Markov numbers

An answer to three conjectures from Aigner

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Outline

1. Markov numbers
2. Aigner's Conjectures
3. m-value
4. Transformations on paths
5. Conclusion

Markov Numbers

Markov triples are the positive integer solutions of the Diophantine equation:

$$x^2 + y^2 + z^2 = 3xyz$$

The first triples are $(1,1,1), (1,1,2)$ and $(1,2,5)$.
In fact we call Markov number the maximal value of the triple, so the first ones are: 1, 2 and 5.

Frobenius' Conjecture

In G.F. Frobenius, *Über die Markoffschen zahlen*, Königliche Akademie der Wissenschaften, 1913 we find the following:

Conjecture

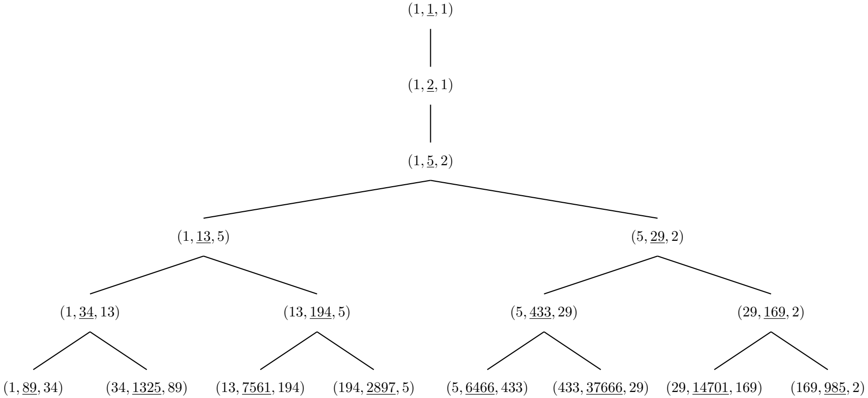
Each Markov number is the maximum of an unique triple.

Recursive construction

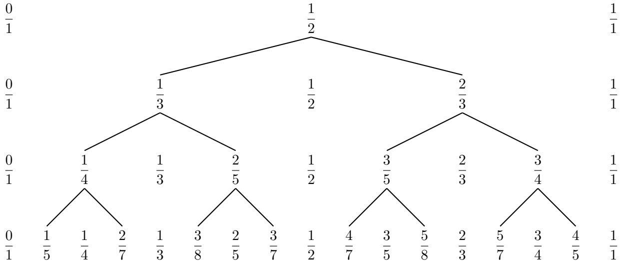
If we put the Markov number in the middle of the triple, we obtain a recursive way of creating new Markov numbers:

$$
\begin{tikzcd}[column sep=small]
 & (\ell, m, r) \arrow[dl, dash] \arrow[dr, dash] & \\
(\ell, 3\ell m - r, m) & & (m, 3rm - \ell, r)
\end{tikzcd}
$$

Markov Tree


(1, 1, 1)
(1, 2, 1)
(1, 5, 2)
(1, 13, 5)
(5, 29, 2)
(1, 34, 13)
(13, 194, 5)
(5, 433, 29)
(29, 169, 2)
(1, 89, 34)
(34, 1325, 89)
(13, 7561, 194)
(194, 2897, 5)
(5, 6466, 433)
(433, 37666, 29)
(29, 14701, 169)
(169, 985, 2)

Stern-Brocot Tree



Definition

Farey fraction: $\dfrac{a}{b} \oplus \dfrac{c}{d} = \dfrac{a+c}{b+d}$

Identification

We can index the Markov numbers (≥ 5) with the Farey fractions in the Stern-Brocot tree. So we refer to the Markov number by the Farey fraction it is associated with, $m_{\frac{p}{q}}$.

