Left Lyndon tree Construction

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Combinatorics on words seminar

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Definitions

A word (string) is a sequence of symbols, e.g. `abbaaba`

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  for words r, s and t and letters a and b with and $a < b$.

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- $u \prec v$ if $u^\infty < v^\infty$ or both $u^\infty = v^\infty$ and $|u| > |v|$.
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$ab < aba$ but $ab^\infty = abab \cdots > (aba)^\infty = abaaba \cdots$

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- `a`, `b`, `ab`, `aab`, `abb`, ...

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- `a`, `b`, `ab`, `aab`, `abb`, ...

Not Lyndon words:

- $w =$ `abbab`: for $u =$ `abb` and $v =$ `ab` we get $w > vu =$ `ababb`.

.

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Binary Lyndon words:

- a, b, ab, aab, abb, ...

Not Lyndon words:

- $w =$ abbab: for $u =$ abb and $v =$ ab we get $w > vu =$ ababb.
- $w =$ abab: for $u =$ ab we get $(\mathtt{ab})^\infty = (\mathtt{abab})^\infty$.

Motivation

- Words are structured by their Lyndon factors.
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  e.g. [*Banai et al.*, 2017], [*C. et al.*, 2012].

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- Can be viewed as a preprocessing step to word algorithms.
- Help discovering maximal periodicities in words.
  e.g. [*Banai et al.*, 2017], [*C. et al.*, 2012].
- Help sorting word suffixes to create a Suffix Array.
  e.g. [*Mantaci et al.*, 2013], [*Louza et al.*, 2019].

Lyndon Suffix Table

Table *LynS* of a word y is defined, for each position j on y, by

$$LynS[j] = \max\{|w| \mid w \text{ Lyndon suffix of } y[0\,..\,j]\}.$$

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Example

Let $y = \texttt{ababbababbabac}$ on the alphabet of constant letters $\{\texttt{a}, \texttt{b}, \dots\}$ ordered as usual $\texttt{a} < \texttt{b} < \cdots$.

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$y[j]$	a	b	a	b	b	a	b	a	b	b	a	b	a	c
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$LynS_y[j]$	1	2	1	2	5									

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$LynS_y[j]$	1	2	1	2	5	1	2	1	2	5	1	2	1	14

Properties of Lyndon Words

- Let z be a word and a a letter for which za is a prefix of a Lyndon word. Let b be a letter with $a < b$. Then zb is a Lyndon word.

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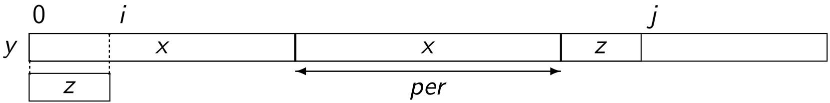
- Let u and v be two Lyndon words with $u < v$, then uv is Lyndon word.

Invariant of *LynS* computation: $w = x^e z$
where x is a Lyndon word and z a proper prefix of x.



If $y[j] > y[i]$ then $y[0 \,.\,.\, j]$ is a Lyndon word **with period** $j + 1$.
[*Duval*, 1983].

