On sets of indefinitely desubstitutable words

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A starting example: infinite binary balanced words 1/5

Definition

A word $\mathbf{w}$ on an alphabet A is balanced if
for all factors u, v of $\mathbf{w}$ with $|u| = |v|$
for each letter a in A, $||u|_a - |v|_a| \leq 1$.

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- Aperiodic words = Sturmian words

Basic example = the Fibonacci word $abaababaabaababaababaabaababaabaab\ldots$

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Example $= ab(abaab)^\omega = (ababa)^\omega$

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- Ultimately periodic words:

Examples = $a^n ba^\omega$, $(ab)^n a(ab)^\omega$, $(abaab)^n aba(abaab)^\omega$, ...

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First desubstitutive step

Hypothesis: $\mathbf{w}_1$ an infinite balanced infinite words over $\{a, b\}$

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- Hence:

$\mathbf{w}_1$ starts with	$\mathbf{w}_1$ does not contain	$\mathbf{w}_1$ can be decomposed
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$$L_a : \begin{cases} a \mapsto a \\ b \mapsto ab \end{cases} \qquad R_b : \begin{cases} a \mapsto ab \\ b \mapsto b \end{cases}$$

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$$\text{Thus } \mathbf{w}_1 = f_1(\mathbf{w}_2) \text{ with } \begin{cases} f_1 \in \{L_a, L_b, R_a, R_b\} \\ \mathbf{w}_2 \in \{a, b\}^\omega \end{cases}$$

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Hypothesis: $\mathbf{w}_1 \in \{a, b\}^\omega$ balanced

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Thus we can iterate the decomposition.

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And so on

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A necessary condition

Infinite sequence of decompositions/desubstitutions

$\mathbf{w} \in \{a, b\}^\omega$ balanced,

$$\Rightarrow \begin{cases} \exists\ (\mathbf{w}_i)_{i\geq 1} \in \{a, b\}^\omega \\ \exists\ (f_i)_{i\geq 1} \in \{L_a, L_b, R_a, R_b\} \end{cases} \text{s.t.} \begin{cases} \mathbf{w}_1 = \mathbf{w} \text{ and} \\ \mathbf{w}_i = f_i(\mathbf{w}_{i+1}) \quad (\forall i \geq 1) \end{cases}$$

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  Let $(\sigma_n)_{n \geq 1}$ be a sequence of substitutions in $\mathcal{S}$.
  $\mathbf{w} \in A^\omega$ is a *limit point of* $(\sigma_n)_{n \geq 1}$ $\quad ((\sigma_n)_{n \geq 1} =$ directive sequence of $\mathbf{w}$)
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  Stab$(\mathcal{S})$ = stable set of $\mathcal{S}$ = set of all limit points of sequences in S^ω

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Equivalently: $\mathbf{w} \in \{a, b\}^\omega$ balanced $\Rightarrow \mathbf{w} \in \mathit{Stab}(\{L_a, L_b, R_a, R_b\})$

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$\Rightarrow$ (well-known) $\mathbf{w}$ is Sturmian, that is, aperiodic balanced

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  $\mathbf{w}$ is balanced.
- Case 3, $\mathbf{w}$ contains finitely many elements of $\{L_a, R_a\}$: similar $\square$

Questions for the end of the talk

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- Does there exist any general link with property preserving morphisms?

Contents

A generalization of fixed points of morphisms

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Proposition [Godelle 2010]

Let f be a substitution (non erasing morphism).
$\mathbf{w} \in Stab(\{f\})$ if and only $\mathbf{w}$ is a fixed point of f^n for some n.

A generalization of fixed points of morphisms

Let $\mathcal{S}$ be a set of substitutions.
$Stab(\mathcal{S})$ is the greatest set X (w.r.t. the inclusion) such that $X = \bigcup_{f \in \mathcal{S}} f(X)$.

Proposition [Godelle 2010]

Let f be a substitution (non erasing morphism).
$\mathbf{w} \in Stab(\{f\})$ if and only $\mathbf{w}$ is a fixed point of f^n for some n.