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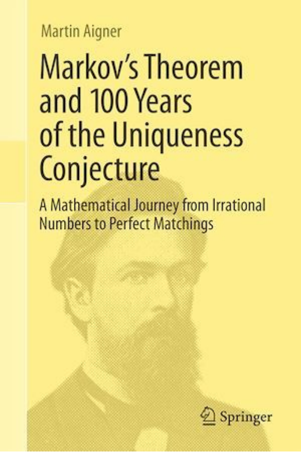
Example

$m_{\frac{1}{2}} = 5$
$m_{\frac{1}{3}} = 13$
$m_{\frac{2}{3}} = 29$

Aigner's Conjectures



Aigner's Conjectures

1. **The fixed numerator conjecture**
   Let p, q and $i \in \mathbb{N}$ such that $i > 0, p < q, \gcd(p, q) = 1$ and $\gcd(p, q+i) = 1$ then $m_{\frac{p}{q}} < m_{\frac{p}{q+i}}$;

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3. **The fixed sum conjecture**
   Let p, q and $i \in \mathbb{N}$ such that $i > 0, p < q, \gcd(p, q) = 1$ and $\gcd(p - i, q + i) = 1$ then $m_{\frac{p}{q}} < m_{\frac{p-i}{q+i}}$.

Aigner's Conjectures

The fixed numerator conjecture was proved last year in M. Rabideau and R. Schiffler, *Continued fractions and orderings on the Markov numbers*, Advances in Mathematics, vol. 370, p. 107231, 2020.

The proof is quite technical and is based on the perfect matchings of snake graphs.

Cohn matrices

Let

$$\begin{aligned} M : \{a, b\}^\star &\to SL_2(\mathbb{Z}) \\ \mathbf{w} &\mapsto M^{\mathbf{w}} \end{aligned}$$

defined by

$$A \stackrel{\text{def}}{=} M^a = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B \stackrel{\text{def}}{=} M^b = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

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Example

$$M(aaba) = M^{aaba} = M^a M^a M^b M^a = \begin{pmatrix} 12 & 17 \\ 19 & 27 \end{pmatrix}$$

Some properties of Cohn matrices

Let $U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Properties

- $AB - BA = 2U$
- $AUA = U$
- $BUB = U$
- $AUB = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$, *which is nonnegative*
- $BUA = \begin{pmatrix} -1 & -3 \\ 0 & -1 \end{pmatrix}$, *which is nonpositive*

m-value

To each word w in $\{a, b\}^\star$ we associate its m-value through the map:

$$m(w) = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot M^w \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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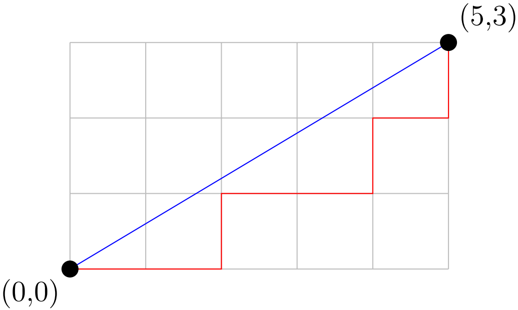
Remark

We chose the name m-value because its an extension of the Markov numbers.

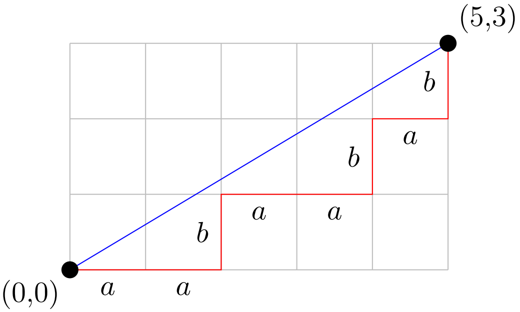
Christoffel words and Markov numbers



We associate to each fraction $\frac{p}{q}$ the path from $(0,0)$ to (q,p) where we run through the integer points and respect the two conditions:

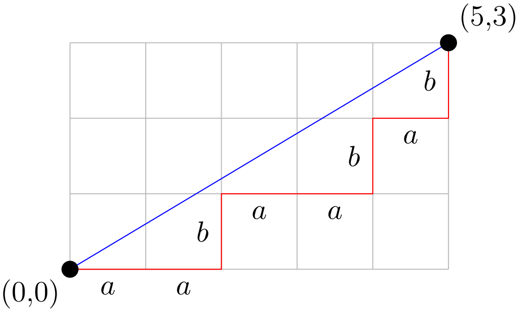
1. we stay below the line segment from $(0,0)$ to (q,p);
2. there is no integer point that lie strictly between the path and the line segment.

