Recent advances around Nivat’s conjecture

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Combinatorics on Words seminar

31 May 2021

Nivat's Conjecture

Pattern Complexity – 1D

$\mathcal{A} = \{$  $\}$

$w \in \mathcal{A}^{\mathbb{Z}}$

Pattern Complexity – 1D

$$P_w(n) = \text{ number of patterns of size } n$$

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$$P_w(1) = 4$$

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$$P_w(4) = 18$$

Pattern Complexity – 1D: Periodicity

$$P_w(1) = 2$$

$$P_w(2) = 3$$

$$P_w(3) = 5$$

$$P_w(4) = 5$$

$$P_w(5) = 5$$

$$P_w(6) = 5$$

$$\vdots$$

Pattern Complexity – 1D: Periodicity

Property

If w is periodic, then $\exists n, P_w(n) \leq n$.

Pattern Complexity – 1D

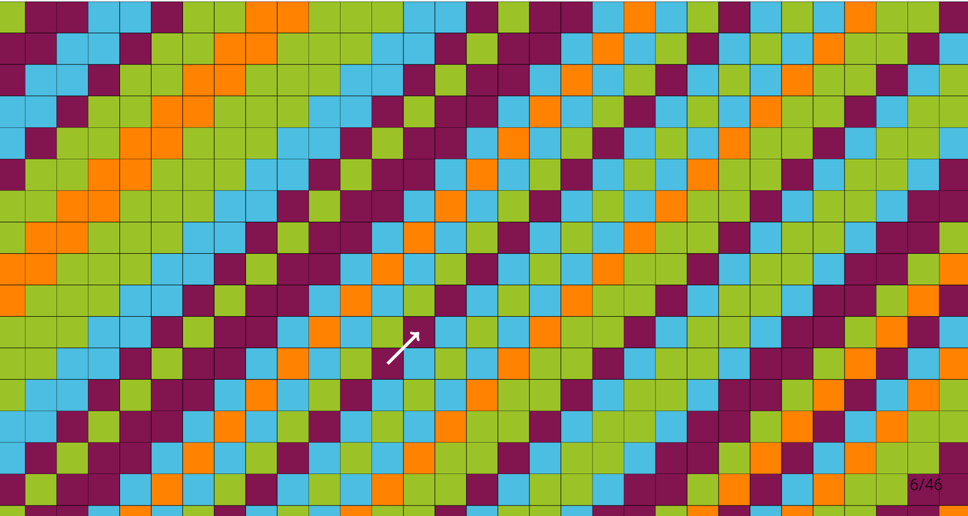
Theorem (Morse & Hedlund, 1938)

$$\forall w \in \mathcal{A}^{\mathbb{Z}},$$
$$\exists n > 0,\ P_w(n) \leq n \quad \Rightarrow \quad w \textit{ periodic}$$

Periodic Configuration – 2D

$c \in \mathcal{A}^{\mathbb{Z}^2}$ is:

- **1-periodic** / **weakly periodic**: $\exists \mathbf{u}, \forall \mathbf{v}, \quad c_{\mathbf{v}-\mathbf{u}} = c_{\mathbf{v}}$



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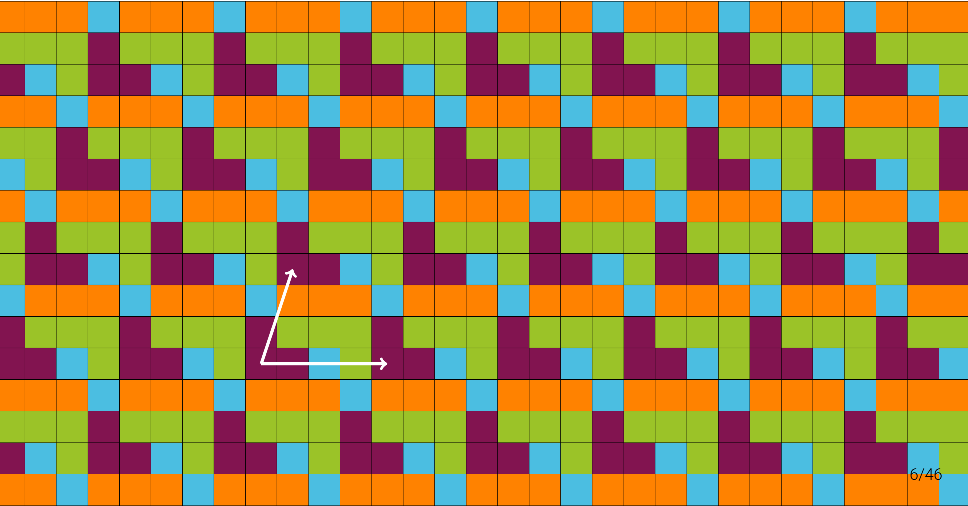
- **2-periodic** / **strongly periodic**: c is 1-periodic along $\mathbf{u}_1, \mathbf{u}_2$ not colinear