Lyndon Suffix Table - Algorithm

LyndonSuffix(y Lyndon word of length n)

```
LynS[0] ← 1
(per, i) ← (1, 0)
for j ← 1 to n − 1 do
        if y[j] ≠ y[i] then                ▷ y[j] > y[i] = y[j − per]
                LynS[j] ← j + 1
                (per, i) ← (j + 1, 0)
        else    LynS[j] ← LynS[i]
                i ← i + 1 mod per
return LynS
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    $i \leftarrow i + 1 \bmod per$
return $LynS$

Proposition

Algorithm LyndonSuffix *computes the Lyndon suffix table of a Lyndon word of length n in time $O(n)$.*

Left Lyndon tree of a Lyndon word

Let y be a Lyndon word.

- Let u be the longest proper Lyndon prefix of y and $y = uv$. Then v is a Lyndon word.
  uv is the **left Lyndon factorisation** of y.
  aaaababbaabaab = aaaababbaab · aab

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  $\mathtt{aaaababbaabaab} = \mathtt{aaaababbaab} \cdot \mathtt{aab}$
- Let v be the longest proper Lyndon suffix of y and $y = uv$. Then u is a Lyndon word. uv is the **right Lyndon factorisation** of y.
  $\mathtt{aaaababbaabaab} = \mathtt{a} \cdot \mathtt{aaababbaabaab}$

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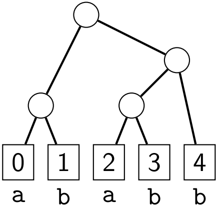
Left Lyndon tree of y:

- Obtained by recursive application of left Lyndon factorisation:
  $\mathtt{ababb} = \mathtt{ab} \cdot \mathtt{abb} = (\mathtt{a} \cdot \mathtt{b}) \cdot (\mathtt{ab} \cdot \mathtt{b}) = (\mathtt{a} \cdot \mathtt{b}) \cdot ((\mathtt{a} \cdot \mathtt{b}) \cdot \mathtt{b})$

Left, right Lyndon trees

Left Lyndon tree of ababb:



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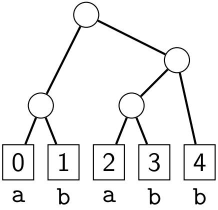

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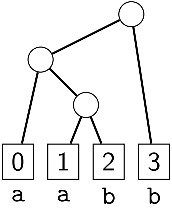

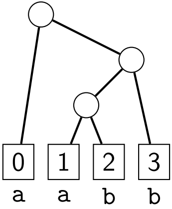

Left Lyndon Tree - Algorithm

```
LeftLyndonTree(y Lyndon word of length n)
(LynS[0], root[0]) ← (1, 0)
(per, i) ← (1, 0)
for j ← 1 to n − 1 do
    root[j] ← j
    if y[j] ≠ y[i] then                ▷ y[j] > y[i] = y[j − per]
        LynS[j] ← j + 1
        (per, i) ← (j + 1, 0)
    else  LynS[j] ← LynS[i]
        i ← i + 1 mod per
    (ℓ, k) ← (1, j − 1)
    while ℓ < LynS[j] do
        q ← new node ≥ n
        (left[q], right[q]) ← (root[k], root[j])
        root[j] ← q
        (ℓ, k) ← (ℓ + LynS[k], k − LynS[k])
return root[n − 1]
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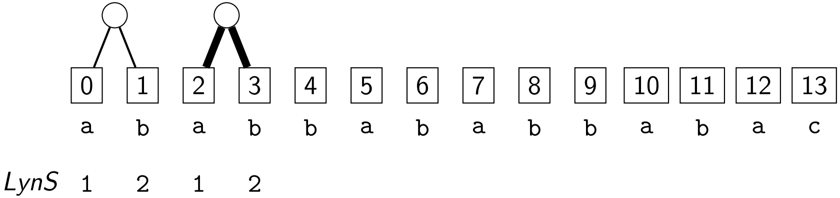
Theorem

Algorithm LeftLyndonTree *builds the left Lyndon tree of a Lyndon word of length n in time $O(n)$.*

Building the tree

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
	a	b	a	b	b	a	b	a	b	b	a	b	a	c
LynS	1	2												

Building the tree



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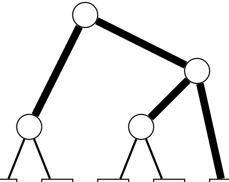

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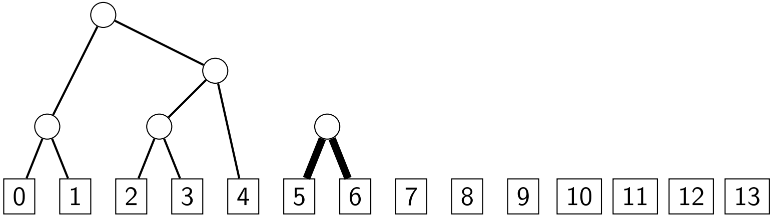
	a	b	a	b	b	a	b	a	b	b	a	b	a	c
LynS	1	2	1	2	5									