Example

$$f : \begin{cases} a \mapsto aba \\ b \mapsto b \end{cases} \qquad \text{(observe } f(ab) = abab\text{)}$$

$Stab(\{f\}) = \{b^\omega\} \cup b^*(ab)^\omega$

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- Many interesting examples of S-adic words:
  - Morphic words, Sturmian words, 3-Interval exchange transformations, Arnoux-Rauzy words, episturmian words...
  - Remark: S-adicity = a necessary condition
    characterizations obtained with additional conditions on the sequence of desubstitutions

S-adicity, definition

Definition (S-adicity or Substitutive-adicity)

An infinite word $\mathbf{w}$ is *Substitutive*-adic

if there exist $\begin{cases} \text{a sequence } (\sigma_n)_{n\geq 1},\ \sigma_n : A_{n+1}^* \to A_n^* \text{ of substitutions} \\ \text{a sequence of letters } (a_n)_{n\geq 1} \end{cases}$

$\mathbf{w} = \lim_{n\to\infty} \sigma_1 \cdots \sigma_n(a_n)$ with *directive sequence* $(\sigma_n)_{n\geq 1}$

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Remark

In the definition of $\mathcal{S}$-adicity, cardinalities of alphabets may not be bounded contrarily to what happens in the definition of a stable set, where the alphabet is fixed.

Links between S-adicity and stable sets 1/5

Let $\mathcal{S}$ be a set of substitutions on a fixed alphabet A.

Assume $\begin{cases} \mathbf{w} = \lim_{n\to\infty} \sigma_1 \cdots \sigma_n(a_n) \\ \text{for all } k,\ \mathbf{w}_k = \lim_{n\to\infty} \sigma_k \cdots \sigma_n(a_n) \text{ exists} \end{cases}$

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Nevertheless $a^\omega = f(a^\omega) = g(b^\omega)$ and $b^\omega = g(a^\omega)$

$$\Rightarrow a^\omega \in Stab(\{f, g\}).$$

Links between S-adicity and stable sets 2/5

Proposition

For any set $\mathcal{S}$ of substitutions on a fixed alphabet A,

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- Construction of suitable desubstituted words.
  Given $\mathbf{w} = \lim_{n\to\infty} \sigma_1 \cdots \sigma_n(a_n)$
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But the only $\{f\}$-adic word is $(ab)^\omega = \lim_{n\to\infty} f^n(a)$

Links between S-adicity and stable sets 3/5

Definitions, with $\mathbf{s} = (\sigma_n)_{n\geq 1}$ a sequence of substitutions

- $Stab(\mathbf{s})$ = set of infinite words that can be indefinitely desubstituted with directive sequence $\mathbf{s}$

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$$\mathrm{StabFin}(f^\omega) = b^+ \text{ and } \mathrm{adic}(f^\omega) = \{(ab)^\omega\}$$

Links between S-adicity and stable sets 4/5

Corollary

If $\mathsf{StabFin}(\mathbf{s}) = \emptyset$ then $Stab(\mathbf{s}) = \mathsf{adic}(\mathbf{s})$.

Links between S-adicity and stable sets 4/5

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If $\mathrm{StabFin}(\mathbf{s}) = \emptyset$ then $\mathit{Stab}(\mathbf{s}) = \mathrm{adic}(\mathbf{s})$.

But the converse does not hold

Example

Remember $L_a(a) = a$ and $L_a(b) = ab$

- $\mathrm{StabFin}(L_a^\omega) = \{a^n \mid n \geq 1\} \neq \emptyset$
- $\mathit{Stab}(L_a^\omega) = \{a^\omega\}$

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Remark

$$\begin{aligned}\text{Set of infinite binary balanced words} \quad &= \mathrm{Stab}(\{L_a, L_b, R_a, R_b\}) \\ &= \mathrm{adic}(\{L_a, L_b, R_a, R_b\})\end{aligned}$$

$$\mathrm{Stab}(\{L_a, L_b, R_a, R_b\}) = \cup \begin{cases} \text{the set of Sturmian words} \\ \bigcup_{f\in\{L_a,L_b,R_a,R_b\}^*} f(\{a^\omega, ab^\omega, ba^\omega, b^\omega\}) \end{cases}$$

Links between S-adicity and stable sets 5/5

Remark. Given a set $\mathcal{S}$ of substitutions on a fixed alphabet A, one can decide whether StabFin($\mathbf{s}$) is empty for all sequences of substitutions over $\mathcal{S}$.

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Let Id_A be the identity on A (of cardinality at least 2).

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Remark (Cassaigne's example). A^ω is a subset of adic words over a finite set of substitutions defined on $A \cup \{\ell\}$ with $\ell \notin A$.