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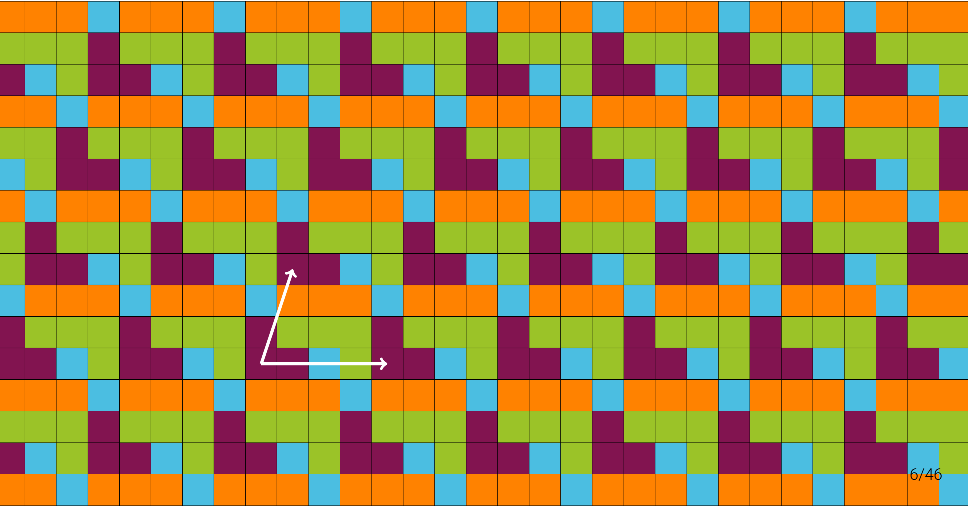
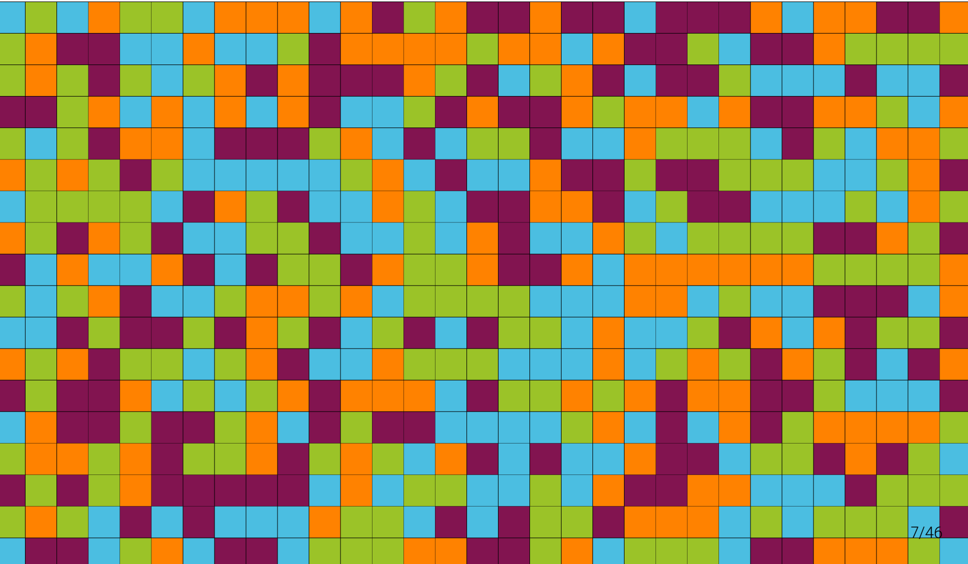
- **2-periodic / strongly periodic**: c is 1-periodic along $\mathbf{u}_1, \mathbf{u}_2$ not colinear $\Rightarrow$ Finitely many different translations of c

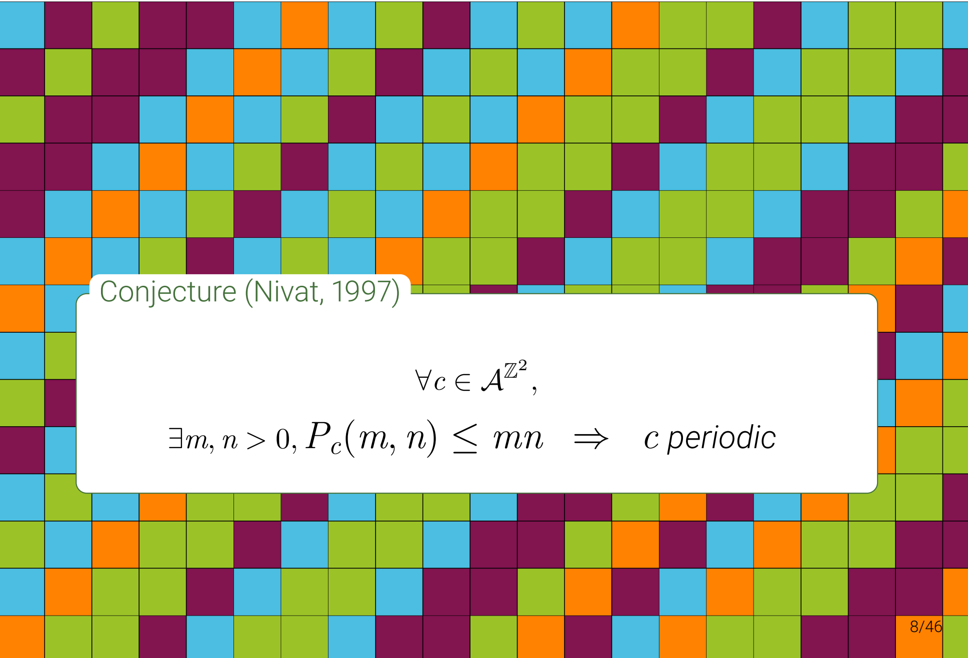


Pattern Complexity – 2D

$$P_c(m,n) = \text{ number of rectangular patterns of size } m \times n$$



2D: Nivat's conjecture



Conjecture (Nivat, 1997)

$$\forall c \in \mathcal{A}^{\mathbb{Z}^2},$$

$$\exists m, n > 0, P_c(m,n) \leq mn \quad \Rightarrow \quad c \textit{ periodic}$$

2D: no equivalence

Property

There exists $c \in \{0,1\}^{\mathbb{Z}^2}$ periodic s.t.

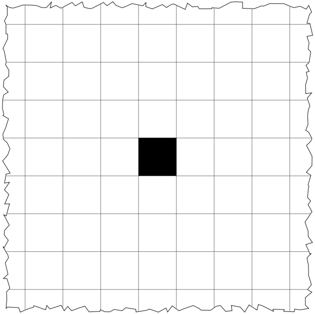
$$P_c(m,n) = 2^{m+n-1}$$

Nivat's conjecture is optimal

Theorem

$$\exists c \in \mathcal{A}^{\mathbb{Z}^2},$$

$$\exists m, n > 0, \quad P_c(m,n) = mn + 1 \quad \textit{and} \quad c \textit{ not periodic}$$



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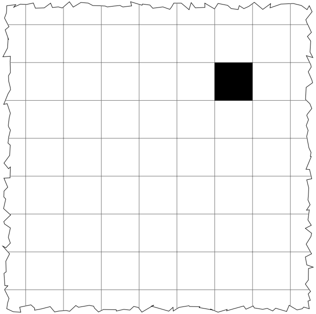

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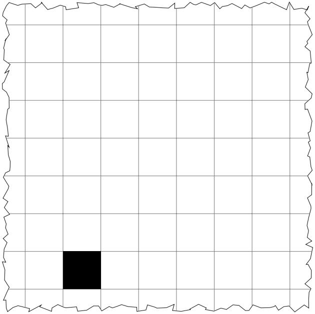