	a	b	a	b	b	a	b	a	b	b	a	b	a	c
LynS	1	2	1	2	5	1	2							

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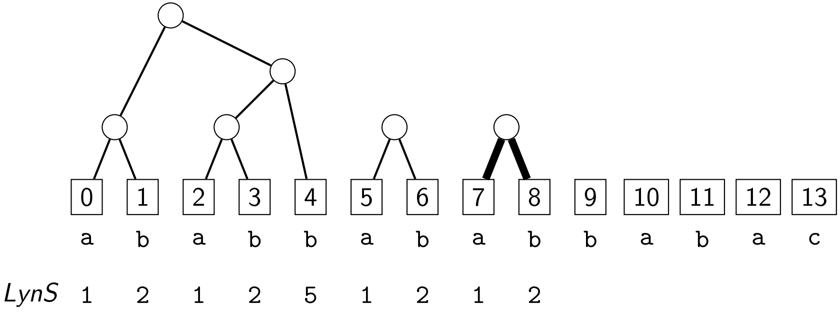

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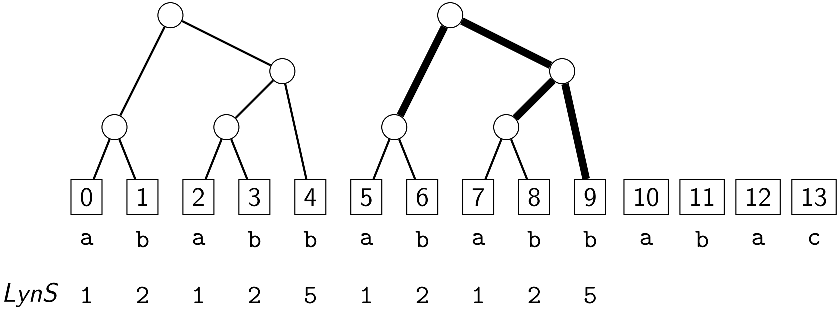

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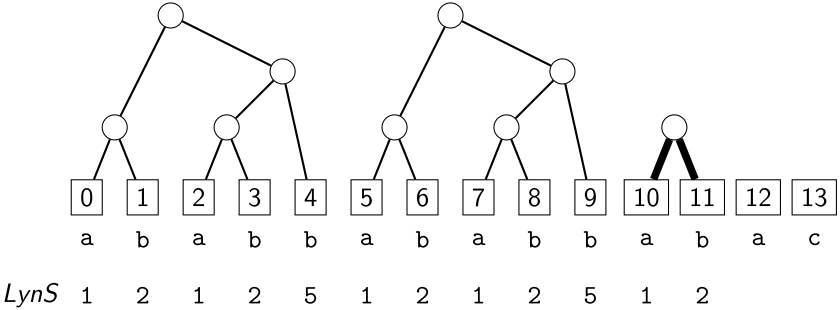

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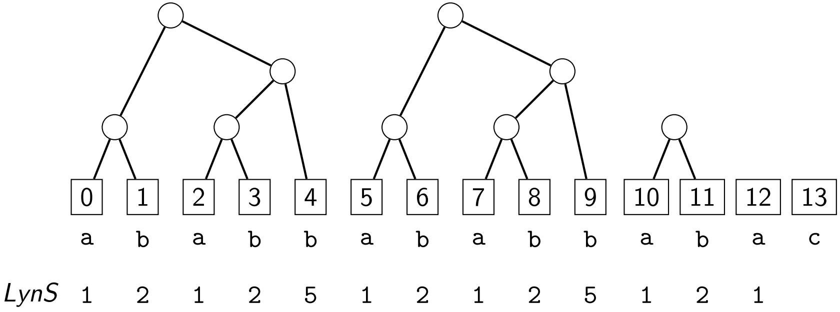

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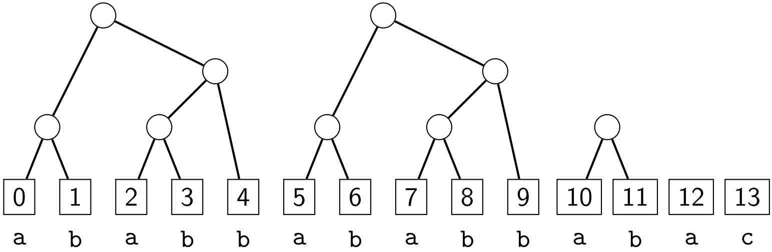