Contents

A link with property preserving morphisms

A basic property

For any $\varphi \in \mathcal{S}$: $\qquad \varphi(Stab(\mathcal{S})) \subseteq Stab(\mathcal{S})$
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Proof

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$$\mu = \begin{cases} a \mapsto ab \\ b \mapsto ba \end{cases} \quad E = \begin{cases} a \mapsto b \\ b \mapsto a \end{cases}$$

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- $Stab(\{\mu, E\}^* \setminus \{E, Id\}) = \{\mathbf{M}, E(\mathbf{M})\}$

$$\mathbf{M} = \text{ Thue-Morse word } = \mu^\omega(a) = abbabaabbaababbab\cdots$$

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- A Sturmian word is an $\{L_a, L_b, R_a, R_b\}$-adic words having a directive sequence with $\begin{cases} \text{infinitely many elements of } \{L_a, R_a\} \text{ and} \\ \text{infinitely many elements of } \{L_b, R_b\}. \end{cases}$

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Example

$\mathbf{s} = L_a R_a L_a R_b L_b R_b R_b R_b L_a R_b L_a L_b R_a L_a R_b L_a R_b \cdots$

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- Property. { Morphisms that preserve Sturmian words } $=$
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Lyndon Sturmian words

Infinite Lyndon word: word smaller than all its suffixes

(w.r.t. the lexicographic order)

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A Sturmian word w is a Lyndon word over $\{a < b\}$ if and only if it can be infinitely decomposed over $\{L_a, R_b\}$ with infinitely many occurrences of L_a and infinitely many occurrences of R_b in the directive sequence.

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Set $\mathcal{S}_{Lynd} = \{L_a^n R_b, R_b^n L_a \mid n \geq 1\}$.

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A word is a Lyndon Sturmian word if and only if it belongs to $Stab(\mathcal{S}_{Lynd})$

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Proposition

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Standard words

Definition (Infinite standard words)

Binary words having all its left special factors as prefixes
and exactly one left special factor of each length.
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To summarize:

- $\{$ binary LSP words $\} = Stab(\{L_a, L_b\})$

LSP infinite words over alphabet with at least 3 letters

[R. 2017-2019]

- Given an alphabet A ($\#A \geq 3$),
  $\exists$ a finite set $\mathcal{S}_{LSP}$ s.t. $\{$ LSP infinite words over $A\} \subsetneq Stab(\mathcal{S}_{LSP})$.

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- A characterization of directive sequences in $Stab(\mathcal{S}_{LSP})$ with a complex condition (allowed paths in a huge automaton/graph).
- No set $\mathcal{S}$ such that $\{$ LSP words over $A\} = Stab(\mathcal{S})$.
  Proof based on a characterization of morphisms preserving LSP infinite words.

Episturmian words

- Words introduced by Droubay, Justin and Pirillo in 2001 as a generalization of Sturmian words on arbitrary alphabets
- Many characteristic properties: *for instance*, infinite words having at most one right special factor of each length, and, whose set of factors is closed under mirror image

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- A generalization of balanced words [R. 2007]
  For a *recurrent* infinite word w, the following assertions are equivalent:
  1. $\mathbf{w}$ is episturmian;
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  Remark. Proof obtained using a desubstitution property of episturmian words

Episturmian words and desubstitutions

$$L_\alpha : \begin{cases} \alpha \mapsto \alpha \\ \beta \mapsto \alpha\beta \text{ for } \beta \neq \alpha \end{cases} \qquad R_\alpha : \begin{cases} \alpha \mapsto \alpha \\ \beta \mapsto \beta\alpha \text{ for } \beta \neq \alpha \end{cases}$$

Characterization using desubstitutions [Justin, Pirillo 2002]

$\mathbf{w}$ is episturmian
if and only if
$\mathbf{w} \in Stab(\{L_\alpha, R_\alpha \mid \alpha \in A\})$ and
it has a sequence $(\mathbf{w}_n)_{n\geq 0}$ of **recurrent** desubstituted words.