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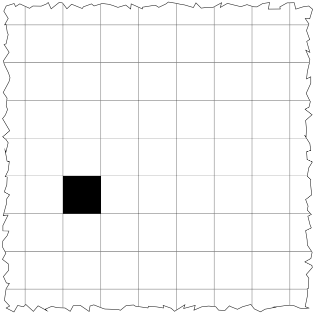

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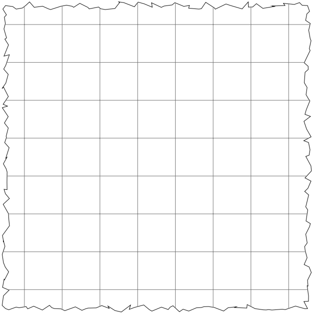

Uniform Recurrence

c is **uniformly recurrent** if all patterns of c appear *everywhere*:

$$\forall p \sqsubset c, \exists n_p, \quad p \text{ appears in all balls of size } n_p$$

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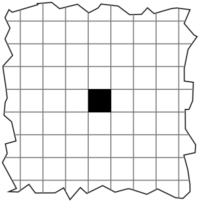

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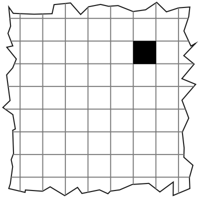

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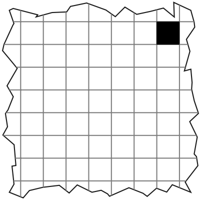

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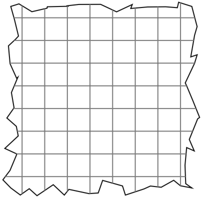


is not uniformly recurrent

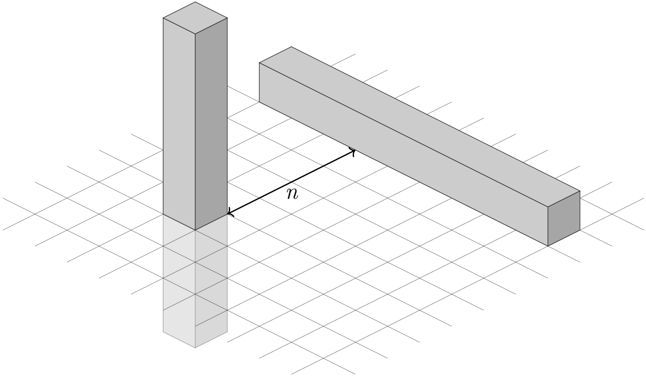
Higher dimension?

The same result does not hold in 3D and above:

$$\exists c \in \mathcal{A}^{\mathbb{Z}^3},$$

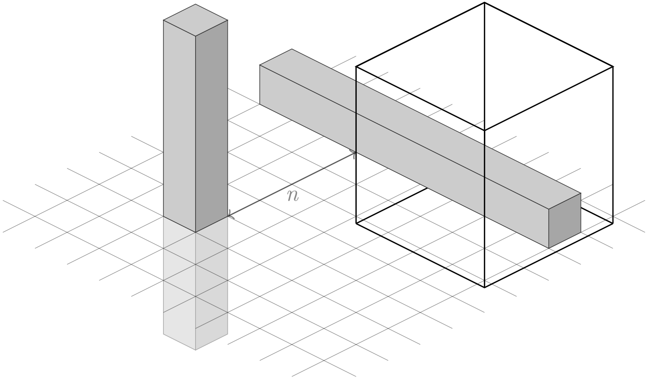
$\exists n > 0, \quad P_c(n,n,n) \leq n^3$ *and* c ***not*** *periodic*

Higher dimension?



$$P_c(n,n,n) = 2n^2 + 1 < n^3$$

Higher dimension?



$$P_c(n,n,n) = \mathbf{n^2} + n^2 + 1 < n^3$$

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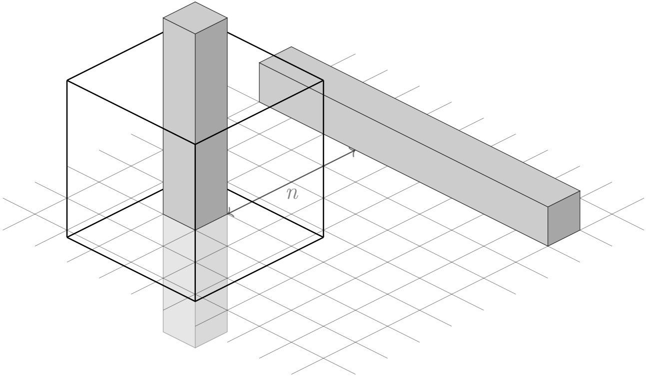

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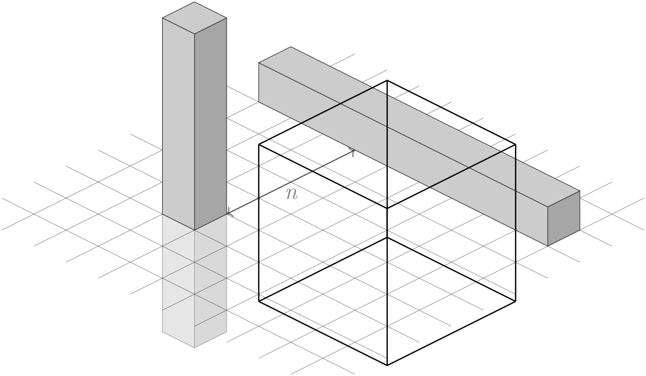