LynS	1	2	1	2	5	1	2	1	2	5	1	2	1	14

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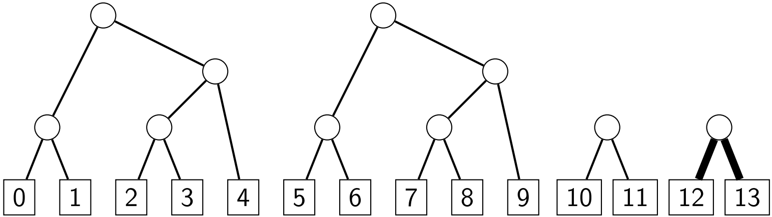

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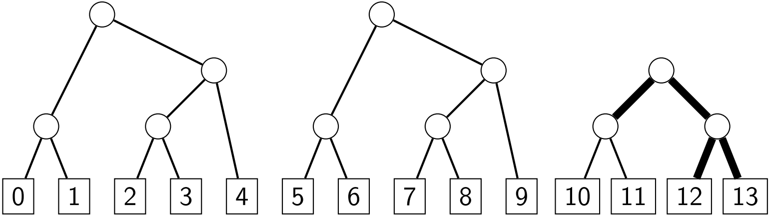

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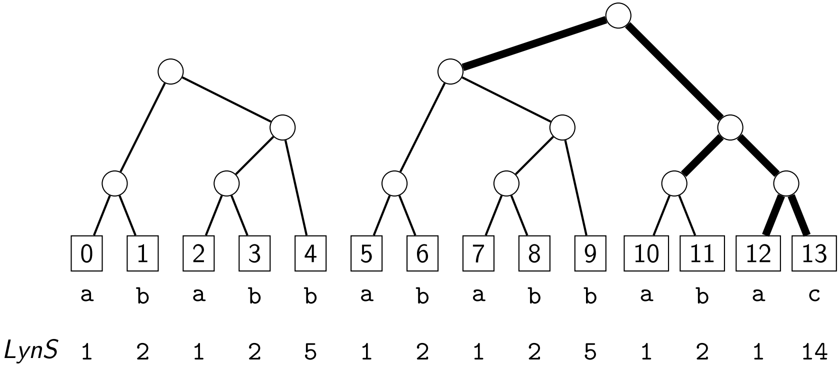

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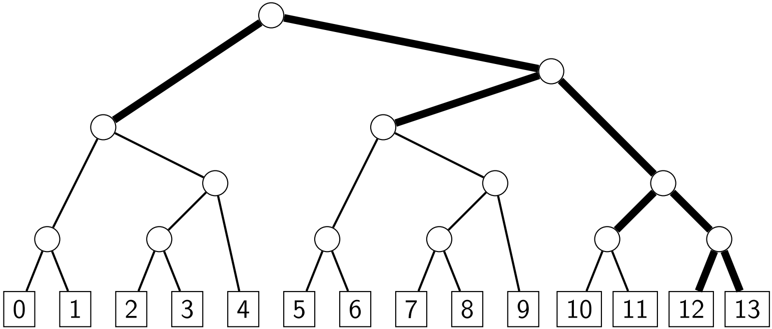

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LynS	1	2	1	2	5	1	2	1	2	5	1	2	1	14

Prefix standard permutation

Ranks according to the infinite ordering $\prec$.

j	$rank[j]$	
0	1	`a. a`
1		`a b. a b. a b. a b. a b. a b`
2		`a b a. a b a`
3		`a b a b. a b a b. a b a b`
4		`a b a b b. a b a b b`
5		`a b a b b a. a b a b b`
6		`a b a b b a b. a b a b b a b`
7		`a b a b b a b a. a b a b b a b a`
8		`a b a b b a b a b. a b a b b a b a b`
9		`a b a b b a b a b b. a b a b b a b a b b`
10		`a b a b b a b a b b a. a b a b b a b a b b`
11		`a b a b b a b a b b a b. a b a b b a b a b`
12		`a b a b b a b a b b a b a. a b a b b a b a`

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1		a b. a b. a b. a b. a b. a b
2	2	a b a. a b a
3		a b a b. a b a b. a b a b
4		a b a b b. a b a b b
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Prefix standard permutation

Ranks according to the infinite ordering $\prec$.