Can be transformed to:

Episturmian words $=$ **recurrent** elements of $Stab(\{L_\alpha, R_\alpha \mid \alpha \in A\})$

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Episturmian words = **recurrent** elements of $Stab(\{L_\alpha, R_\alpha \mid \alpha \in A\})$

Remark: elements of $Stab(\{L_\alpha, R_\alpha \mid \alpha \in A\})$ not episturmian =
$f(ba^\omega)$ with $a, b \in A$, $f \in \{L_\alpha, R_\alpha \mid \alpha \in A\}^*$

Episturmian words and stable sets

Let:

- $\mathcal{L} = \{L_\alpha \mid \alpha \in A\}$
- $\mathcal{R} = \{L_\alpha \mid \alpha \in A\}$

Proposition

set of episturmian words $= Stab(\mathcal{R}^*\mathcal{L})$

Idea of the proof = in any infinite desubstitution of a recurrent element of $Stab(\mathcal{L} \cup \mathcal{R})$, infinitely many elements of $\mathcal{L}$ occur.

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Proposition

There is no finite set $\mathcal{S}$ of substitutions such that the set of episturmian words is $Stab(\mathcal{S})$.

Proof:

- Similarly as for Sturmian sets
- Need the characterization of morphisms that preserve episturmian words.

Morphisms that preserve episturmian words

A morphism preserves episturmian words on A

if and only if

it is a composition of elements of the following sets

- $\mathcal{L} = \{L_\alpha \mid \alpha \in A\}$
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- set of permutations: $\{f|f(A) = A\}$
- $\{\pi_a | a \in A, \forall b, \pi_a(b) \in a^+\}$ $(\forall \mathbf{w}, \pi_a(\mathbf{w}) = a^\omega)$

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Remark. Morphisms π_a do not occur usually: a^ω is also directed by L_a^ω $(R_a^\omega, \ldots)$

Sub-families of Episturmian words

- Standard episturmian words = episturmian + all left special factors as prefixes

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  - The set of A-strict episturmian words is $Stab(\mathcal{S}_{strictepi})$ with $\mathcal{S}_{strictepi} = (\mathcal{L} \cup \mathcal{R})^* \mathcal{L}(\mathcal{L} \cup \mathcal{R}) \cap \cap_{\alpha \in A} (\mathcal{L} \cup \mathcal{R})^* \{L_\alpha, R_\alpha\}(\mathcal{L} \cup \mathcal{R})^*$
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- *A-strict epistandard words* (epistandard = standard episturmian)

  - The set of A-strict epistandard words is the stable sets of $\mathcal{S}_{strictepi} \cap \mathcal{L}^*$.
  - It is not the stable set of a finite set of substitutions.

Contents

Conclusion (1/3)

The main studied problem

For which known families $\mathcal{F}$ of words,
does there exists a set $\mathcal{S}$ of substitutions such that

$$\mathcal{F} = Stab(\mathcal{S}) \ ?$$

Answers of this talk

finite sets $\mathcal{S}$	only infinite sets $\mathcal{S}$	no set
A^ω		overlap-free words
balanced finite words	Sturmian words	
LSP binary words		LSP words $\#A \geq 3$
	Lyndon Sturmian words	
standard episturmian words	standard Sturmian words	
	episturmian words	
	strict episturmian words	
	strict epistandard words	

Question

Others ?

Conclusion (2/3)

Remark

- For each of the previous families $\mathcal{F}$ for which there is only infinite sets $\mathcal{S}$ s.t. $\mathcal{F} = Stab(\mathcal{S})$, there exists a characterization of $\mathcal{F}$ as a *subset* of the stable set of a finite set of substitutions.
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  - For LSP Words over alphabets of cardinality at least 3 [Richomme 2019]

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  - For many characterizations of families of words using S-adicity [. . .]

Conclusion (3/3)

Similar question for $\mathcal{S}$-adicity

For which known families $\mathcal{F}$ of words, does there exists $\mathcal{S}$ of substitutions such that

$$\mathcal{F} = adic(\mathcal{S}) \text{ ?}$$

With the same set of substitutions than for stable sets:

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LSP binary words	Lyndon Sturmian words	
Standard episturmian words	Standard Sturmian words	overlap-free words
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Problems

- Others ?
- If $\mathcal{F}$ is $\mathcal{S}$-adic for some finite set $\mathcal{S}$,
  does it implies that $\mathcal{F} = Stab(\mathcal{S}')$ for some finite set $\mathcal{S}'$?

Thanks for your attention!

Reference: On sets of indefinitely desubstitutable words, Theoretical Computer Science 857, 97-113, 2021