$$P_c(n, n, n) = n^2 + n^2 + \mathbf{1} < n^3$$

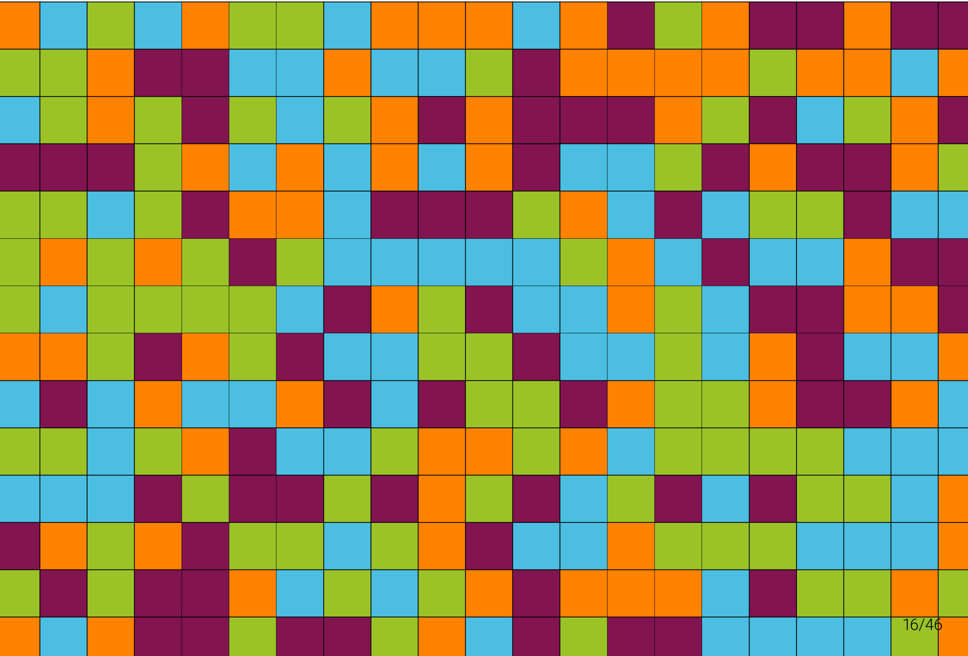
Getting closer to the conjecture

- $P_c(2,n) \leq 2n$ [Sanders & Tijdeman 2002]
- $P_c(3,n) \leq 3n$ [Cyr & Kra, 2016]

- $P_c(m,n) \leq \frac{mn}{144}$ [Epifanio, Koskas & Mignosi, 2003]
- $P_c(m,n) \leq \frac{mn}{16}$ [Quas & Zamboni, 2004]
- $P_c(m,n) \leq \frac{mn}{2}$ [Cyr & Kra, 2015]
- $P_c(m,n) \leq \frac{mn}{2} + |\mathcal{A}| - 1$ [Colle & Garibaldi, 2019]

Algebraic Tools

Configurations are (Laurent) series



Configurations are (Laurent) series

3	1	1	1	0	3	1	2	1	1	0	1	1	0	2	3	2	1	1	2	3
1	1	2	0	0	0	1	2	0	1	3	0	0	1	0	2	1	3	3	2	1
2	1	1	0	0	1	1	1	0	0	3	0	1	1	3	0	3	1	1	1	3
1	3	0	1	1	3	3	1	0	2	1	3	2	0	0	2	0	2	0	0	3
3	0	0	1	1	0	1	0	3	2	1	3	1	3	0	3	2	3	2	0	1
2	1	1	2	3	1	2	0	2	1	2	1	2	1	1	2	0	0	2	2	1
1	0	3	1	3	2	3	0	2	3	3	0	2	0	2	1	2	2	3	1	2
2	2	2	3	1	1	1	0	3	1	2	2	0	3	1	3	0	2	1	1	0
1	1	1	0	3	1	2	3	2	3	0	3	3	1	0	1	1	0	3	2	0
0	3	0	3	2	3	3	1	1	3	1	0	2	0	1	2	1	0	1	0	2
1	0	1	2	1	3	2	3	1	2	1	2	0	0	1	0	2	3	3	2	2
3	1	1	1	0	2	2	1	3	1	0	1	2	0	2	3	3	0	3	2	0
1	1	3	0	2	0	3	0	0	1	1	0	0	2	0	3	3	3	2	1	3
2	2	1	3	3	1	2	2	3	0	0	2	2	0	2	0	3	0	0	0	2

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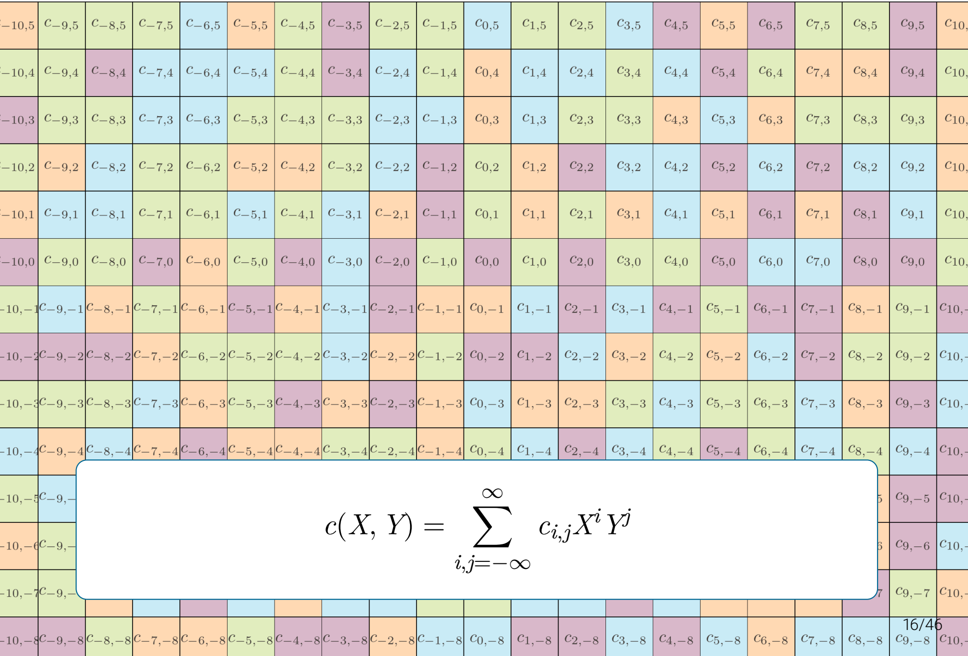