Christoffel words and Markov numbers



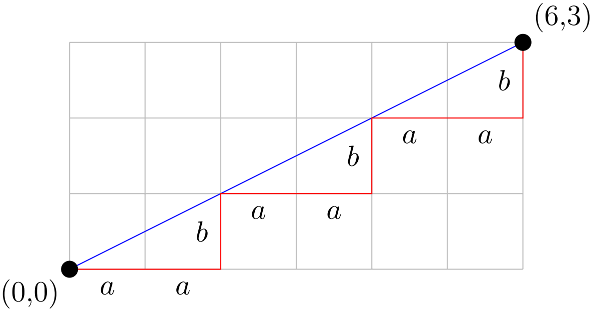
We associate to each path the word obtained when we replace each horizontal step by a and each vertical step by b.
For example: $c(3,5) = aabaabab = c_{\frac{3}{5}}$.

Christoffel words and Markov numbers



Looking at the m-value of this word we get $m(c(3,5)) = 433 = m_{\frac{3}{5}}$.

Power of a Christoffel word and m-values



When n and d are not relatively primes, the path we obtain (with a similar construction) leads to a power of a Christoffel word. Here we get $m(\mathfrak{c}(3,6)) = 1120$.

Generalized Christoffel words and m-values

144	233	507	1120	2523	5741	13860
55	89	194	433	985	2378	5741
21	34	75	169	408	985	2523
8	13	29	70	169	433	1120
3	5	12	29	75	194	507
1	2	5	13	34	89	233
0	1	3	8	21	55	144

m-values for $n, d \in [0; 6]^2$

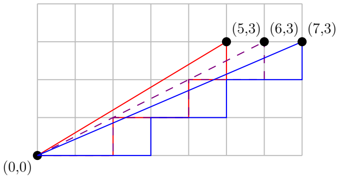
Remarks

1. there is a symmetry along the diagonal;
2. the bottom line contains only the even-indexed Fibonacci numbers ($F_{n+2} = F_{n+1} + F_n$);
3. the diagonal contains only the even-indexed Pell numbers ($P_{n+2} = 2P_{n+1} + P_n$);

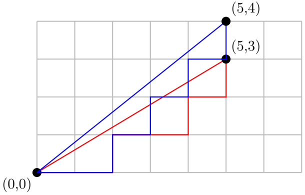
Idea

The main idea of our work was to use transformations on paths from one path to another and see if we could get a monotonous chain of inequalities on the associated m-values.

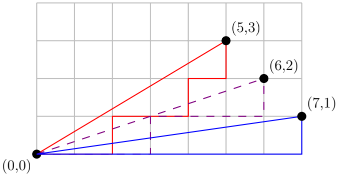
Idea



Fixed numerator conjecture



Fixed denominator conjecture



Fixed sum conjecture

Flips

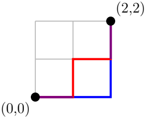
Let $\mathbf{w} \in \{a,b\}^\star$, we note $\overline{\mathbf{w}}$ the reversal of $\mathbf{w}$.

Lemma

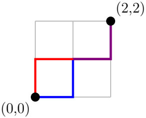
Let $(\mathbf{w}_1, \mathbf{w}_2) \in (\{a,b\}^\star)$ and $\mathbf{u}$ being the largest common prefix of $\mathbf{w}_1$ and $\mathbf{w}_2$. One of these cases occurs:

1. *If $\mathbf{w}_1 = \mathbf{u}a\mathbf{u}_1$ and $\mathbf{w}_2 = \mathbf{u}b\mathbf{u}_2$, then* $m(\overline{\mathbf{w}_1}ab\mathbf{w}_2) \geq m(\overline{\mathbf{w}_1}ba\mathbf{w}_2)$.
2. *If $\mathbf{w}_1 = \mathbf{u}b\mathbf{u}_1$ and $\mathbf{w}_2 = \mathbf{u}a\mathbf{u}_2$, then* $m(\overline{\mathbf{w}_1}ab\mathbf{w}_2) < m(\overline{\mathbf{w}_1}ba\mathbf{w}_2)$.
3. *If $\mathbf{w}_1 = \mathbf{u}$ or $\mathbf{w}_2 = \mathbf{u}$, then $m(\overline{\mathbf{w}_1}ab\mathbf{w}_2) < m(\overline{\mathbf{w}_1}ba\mathbf{w}_2)$.*
4. *Moreover, ($\mathbf{w}_1 = \mathbf{u}a$ and $\mathbf{w}_2 = \mathbf{u}b$) if and only if* $m(\overline{\mathbf{w}_1}ab\mathbf{w}_2) = m(\overline{\mathbf{w}_1}ba\mathbf{w}_2)$.

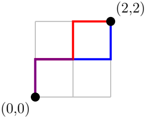
Flips



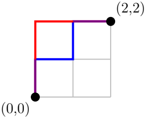
$m(aabb) = m(abab)$ by (4)



$m(abab) < m(baab)$ by (3)



$m(baab) < m(baba)$ by (3)



$m(baba) < m(bbaa)$ by (2)

$$m(aabb) = m(abab) = 12$$
$$m(baab) = 18$$
$$m(baba) = 24$$
$$m(bbaa) = 30$$

Composite flips

However it is not sufficient because we are interested in words of the form: $a\mathbf{p}ab\mathbf{p}ab\mathbf{p}b$, where $\mathbf{p}$ is a palindorme.
And whatever the order of the flips chosen, we will never have a monotonous chain of inequalities for the m-values, because:

$$m(a\mathbf{p}ab\mathbf{p}ab\mathbf{p}b) = m(a\mathbf{p}ba\mathbf{p}ba\mathbf{p}b).$$

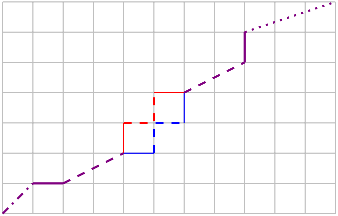
Composite flips

Lemma

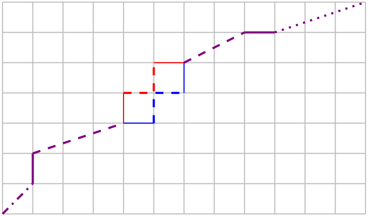
Let $\mathbf{z} \in \{a,b\}^\star$*. Let* $(\mathbf{w}_1, \mathbf{w}_2) \in (\{a,b\}^\star)$ *and* $\mathbf{u}$ *being the largest common prefix of* $\mathbf{w}_1$ *and* $\mathbf{w}_2$*. One of the first three cases occurs:*

1. *If* $\mathbf{w}_1 = \mathbf{u}a\mathbf{u}_1$ *and* $\mathbf{w}_2 = \mathbf{u}b\mathbf{u}_2$*, then* $m(\overline{\mathbf{w}_1}a\mathbf{z}b\mathbf{w}_2) \geq m(\overline{\mathbf{w}_1}b\overline{\mathbf{z}}a\mathbf{w}_2)$.
2. *If* $\mathbf{w}_1 = \mathbf{u}b\mathbf{u}_1$ *and* $\mathbf{w}_2 = \mathbf{u}a\mathbf{u}_2$*, then* $m(\overline{\mathbf{w}_1}a\mathbf{z}b\mathbf{w}_2) < m(\overline{\mathbf{w}_1}b\overline{\mathbf{z}}a\mathbf{w}_2)$.
3. *If* $\mathbf{w}_1 = \mathbf{u}$ *or* $\mathbf{w}_2 = \mathbf{u}$*, then* $m(\overline{\mathbf{w}_1}a\mathbf{z}b\mathbf{w}_2) < m(\overline{\mathbf{w}_1}b\overline{\mathbf{z}}a\mathbf{w}_2)$.
4. *Moreover, (*$\mathbf{w}_1 = \mathbf{u}a$ *and* $\mathbf{w}_2 = \mathbf{u}b$*) if and only if* $m(\overline{\mathbf{w}_1}a\mathbf{z}b\mathbf{w}_2) = m(\overline{\mathbf{w}_1}b\overline{\mathbf{z}}a\mathbf{w}_2)$.

