdots x_n) = \det(X).}$$

Cyclic continuants of Motzkin-Straus (1956)

The following cyclic analogues of K and $\dot{K}$ are well defined on cyclic words (circular words/necklaces...) :

$$\boxed{K^{\circlearrowright}(x_1 x_2 \cdots x_n) = K(x_1 x_2 \cdots x_n) + K(x_2 \cdots x_{n-1})}$$

$$\boxed{\dot{K}^{\circlearrowright}(x_1 x_2 \cdots x_n) = \dot{K}(x_1 x_2 \cdots x_n) - \dot{K}(x_2 \cdots x_{n-1})}$$

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$$\dot{K}^{\circlearrowright}(x_1 x_2 \cdots x_n) = \dot{K}(x_1 x_2 \cdots x_n) - \dot{K}(x_2 \cdots x_{n-1})$$

Remark

The cyclic continuant $K^{\circlearrowright}$ also appears in a 2008 paper by J. Berstel, L. Boasson, O. Carton, under the name *circular continuant*, in connection with Hopcroft's automaton minimisation algorithm.

Problem

Given

$$x = a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}$$

$1 \leq a_1 < a_2 < \cdots < a_k$ and $n_1 + n_2 + \cdots + n_k = n$

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Problem (C.A. Nicol, $\leq$ 1955)

Describe the extremal (maximising/minimising) arrangements for $K(\cdot)$.

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Problem

Describe the extremal arrangements for $\dot{K}(\cdot)$.

Problem (Ramharter 83)

Describe the extremal arrangements for $K^{\circlearrowright}(\cdot)$ and $\dot{K}^{\circlearrowright}(\cdot)$.

Ramharter's theorem 1983

- Ramharter found both the maximising and minimising arrangements for the regular continuant $K(\cdot)$.

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- Ramharter found both the maximising and minimising arrangements for the regular continuant $K(\cdot)$.
- He also found the minimising arrangement for $\dot{K}(\cdot)$.
- In all three cases, the extremal arrangements are unique (up to reversal) and independent of the actual values of the $+$'ve integers $a_1, a_2, \ldots, a_k$.

Ramharter's theorem 1983

Example : If $x = a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}$ with $1 \leq a_1 < a_2 < \cdots < a_k$ then

- maximising arrangement for $K(\cdot)$ is unique up to reversal and is given by :

$$\boxed{a_k L_{k-1} a_{k-2} L_{k-3} \cdots a_1^{n_1} \cdots a_{k-3} L_{k-2} a_{k-1} L_k}$$

$L_i = a_i^{n_i - 1}$ (leftovers).

2233333555888 $\mapsto$ 8553223333588.

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  - *"There is an infinity of essentially different patterns."*
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- The maximising arrangement for $\dot{K}(\cdot)$ in the binary case $x = a_1^{n_1} a_2^{n_2}$ is unique and independent on the actual choice of $+$'ve integers a_1 and a_2.

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    - Palindromic (binary) maximising arrangements are in 1-1 correspondence with the extremal cases of the Fine and Wilf theorem with co-prime periods p and q.

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  - Palindromic (binary) maximising arrangements are in 1-1 correspondence with the extremal cases of the Fine and Wilf theorem with co-prime periods p and q.
- Ramharter conjectured that for general $a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}$ with $2 \leq a_1 < a_2 < \cdots < a_k$, the maximising arrangement for $\dot{K}(\cdot)$ is unique and independent of the actual values of the $+$'ve integers a_i.

Ramharter's key observations

Theorem (1, Ramharter 83)

Let $x = x_1 x_2 \cdots x_n \ (x_i \geq 2)$.

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Let $x = x_1 x_2 \cdots x_n \ (x_i \geq 2)$*. Suppose* $x = u^* v w$ *with* $v \neq v^*$ *and* $u \neq w$*. If* $v \prec v^*$ *and* $u \prec w$

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Theorem (2, Ramharter 83)

Let $x = x_1 x_2 \cdots x_n$ $(x_i \geq 2)$. Suppose $x = u^* v w$ with $v \neq v^*$ and $u \neq w$. If $v \prec_{alt} v^*$ and $u \prec_{alt} w$ (or $v \succ_{alt} v^*$ and $u \succ_{alt} w$), then $K(u^* v^* w) < K(u^* v w)$.

$$(K, \prec_{alt}) \quad (\dot{K}, \prec)$$

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$(n_2, n_3, n_4, n_5) = (3, 6, 4, 6)$.

- $x = 5543324533324545235$

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  $\dot{K}(5543325454233354235) > \dot{K}(5543324533324545235)$
  $3024689366 > 3022832894$.