$$c(X, Y) = \sum_{i,j=-\infty}^{\infty} c_{i,j} X^i Y^j$$

Operations: Sum

$$c + d = \sum_{i,j=-\infty}^{\infty} (c_{i,j} + d_{i,j}) \, X^i Y^j$$

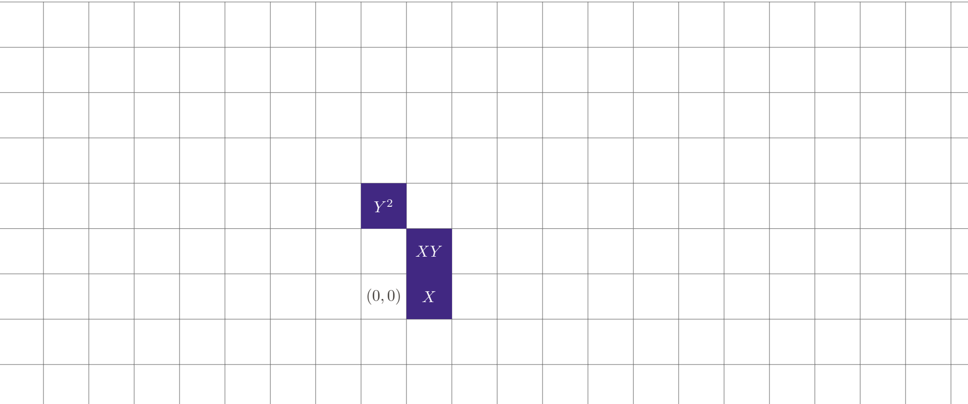
Formal sum $\leftrightarrow$ Sum of configurations

Operations: Multiplication

$$X^a Y^b c = \sum_{i,j=-\infty}^{\infty} c_{i,j} X^{i+a} Y^{j+b} = \sum_{i,j=-\infty}^{\infty} c_{i-a,j-b} X^i Y^j$$

Multiplication by $X^a Y^b \leftrightarrow$ Translation of vector (a, b)

(Parenthesis: see your polynomials like you never did)



$$X + XY + Y^2$$

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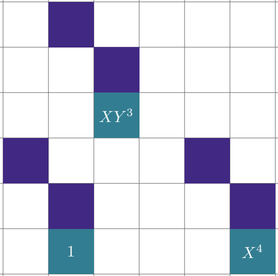

$$(X + XY + Y^2)(1 + X^4 + XY^3)$$

Expressing periodicity

c periodic of period (a, b)

$$\Leftrightarrow$$

$$c = X^a Y^b c$$

$$\Leftrightarrow$$

$$(X^a Y^b - 1)c = 0$$

$$\Leftrightarrow$$

$$(X^a Y^b - 1) \in \mathsf{Ann}(c)$$

Polynomial Ideal

$$\mathsf{Ann}(c) = \{p \mid pc = 0\} \subset \mathbb{R}[X^{\pm}, Y^{\pm}]$$

$$c \text{ periodic } \Leftrightarrow \exists a, b \in \mathbb{Z}\backslash\{\mathbf{0}\}, (X^a Y^b - 1) \in \mathsf{Ann}(c)$$

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$$c \text{ periodic } \Leftrightarrow \exists a, b \in \mathbb{Z}\backslash\{\mathbf{0}\}, (X^a Y^b - 1) \in \mathrm{Ann}(c)$$

$\mathrm{Ann}(c)$ is a **polynomial ideal**
$\rightarrow$ a lot of tools to understand its structure !

First results

c of low complexity : $\exists m, n, P_c(m, n) \leq mn$

Theorem (Kari & Szabados, 2015)

$\exists p \neq 0 \in \mathrm{Ann}(c)$

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Theorem (Kari & Szabados, 2015)

$\exists a_1, b_1, a_2, b_2 \ldots, a_r, b_r \in \mathbb{Z}$

$$\left(X^{a_1} Y^{b_1} - 1\right)\left(X^{a_2} Y^{b_2} - 1\right)\cdots\left(X^{a_r} Y^{b_r} - 1\right) \in \mathsf{Ann}(c)$$

Periodic Decomposition

c of low complexity : $\exists m, n, P_c(m, n) \leq mn$

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Theorem (Kari & Szabados, 2015)

There exist periodic $c_1, \ldots, c_r$,

$$c = c_1 + \cdots + c_r$$

Periodic Decomposition: any dimension !

c in dimension d with a non-trivial annihilator

Theorem (Kari & Szabados, 2015)

$\exists \mathbf{u}_1, \ldots, \mathbf{u}_r \in \mathbb{Z}^d$

$$(X^{\mathbf{u}_1} - 1)(X^{\mathbf{u}_2} - 1) \cdots (X^{\mathbf{u}_r} - 1) \in \mathrm{Ann}(c)$$

Theorem (Kari & Szabados, 2015)

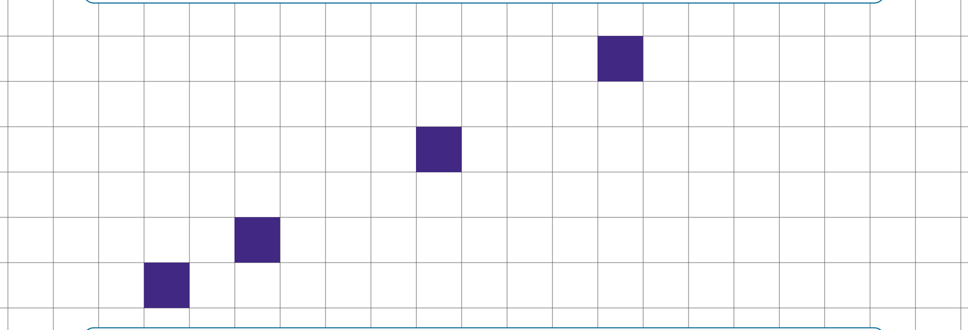
There exist periodic $c_1, \ldots, c_r$,

$$c = c_1 + \cdots + c_r$$

Line Polynomial

p is a **line polynomial** if $\exists a, b, c, d$,

$$p = X^c Y^d \sum_i p_i X^{ia} Y^{ib}$$



$$1 + X^2 Y + X^6 Y^3 + X^{10} Y^5$$

Annihilator Ideal Decomposition (2D)

Theorem (Kari & Szabados, 2015)

$c \in \mathcal{A}^{\mathbb{Z}^2}$ w. non-trivial annihilator. Then there are line polynomials $\phi_1, \ldots \phi_r$ and ideal H s.t.

$$\mathrm{Ann}(c) = \phi_1 \cdots \phi_r H,$$

H intersection of maximal ideals, $\langle \phi_1 \cdots \phi_r \rangle$ and H co-maximal, $\phi_1, \ldots, \phi_r, H$, unique

Periodic Decomposition

Theorem (Kari & Szabados, 2015)

If $c = c_1 + c_2$, c_1 and c_2 periodic, and $P_c(m,n) \leq mn$, then c is periodic.