Composite flips

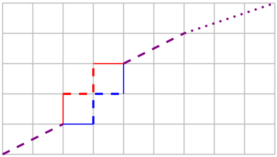


$m(\overline{\mathbf{w}_1}a\mathbf{z}b\mathbf{w}_2) \geq m(\overline{\mathbf{w}_1}b\overline{\mathbf{z}}a\mathbf{w}_2)$

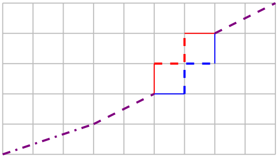


$m(\overline{\mathbf{w}_1}a\mathbf{z}b\mathbf{w}_2) < m(\overline{\mathbf{w}_1}b\overline{\mathbf{z}}a\mathbf{w}_2)$

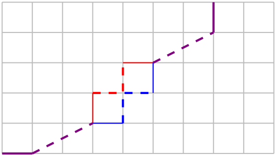
Composite flips



$m(\overline{\mathbf{u}}a\mathbf{z}b\mathbf{u}\mathbf{v}_2) < m(\overline{\mathbf{u}}b\overline{\mathbf{z}}a\mathbf{u}\mathbf{v}_2)$

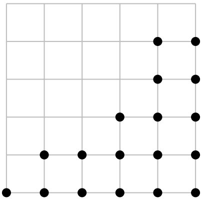


$m(\overline{\mathbf{u}\mathbf{v}_1}a\mathbf{z}b\mathbf{u}) < m(\overline{\mathbf{u}\mathbf{v}_1}b\overline{\mathbf{z}}a\mathbf{u})$

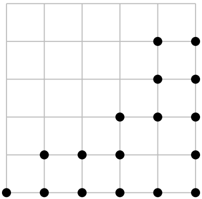


$m(\overline{\mathbf{u}a}a\mathbf{z}b\mathbf{u}b) = m(\overline{\mathbf{u}a}b\overline{\mathbf{z}}a\mathbf{u}b)$

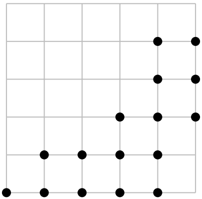
Packed sets



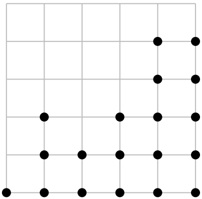
Packed set



Not packed set



Not packed set



Not packed set

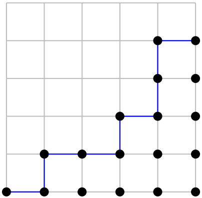
Below sets: definition

We define $\mathrm{Bel}(\mathbf{w})$ as the set of points below the word $\mathbf{w}$. More formally,

$$\mathrm{Bel}(\mathbf{w}) = \{(u, v) \in [d] \times \mathbb{N} \mid \exists p \geq v \text{ such that } \mathbf{w} \text{ goes through } (u, p)\}.$$

Given a packed set S, let $\mathrm{Hull}(S)$ be the upper hull of S,

$$\mathrm{Hull}(S) = \{(x, y) \in S \mid (x-1, y+1) \in S\}.$$



Below set of *abaababba*

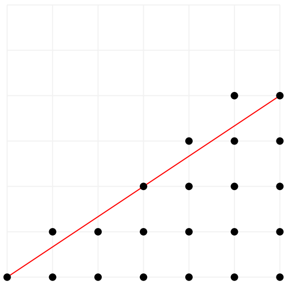
Below sets: properties

There is a bijection between packed sets and words in $\{a, b\}^\star$:

$$\{packed\ sets\} \underset{\text{Bel}}{\overset{\text{Hull}}{\rightleftarrows}} \{a, b\}^\star$$

In particular, the Christofffel and pseudo-Christoffel words are the upper hull of the below sets of the triangles delimited by the vertices $(0, 0)$, (n, d) and $(0, d)$, which will be named $T_{n,d}$.