j	$rank[j]$	
0	1	a a
1	4	a b. a b. a b. a b. a b. a b
2	2	a b a. a b a
3	3	a b a b. a b a b. a b a b
4		a b a b b. a b a b b
5		a b a b b a. a b a b b
6		a b a b b a b. a b a b b a b
7		a b a b b a b a. a b a b b a b a
8		a b a b b a b a b. a b a b b a b a b
9		a b a b b a b a b b. a b a b b a b a b b
10		a b a b b a b a b b a. a b a b b a b a b b
11		a b a b b a b a b b a b. a b a b b a b a b
12		a b a b b a b a b b a b a. a b a b b a b a

Prefix standard permutation

Ranks according to the infinite ordering $\prec$.

j	$rank[j]$	
0	1	a a
1	4	a b. a b. a b. a b. a b. a b
2	2	a b a. a b a
3	3	a b a b. a b a b. a b a b
4		a b a b b. a b a b b
5	5	a b a b b a. a b a b b
6		a b a b b a b. a b a b b a b
7		a b a b b a b a. a b a b b a b a
8		a b a b b a b a b. a b a b b a b a b
9		a b a b b a b a b b. a b a b b a b a b b
10		a b a b b a b a b b a. a b a b b a b a b b
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4		a b a b b. a b a b b
5	5	a b a b b a. a b a b b
6	8	a b a b b a b. a b a b b a b
7	6	a b a b b a b a. a b a b b a b a
8	7	a b a b b a b a b. a b a b b a b a b
9		a b a b b a b a b b. a b a b b a b a b b
10		a b a b b a b a b b a. a b a b b a b a b b
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5	5	a b a b b a. a b a b b
6	8	a b a b b a b. a b a b b a b
7	6	a b a b b a b a. a b a b b a b a
8	7	a b a b b a b a b. a b a b b a b a b
9		a b a b b a b a b b. a b a b b a b a b b
10	9	a b a b b a b a b b a. a b a b b a b a b b
11	11	a b a b b a b a b b a b. a b a b b a b a b
12	10	a b a b b a b a b b a b a. a b a b b a b a

Prefix standard permutation

Ranks according to the infinite ordering $\prec$.

j	$rank[j]$	
0	1	a a
1	4	a b. a b. a b. a b. a b. a b
2	2	a b a. a b a
3	3	a b a b. a b a b. a b a b
4	13	a b a b b. a b a b b
5	5	a b a b b a. a b a b b
6	8	a b a b b a b. a b a b b a b
7	6	a b a b b a b a. a b a b b a b a
8	7	a b a b b a b a b. a b a b b a b a b
9	12	a b a b b a b a b b. a b a b b a b a b b
10	9	a b a b b a b a b b a. a b a b b a b a b b
11	11	a b a b b a b a b b a b. a b a b b a b a b
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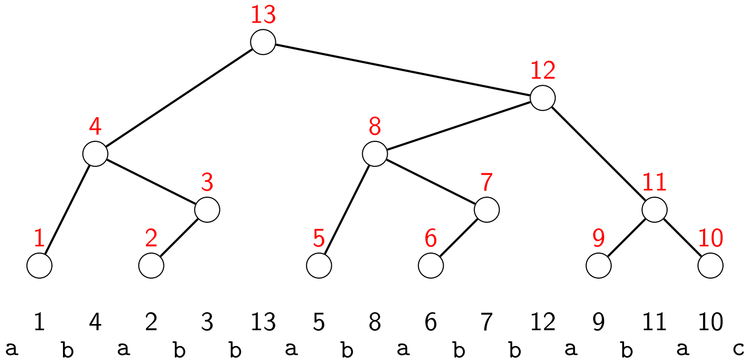
Cartesian tree of prefix ranks