Theorem (Cyr & Kra, 2015)

If $\exists m, n \in \mathbb{N}, P_c(m,n) \leq \frac{mn}{2}$, then c is periodic.

Asymptotic Version

Theorem (Kari & Szabados, 2015)

If $\exists$ infinitely many $m, n \in \mathbb{N}, P_c(m, n) \leq mn$, then c is periodic.

Uniformly Recurrent Case

Uniformly recurrent configurations

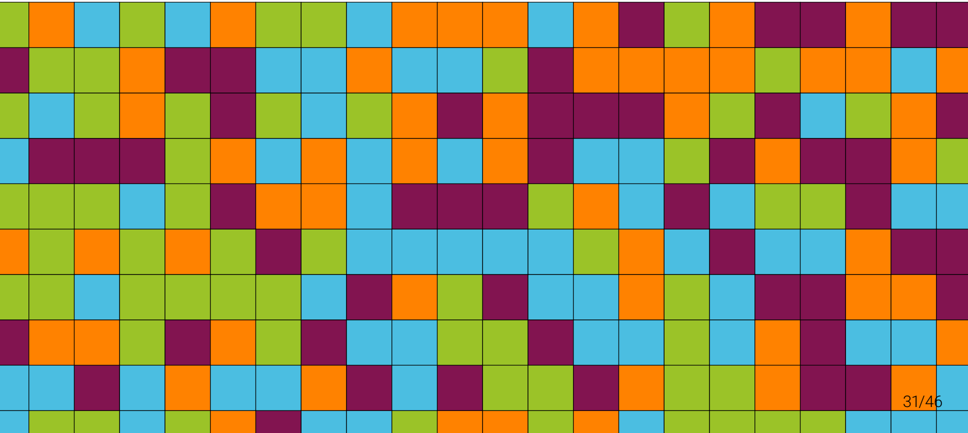
Theorem (Kari & M. 2020)

Nivat's conjecture holds for uniformly recurrent configurations

Subshifts of Finite Type – Configurations

Finite alphabet: $\mathcal{A} = \Big\{$  $\Big\}$

Configuration: $c \in \mathcal{A}^{\mathbb{Z}^2}$



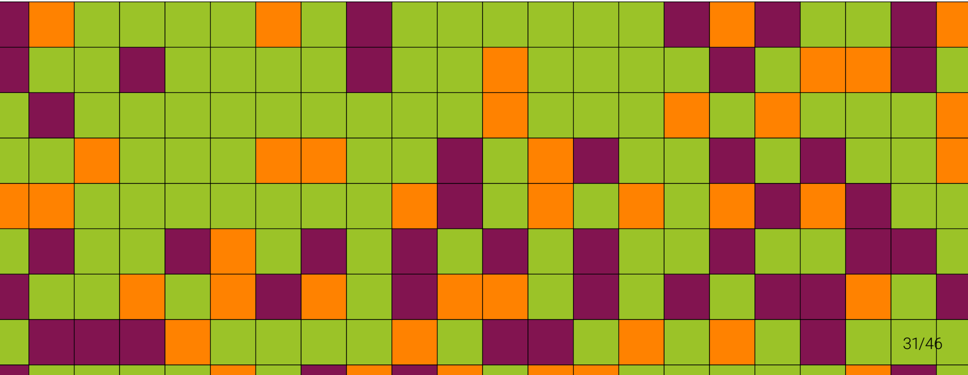
Subshifts of Finite Type

Finite alphabet: $\mathcal{A} = \{$  $\}$

Set of forbidden patterns: $F = \{$  $\}$

Subshift :

$$X_F = \{c \in \mathcal{A}^{\mathbb{Z}^2} \mid \forall m \in F, m \text{ does not appear in } c\}$$



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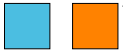
Set of forbidden patterns: $F = \{$  $\}$

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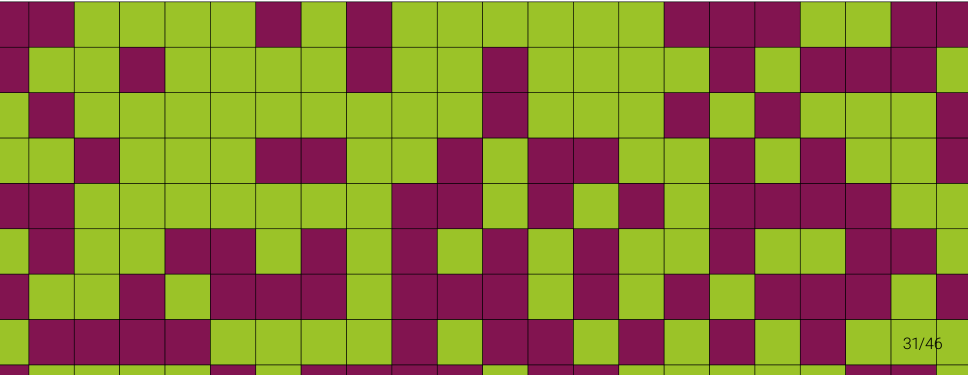

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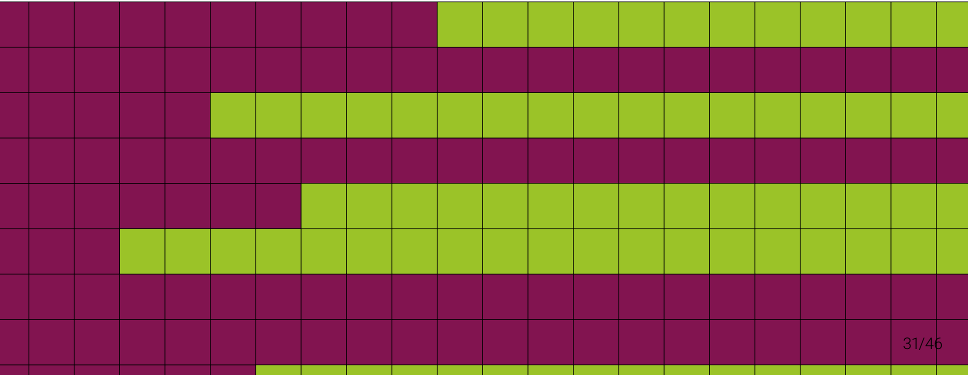