Below sets: flattening

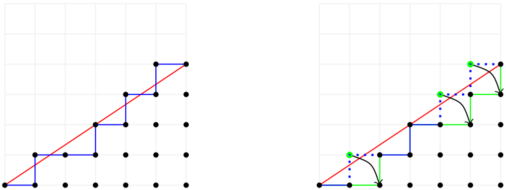


Flattening

Below sets: flattening

We define

$$\mathrm{Flat}(\mathbf{w}) \stackrel{\mathrm{def}}{=} \mathrm{Hull}\left(\mathrm{Bel}(\mathbf{w}) \cap \delta_{\mathbf{w}}^{-1}(]-\infty, 0])\right)$$
$$= \mathrm{Hull}\left(\mathrm{Bel}(\mathbf{w}) \cap T_{n,d}\right)$$



Flattening of a word: $abaabababa \mapsto aababaabab$

So we have

$$\mathrm{Bel}(\mathrm{Flat}(\mathbf{w})) = \mathrm{Bel}(\mathbf{w}) \cap \delta_{\mathbf{w}}^{-1}(]-\infty, 0]).$$

Below sets: flattening

Let $\mathbf{w} \in \{a, b\}^\star$ be a path in $\mathbb{N}^2$ from $(0,0)$ to (d,n) with $d \geq 1$. To each point $(x,y) \in [d] \times \mathbb{N}$, we associate its *algebraic vertical distance* to the line linking $(0,0)$ and (d,n):

$$\begin{aligned} \delta_{\mathbf{w}} : [d] \times \mathbb{N} &\to \mathbb{R} \\ (x,y) &\mapsto \frac{dy - nx}{d} \end{aligned}$$

(we can extend this definition for the words $\mathbf{w} \in \{b\}^\star$ by $\delta_{\mathbf{w}} : [0] \times \mathbb{N} \to \mathbb{R}, (0,y) \mapsto y$).

Proofs of fixed numerator and denominator conjectures

Lemma

Let $\mathbf{w} \in \{a,b\}^\star$. *Then*

$$m(\mathrm{Flat}(\mathbf{w})) \leq m(\mathbf{w}),$$

and the equality stands only if $\mathrm{Flat}(\mathbf{w}) = \mathbf{w}$.

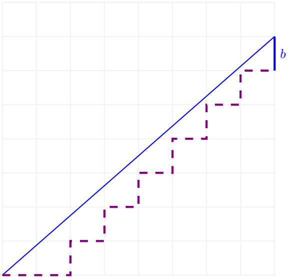
Proposition

Let $(n,d) \in (\mathbb{N} \setminus \{0\})^2$. *We have* $m(c(n,d)) < m(c(n+1,d))$.

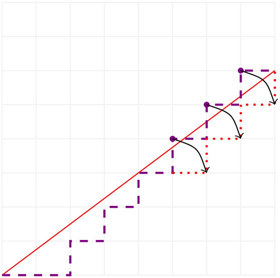
Corollary

Let $(n,d) \in (\mathbb{N} \setminus \{0\})^2$. *We have* $m(c(n,d)) < m(c(n,d+1))$.

Proofs of fixed numerator and denominator conjectures



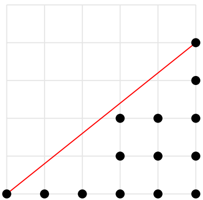
$m(c(n+1,d)) = m(\mathbf{w}b) > m(\mathbf{w})$

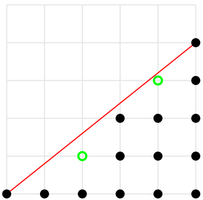


$m(\mathbf{w}) > m(\mathrm{Flat}(\mathbf{w})) = m(c(n,d))$

Visualisation of the proof

Below sets: lifting





Lifting

Below sets: lifting

The *lifting* operation consists in adding to $\mathrm{Bel}(\mathbf{w})$ all points which stand strictly below the line. We define

$$\mathrm{Lift}(\mathbf{w}) \stackrel{\text{def}}{=} \mathrm{Hull}\left(\mathrm{Bel}(\mathbf{w}) \cup \delta_{\mathbf{w}}^{-1}(]-\infty, 0[)\right).$$