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  $\dot{K}(5543325454233354235) > \dot{K}(5543324533324545235)$
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- $x' = 5543325454233354235$

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  $\dot{K}(554332\,5454233354\,235) > \dot{K}(554332\,4533324545\,235)$
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- $x' = 5543325454233354235 = 554332545423335\,4235$

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- $x = 554332453332454523 5 = 554332\textcolor{red}{4533324545}235$
  $\dot{K}(554332\textcolor{red}{5454233354}235) > \dot{K}(554332\textcolor{red}{4533324545}235)$
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- $x' = 5543325454233354235 = 554332545423335\textcolor{red}{4235}$
  $\dot{K}(554332545423335\textcolor{red}{5324}) > \dot{K}(554332545423335\textcolor{red}{4235})$
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  $\dot{K}(5543325454233355324) > \dot{K}(5543325454233354235)$
  $3078516614 > 3024689366$.
- $x_{max} = 2535253534435344352 : \dot{K}_{max} = 4823503656$.
  $x_{min} = 5554433322233344555 : \dot{K}_{min} = 1888985692$.

Directed graph construction (version 1)

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- For $y, z \in \Pi(x)$, put a directed edge $y \to z$ whenever $y = u^* v w$, $z = u^* v^* w$ with $v \prec v^*$ and $u \prec w$ (or $v \succ v^*$ and $u \succ w$).

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- This construction factors to the quotient $\mathfrak{X}(x) = \Pi(x)/*$ and defines a directed graph $\vec{\mathcal{G}}(x)$ with vertex set $\mathfrak{X}(x)$.

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Theorem (Ramharter 83)

The directed graph $\dot{\mathcal{G}}(x)$ is acyclic and has a unique vertex with in-degree zero (and hence in particular $\dot{\mathcal{G}}(x)$ is connected as a graph). Thus the minimising arrangement for $\dot{K}(\cdot)$ is unique.

Directed graph construction (version 2)

Let $\mathbb{A}$ be an ordered (abstract) alphabet and $x = x_1 x_2 \cdots x_n \in \mathbb{A}^+$.

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Theorem (Ramharter '83)

The directed graph $\mathcal{G}(x)$ is acyclic and has a unique vertex with in-degree zero and a unique vertex with out-degree zero. Thus both extremal arrangements for $K(\cdot)$ are unique.

Directed graph construction (exotic version a la DRR)

Let $\mathbb{A}$ be an ordered (abstract) alphabet and $x = x_1 x_2 \cdots x_n \in \mathbb{A}^+$. To each $\alpha \in \{0,1\}^{\mathbb{N}} \mapsto \preceq_\alpha$ on $\mathbb{A}^*$.
Eg. $\alpha = 0^\omega \mapsto \preceq$ and $\alpha = (01)^\omega \mapsto \preceq_{alt}$.

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- This construction factors to the quotient $\mathfrak{X}(x) = \Pi(x)/*$ and defines a directed graph $\mathcal{G}_\alpha(x)$.

Theorem

Let $\alpha \in \{0,1\}^{\mathbb{N}}$. The directed graph $\mathcal{G}_\alpha(x)$ is acyclic for each $x \in \mathbb{A}^+$ iff $\alpha = 0^\omega$ or $\alpha = (01)^\omega$.

Cyclic versions of Theorems 1 & 2

Theorem ($1^{\circlearrowright}$)

Let $x = x_1 x_2 \cdots x_n$ $x_i \geq 2$ be a cyclic word. Suppose $x = uv$ with $u \neq u^*$ and $v \neq v^*$. If $u \prec u^*$ and $v \succ v^*$ (or $u \succ u^*$ and $v \prec v^*$), then $\dot{K}^{\circlearrowright}(u^* v) > \dot{K}^{\circlearrowright}(uv)$.

Theorem ($2^{\circlearrowright}$)

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Same story in the cyclic case

Let $x \in \mathbb{A}^{\circlearrowright}$ be a cyclic word over an ordered alphabet $\mathbb{A}$.

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The directed graph $\mathcal{G}^{\circlearrowright}(x)$ has a unique vertex with in-degree 0 and a unique vertex with out degree 0. In particular, if $x = a_1^{n_1} \cdots a_k^{n_k}$ ($a_i \geq 2$), then both extremal arrangements for $K^{\circlearrowright}(\cdot)$ are unique (up to reversal) and independent of the values of the a_i.

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Singular words - Linear version

Definition

Let $\mathbb{A}$ be an ordered alphabet and let $x \in \mathbb{A}^+ \cup \mathbb{A}^{\mathbb{N}} \cup \mathbb{A}^{\mathbb{Z}}$. We say x is *singular* if for all factorisations $x = u^* vw$ $(v \in \mathbb{A}^+)$ with $v \neq v^*$ and $u \neq w$ we have $v \prec v^*$ iff $w \prec u$.