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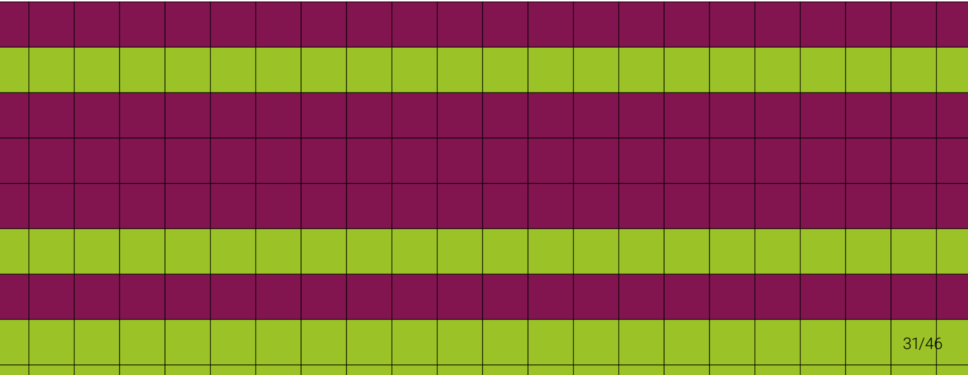

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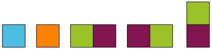
Set of forbidden patterns: $F = \{$  $\}$

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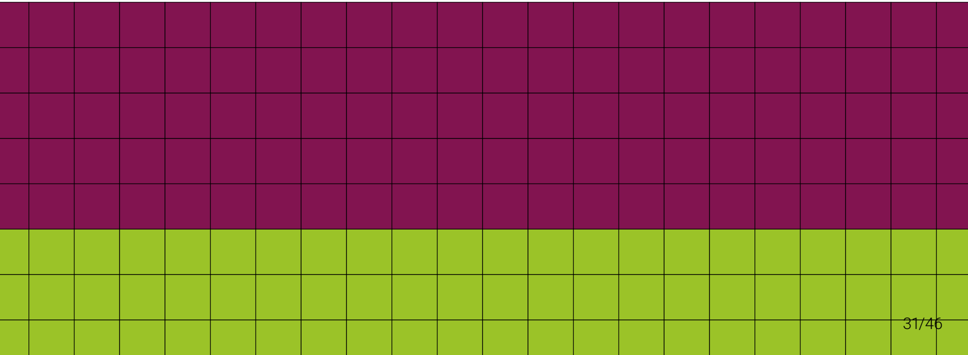
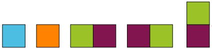


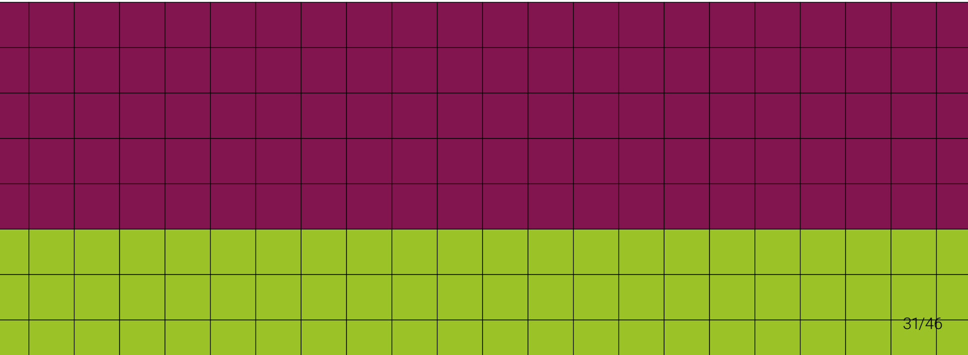
Subshifts of Finite Type

Finite alphabet: $\mathcal{A} = \{$  $\}$

Finite Set of forbidden patterns: $F = \{$  $\}$

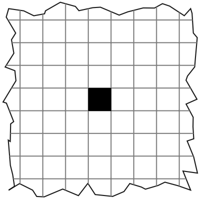
Subshift of Finite Type (SFT):

$$X_F = \{c \in \mathcal{A}^{\mathbb{Z}^2} \mid \forall m \in F, m \text{ does not appear in } c\}$$



Orbits

Orbit of c: $\mathcal{O}(c) = \{\mathbf{u} \cdot c \mid \mathbf{u} \in \mathbb{Z}^2\}$



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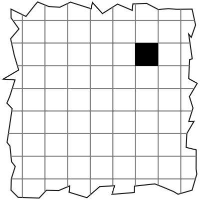

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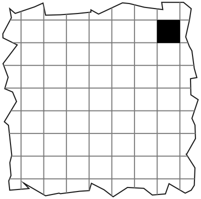

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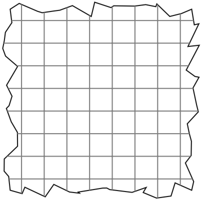
Orbit closure of c: $\overline{\mathcal{O}(c)}$, topological closure of $\mathcal{O}(c)$ (a subshift)



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c unifomly recurrent $\Leftrightarrow \forall d \in \overline{\mathcal{O}(c)}, \overline{\mathcal{O}(d)} = \overline{\mathcal{O}(c)}$

Main Theorem

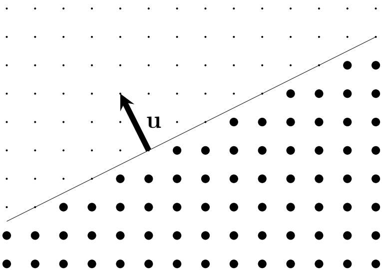
Theorem (Kari & M. 2020)

c such that $\exists m, n, P_c(m,n) \leq mn$,

$$\exists d \in \overline{\mathcal{O}(c)} \text{ which is periodic}$$

Key: **determinism**

Determinism



Half-plane $H_{(-1,2)}$

X is $\mathbf{u}$-deterministic:

$$\forall c, c' \in X,$$

$$c|_{H_\mathbf{u}} = c'|_{H_\mathbf{u}} \implies c = c'$$

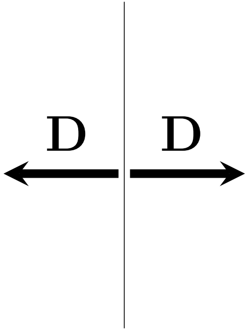
Why Determinism? (1)