Lifting of a word: $aaabbaabb \mapsto aabababab$

So we have

$$\mathrm{Bel}(\mathrm{Lift}(\mathbf{w})) = \mathrm{Bel}(\mathbf{w}) \cup \delta_{\mathbf{w}}^{-1}(]-\infty, 0[).$$

Minimality of the Christoffel words for the m-value

Lemma

Let $\mathbf{w} \in \{a,b\}^\star$. *Then*

$$m(\mathrm{Lift}(\mathbf{w})) \leq m(\mathbf{w}).$$

Proposition

Let $(d,n) \in \mathbb{N}^2$ *with* $\eta = \gcd(d,n)$. *Let* $\mathbf{w}$ *be a path which ends at* (d,n). *Then,*

$$m(\mathbf{w}) \geq m(c(n,d)).$$

Moreover, it is an equality if and only if

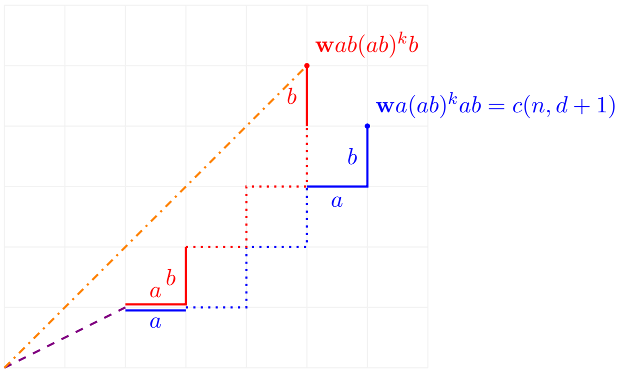
$$\mathbf{w} = c(n,d) = (a\mathbf{p}_{n/d}b)^\eta \text{ or } \begin{cases} \eta \geq 2 \\ \mathbf{w} = (a\mathbf{p}_{n/d}a)(b\mathbf{p}_{n/d}a)^{\eta-2}(b\mathbf{p}_{n/d}b). \end{cases}$$

Proof of the fixed sum conjecture

Proposition

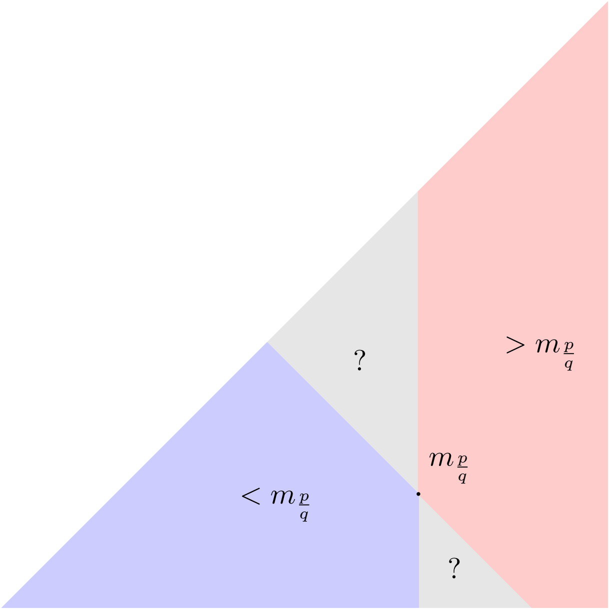
Let $(n, d) \in (\mathbb{N} \setminus \{0\})^2$ *such that* $d \geq n$. *We have* $m(c(n, d)) < m(c(n-1, d+1))$.

Proof of fixed sum conjectures



$$m(c(n+1, d)) = m(\mathrm{Lift}(\mathbf{w}ab(ab)^k b)) \leq m(\mathbf{w}ab(ab)^k b) < m(\mathbf{w}a(ab)^k ab) = m(c(n, d+1))$$

Conclusion



Thanks

Thank you for your attention.

References

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