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Remark

Let $x \in \mathbb{A}^+ \cup \mathbb{A}^{\mathbb{Z}}$. Then x is singular iff x^* is singular.

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Lemma

$x \in \mathbb{A}^+$ is singular iff $x^\infty \in \mathbb{A}^{\circlearrowright}$ is (cyclic) singular.

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- $x \in \mathbb{A}^{+}$ is singular iff x or x^* is of the form b^n, ab^n or ava where v is a bispecial Sturmian word (equiv: $a'vb'$ is a power of a Christoffel word $\mathbb{A} = \{a', b'\}$, (G. Fici, 2014)).

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- $w = abbabbb \cdot abbabbb \cdot abbabba$.

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Bi-infinite binary singular words & Markoff property

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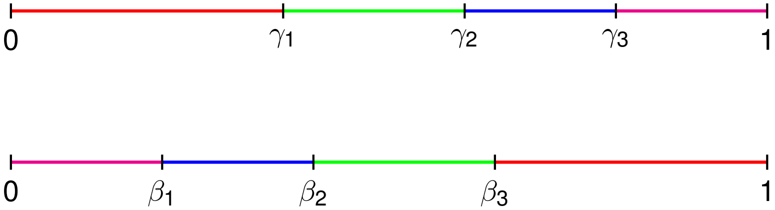
Interval exchange transformations



$i.d.o.c. \Leftrightarrow$ the $k-1$ sets $\{T^{-n}(\gamma_i) : n \geq 0\}$ are infinite & disjoint.

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Lemma

Assume $x \in \mathbb{A}_k^{\mathbb{Z}}$ with $L(x)$ is symmetric. Then x is singular iff $L(x)$ satisfies the symmetric order condition.

Singular words & symmetric interval exchange transformations

Theorem (DEZ)

Let $\mathbb{A}_k = \{1, 2, \ldots, k\}$ $(k \geq 2)$ *and let* $x \in \mathbb{A}_k^{\mathbb{Z}}$ *be uniformly recurrent. Then the following are equivalent :*

1. x *is singular and* $L(x)$ *is symmetric.*
2. $L(x)$ *is the language of a symmetric* k*-interval exchange transformation.*

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Each abelian class over an ordered ternary alphabet contains a unique (up to reversal) singular word. Thus if $x = a_1^{n_1} a_2^{n_2} a_3^{n_3}$ with $2 \leq a_1 < a_2 < a_3$. Then the maximising arrangement for $K(\cdot)$ is unique and independent of the values of the a_i.

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Non-commutative variant of the Euclidean algorithm

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be given with $\alpha_i > 0$. *(Ordered Parikh vector of a cyclic word or a symmetric k-i.e.t. with $|I_j| = \alpha_j$).*

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Applied to a Parikh vector α, this state corresponds to the critical state of the algorithm, and may give rise to multiple cyclic singular words having the same Parikh vector.
Ex : $(1, 2, 1, 2) \leftrightarrow (1, 2, 1, 2, 1)^{\circlearrowright}$ $\quad \delta_c = 0$.

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Ex : $(1, 2, 1, 2) \leftrightarrow (1, 2, 1, 2, 1)^{\circlearrowleft}$ $\delta_c = 0$.

If only cases 1.(a) or 1.(b) occur, then there exists a unique cyclic singular word having the prescribed Parikh vector and hence a unique global maximum for $\dot{K}^{\circlearrowleft}(\cdot)$ or for $\dot{K}(\cdot)$.
Ex : $(2, 3, 4, 3)$.

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- Let $\mathcal{T}$ be the symmetric 3-i.e.t. with interval lengths

$$\alpha = \left(\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{3} - 1, 2 - \sqrt{3}\right).$$

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- Let $\mathcal{T}$ be the symmetric 3-i.e.t. with interval lengths

$$\alpha = \left(\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{3} - 1, 2 - \sqrt{3}\right).$$

- Let $x \in \{a, b, c\}^{\mathbb{N}}$ be the f.p. of $\tau : a \mapsto aca,\ b \mapsto acabab,\ c \mapsto acab.$

Conclusions

Associated to each symmetric k-i.e.t. (i.d.o.c.) is an infinite directive word on $\{1, 2, \ldots, k\}$ (as for A.R. sequences)

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$$\mathrm{drop}_c \circ \mathrm{drop}_b \circ \mathrm{drop}_a(x) = x.$$

THE END

Thank you for your attention !