Lemma (Corollary from [Boyle & Lind 1997])

$\overline{\mathcal{O}(c)}$ is deterministic in all directions

$$\Rightarrow$$

c is (two-)periodic

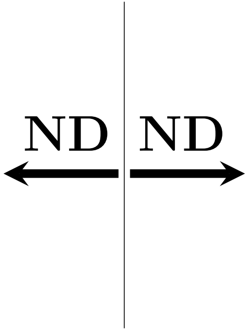


Why Determinism? (2)

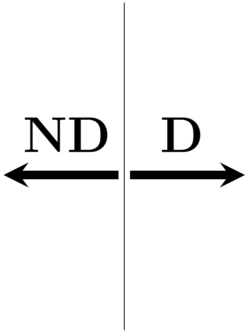
Lemma (Cyr & Kra 2015)

c of low complexity and uniformly recurrent.

$$\exists \mathbf{u}, \overline{\mathcal{O}(c)} \text{ non-deterministic for } \mathbf{u} \text{ and } -\mathbf{u},$$
$$\Rightarrow$$
c *is periodic*



One-sided Determinism



Last case: **One-sided Determinism**

.. can be eliminated

Theorem (Kari & M. 2020)

c 2D with non-trivial annihilator.

$\exists d \in \overline{\mathcal{O}(c)}$ such that $\overline{\mathcal{O}(d)}$ **has no direction of one-sided determinism**.

Main Theorem

Theorem (Kari & M. 2020)

c such that $\exists m, n, P_c(m,n) \leq mn$,

$$\exists d \in \overline{\mathcal{O}(c)} \text{ which is periodic}$$

Proof sketch:

- "Eliminate" all one-sided deterministic directions $\to d \in \overline{\mathcal{O}(c)}$
- d is periodic ! ([Boyle & Lind 1997] + [Cyr & Kra, 2015])

Consequence 1: Uniform Recurrence

Corollary (Kari & M. 2020)

Nivat's conjecture holds for c uniformly recurrent

Proof.

$c \in \mathcal{A}^{\mathbb{Z}^2}, P_c(m,n) \leq mn$.
There exists $d \in \overline{\mathcal{O}(c)}$ periodic.
c uniformly recurrent $\Rightarrow \overline{\mathcal{O}(c)} = \overline{\mathcal{O}(d)}$.
$\overline{\mathcal{O}(d)}$ contains only periodic configurations $\Rightarrow$ c periodic. □

What's next ?

Nivat's Conjecture

$\rightarrow$ Nivat's Conjecture
Non uniformly recurrent configurations: contains arbitrarily large periodic regions

What is the geometry of these regions?

Nivat's Conjecture

$\rightarrow$ Nivat's Conjecture
Non uniformly recurrent configurations: contains arbitrarily large periodic regions

What is the geometry of these regions?

$\rightarrow$ What complexity can have aperiodic SFTs?

What complexity can have aperiodic SFTs?

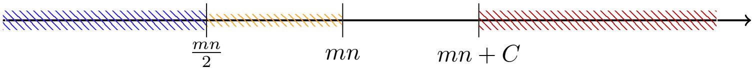
Theorem (Kari & M. 2020)

For all **aperiodic** subshifts X, $\forall c \in X, \forall m, n \in \mathbb{N}$,

$$P_c(m,n) > mn$$

What complexity can have aperiodic SFTs?

$\forall X, \forall c \in X,\ c$ periodic $\quad \forall X, \exists c \in X,\ c$ periodic $\quad \exists X, \forall c \in X,\ c$ not periodic



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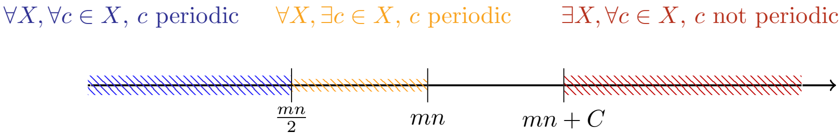
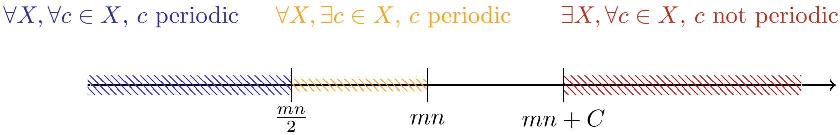


What is the smallest possible C? (we know $C > 0$)

What complexity can have aperiodic SFTs?



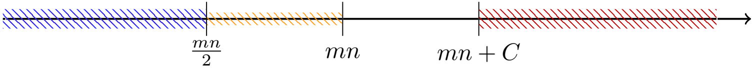
What is the smallest possible C? (we know $C > 0$)

Conjecture

$C > 1$

What complexity can have aperiodic SFTs?

$\forall X, \forall c \in X$, c periodic $\quad \forall X, \exists c \in X$, c periodic $\quad \exists X, \forall c \in X$, c not periodic

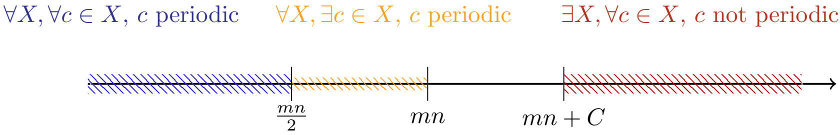


Theorem (M. & Petit-Jean 2021)

$\sigma : \mathcal{A} \to \mathcal{A}^{\{1,\dots,n\}^2}$ primitive substitution with **determining position**. If X^σ is aperiodic, then $\exists C > 1, \forall c \in X^\sigma, \forall n$,

$$P_c(n, n) \geq Cn^2$$

What complexity can have aperiodic SFTs?



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Conjecture

True without the *determining position* assumption

Thank you !