Nonchalant Procedure

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Let X, Y, Z be words and let $U = XYZ$. Every word X, Y and Z is called **a factor** of a word U.

A word XX is called **a square** for every nonempty word X.

A word U is **square-free** if it does not contain a square as a factor.

`repetition` `recreation`

Let AB be a word and let x be a letter. The word AxB is called **an extension** of the word AB.

For example, all extensions of the word 12 over alphabet $\{1, 2\}$ are

$$\underline{1}12, \underline{2}12, 1\underline{2}2, 12\underline{1}.$$

The square-free word is *extremal* if every of its extensions is not square-free.

Extremal words over alphabet $\{1\}$:

$$1$$

Extremal words over alphabet $\{1, 2\}$:

$$121,\ 212$$

Extremal words over alphabet $\{1, 2, 3\}$:

$$1231213231232123121323123,\ \ldots$$

Theorem (Grytczuk, Kordulewski, Niewiadomski, 2019)

There exist infinitely many extremal square-free words over the alphabet $\{1, 2, 3\}$.

Theorem (Mol, Rampersad, 2020)

There exists a square-free extremal word of length n over the alphabet $\{1, 2, 3\}$ *if and only if*

$$n \in \{25, 41, 48, 50, 63, 71, 72, 77, 79, 81, 83, 84, 85\} \cup \\ \cup \{m : m \geqslant 87\}.$$

Nonchalant procedure

- The first element of the sequence is the word 1.
- Every following element is the square-free extension of the previous element by inserting at the rightmost possible position the least possible letter.

Originally, the procedure was considered over the alphabet $\{1, 2, 3\}$.

$$1 \longrightarrow 1\underline{2} \longrightarrow 12\underline{1} \longrightarrow 121\underline{3} \longrightarrow 1213\underline{1} \longrightarrow 12131\underline{2} \longrightarrow$$

$$\longrightarrow 121312\underline{1} \longrightarrow 121312\underline{3}1 \longrightarrow 12131231\underline{3} \longrightarrow \dots$$

Conjecture (Grytczuk, Kordulewski, Niewiadomski, 2019)

The sequence of the nonchalant words over the alphabet $\{1, 2, 3\}$ *is infinite.*

Some facts about the nonchalant procedure:

- The procedure does not stop in 10 000 iterations.
- The lenghts of the longest suffixes for the single entry: 20 (twice; 4436, 6628), 15 (once; 143).
- The number of internal positions on which the nonchalant words are square-free extendable is "increasing" in such manner that if new maximal value n is obtained, then it never goes below $n-2$ (1 000 iterations, max = 75).

Some results "inspired" by the nonchalant procedure (Grytczuk, Kordulewski, P., 2021):

- There exists a square-free word over the alphabet $\{1, 2, 3, 4\}$ which is extendable neither at the last nor the penultimate position.
- For every pair of natural numbers t and k such that $k \geqslant 4$ and $1 \leqslant t < k$ there exists a square-free word W over the alphabet $\{1, 2, \ldots, \mathtt{k}\}$, which is not extendable at the first t internal positions.
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  k: $(k+1)2^{(k-1)(k-2)} - 1$

Natural modification of the nonchalant procedure:

- The first element of the sequence is the word 1.
- Every following element is the square-free extension of the previous element by inserting the least possible letter at the rightmost possible position.

$$1 \to 1\underline{2} \to 12\underline{1} \to 121\underline{3} \to 1213\underline{1} \to 12131\underline{2} \to$$

$$\to 121312\underline{1} \to 121312\underline{3}1 \to 12131231\underline{3} \to \ldots \to$$

$$\to 121312313212312131231321\underline{2} \to \ldots$$

Nonchalant procedure:

$$121312313212312131231321 2 \rightarrow$$

$$\rightarrow 1213123132123121312313231 2$$

Modified nonchalant procedure:

$$1213123132123121312313212 \rightarrow$$

$$\rightarrow 1213123132123121321231321 2$$

A prime number p is **an extremal prime number**, if every nontrivial extension of p is a composite number.

Extremal primes not greater than 10^7:

$$369\,293,\ \ 3\,823\,867,\ \ 5\,364\,431,\ \ 5\,409\,259,$$

$$7\,904\,521,\ \ 8\,309\,369,\ \ 9\,387\,527,\ \ 9\,510\,341.$$

For example:

$$83090369 = 113 \cdot 457 \cdot 1609,\ \ 82309369 = 2729 \cdot 30161$$

$$18309369 = 3 \cdot 13 \cdot 19 \cdot 24709,\ \ 83093699 = 13 \cdot 211 \cdot 30293$$

Conjecture

The set of all extremal primes is infinite.

Similar concepts:

- *digitally delicate primes*, e.g. 294 001
  infinitely many: Erdős, 1978
  positive proportion: Tao, 2011
- *composites that remain composite after inserting a single digit* (r.p. to 10), e.g. 25011
  infinitely many: Filaseta, Kozek, Nicol, Selfridge, 2010
- *minimal primes*,
  Shallit, 2001, full set:

$$\{2, 3, 5, 7, 11, 19, 41, 61, 89, 409, 449, 499, 881, 991, 6469, 6949, 9001, 9049, 9649, 9949, 60649, 666649, 946669, 60000049, 66000049, 66600049\}$$

The nonchalant procedure for the primes

- The first element of the sequence is the number 2.
- Every following prime number is the non-trivial extension of the previous prime number by inserting at the rightmost possible position the least possible digit.

$$2,\ 2\underline{3},\ 23\underline{3},\ 233\underline{3},\ 2333\underline{3},\ 2333\underline{2}3,\ 233323\underline{1},\ 233323\underline{0}1,$$

$$23332300\underline{1},\ 233323001\underline{9},\ 23332\underline{0}30019,\ldots$$

Conjecture (Grytczuk, 2021)

The sequence of the nonchalant primes is infinite.

Conjecture (P., 2021)

The sequence of the nonchalant primes is finite.

base	2	3	4	5	6	7	8	9
length	36	14	25	114	>1270	36	>500	924

Thank you for your attention.