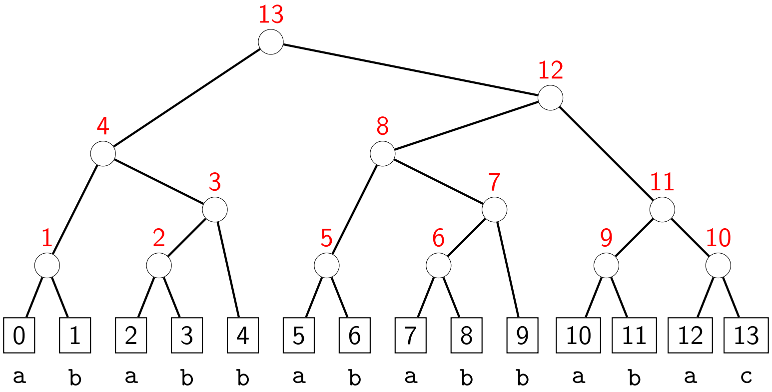
j	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$y[j]$	a	b	a	b	b	a	b	a	b	b	a	b	a	c
rank[j]	1	4	2	3	13	5	8	6	7	12	9	11	10	



Prefix ranks and Left Lyndon tree

Theorem (Dolce, Restivo, Reutenauer, 2019)

The tree of internal nodes of the left Lyndon tree of a Lyndon word y in which nodes are labelled by the ranks of proper prefixes of y sorted according to the infinite order is the Cartesian tree of the ranks.



Prefix sorting

Theorem

Sorting the proper prefixes of a Lyndon word of length n according to the infinite order $\prec$ can be done in time $O(n)$ in the letter-comparison model.

Proof: In Algorithm $\textsc{LeftLyndonTree}$ list prefixes associated to internal nodes instead of building the tree.

Properties of Prefix standard permutation

A Lyndon word is either a singleton or $x^k zb$ where $k > 0$, z is a proper prefix of x and b is a letter bigger than what follows the occurrence of z in x.
Grouping the prefixes: for $e = 0, 1, \cdots, k$, let
$P_e = \{x^e u | e|x| < |u| < min\{(e+1)|x|, |y|\}\}$

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Our example

$y = ababbababbabac = (ababb)^2abac$
$P_0 = \{a, ab, aba, abab\}$
$P_1 = \{ababba, ababbab, ababbaba, ababbabab\}$
$P_2 = \{ababbababba, ababbababbab, ababbababbaba\}$
$x = ababb, x^2 = ababbababb$

- sort P_0 (recursively)
- same relative order for P_1 and P_2
- $x_2 \prec x$

Prefix standard permutation

Ranks according to the infinite ordering $\prec$.

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$y[j]$	a	b	a	b	b	a	b	a	b	b	a	b	a	c
rank[j]	1	4	2	3		5	8	6	7					

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Ranks according to the infinite ordering $\prec$.

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$y[j]$	a	b	a	b	b	a	b	a	b	b	a	b	a	c
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j	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$y[j]$	a	b	a	b	b	a	b	a	b	b	a	b	a	c
rank$[j]$	1	4	2	3	13	5	8	6	7	12	9	11	10	

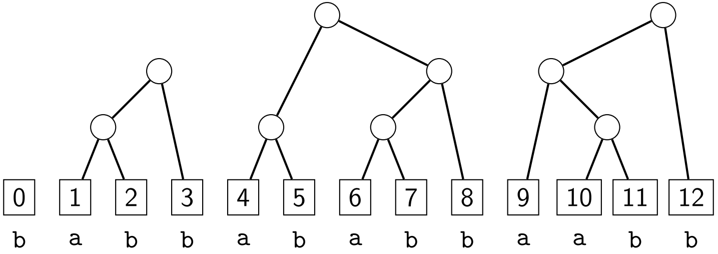
Lyndon Forest: input is any non-empty word

Algorithms extend to factors of the Lyndon factorisation of a non-empty word (algorithm by [*Duval*, 1983]).

Example

The Lyndon suffix table of $y =$ babbababbaabb is as follows.

j	0	1	2	3	4	5	6	7	8	9	10	11	12
$y[j]$	b	a	b	b	a	b	a	b	b	a	a	b	b
$LynS[j]$	1	1	2	3	1	2	1	2	5	1	1	3	4



Works on hand

Reverse engineering

- On a binary alphabet, function psp is one-to-one.
- Given a permutation p of $\{0, 1, \ldots, n-2\}$, the word y of length n for which $\mathrm{psp}(y) = p$ can be found in linear time.
- How much can the right Lyndon tree of y help sort its suffixes?

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Relation between left and right Lyndon trees