On Minimal Critical Exponent of Balanced Sequences

Ľubomíra Dvořáková
joint work with Daniela Opočenská, Edita Pelantová and Arseny M. Shur

FNSPE Czech Technical University in Prague

January, 10, 2022

Rational powers

Definition

Let $e \in \mathbb{Q}$. A word z is an e-th power of a word u if z is a prefix of $u^\omega = uuuuu\ldots$ and $e = \frac{|z|}{|u|}$. We write $z = u^e$.

Example

$$\begin{aligned}
\texttt{abbabb} &= (\texttt{abb})^2\\
\texttt{abbcabbcabbc} &= (\texttt{abbc})^3\\
\texttt{abbabbab} &= (\texttt{abb})^{8/3}\\
\texttt{starosta} &= (\texttt{staro})^{8/5}
\end{aligned}$$

Critical exponent

Definition

Let $\mathbf{u}$ *be a sequence. The* critical exponent *of* $\mathbf{u}$
$E(\mathbf{u}) = \sup\{e \in \mathbb{Q} : u^e$ *is a non-empty factor of* $\mathbf{u}\}$.

Example

The Thue–Morse sequence $\mathbf{u}_{TM} = \mathtt{abbabaabbaababbabaab}\ldots$
$\mathbf{u}_{TM} = \varphi(\mathbf{u}_{TM})$, where $\varphi : \mathtt{a} \to \mathtt{ab}, \quad \mathtt{b} \to \mathtt{ba}$
$\mathbf{u}_{TM}$ does not contain overlaps: $xwxwx$, where w is a factor and x is a letter. Hence $E(\mathbf{u}_{TM}) = 2$.

Minimal critical exponent

Dejean's theorem (conjecture), 1972 – 2011:
(proven by Dejean, Pansiot, Moulin Ollagnier, Mohammad-Noori, Carpi, Currie, Rampersad, Rao)
the least critical exponent of sequences over an alphabet of size d:

- 2 for $d = 2$;
- $7/4$ for $d = 3$;
- $7/5$ for $d = 4$;
- $\frac{d}{d-1}$ for $d \geq 5$.

Conjecture for balanced sequences

- **Rampersad, Shallit, Vandomme**, 2019:
  the least critical exponent of balanced sequences over an alphabet of size d equals $\frac{d-2}{d-3}$ for $d \geq 5$
  - proven for $5 \leq d \leq 8$
- **Dolce, D., Pelantová**, 2021:
  - proven for $9 \leq d \leq 10$
  - disproven: new bound $\frac{d-1}{d-2}$ for $11 \leq d \leq 12$

Conjecture for balanced sequences

- **Rampersad, Shallit, Vandomme**, 2019:
  the least critical exponent of balanced sequences over an alphabet of size d equals $\frac{d-2}{d-3}$ for $d \geq 5$
  - proven for $5 \leq d \leq 8$
- **Dolce, D., Pelantová**, 2021:
  - proven for $9 \leq d \leq 10$
  - disproven: new bound $\frac{d-1}{d-2}$ for $11 \leq d \leq 12$
- **D., Opočenská, Pelantová, Shur**, 2021:
  - new bound $\frac{d-1}{d-2}$ for $d \geq 12, d$ even
  - new conjecture: $\frac{d-1}{d-2}$ for $d \geq 11$

Conjecture for balanced sequences

- **Rampersad, Shallit, Vandomme**, 2019:
  the least critical exponent of balanced sequences over an alphabet of size d equals $\frac{d-2}{d-3}$ for $d \geq 5$
  - proven for $5 \leq d \leq 8$
- **Dolce, D., Pelantová**, 2021:
  - proven for $9 \leq d \leq 10$
  - disproven: new bound $\frac{d-1}{d-2}$ for $11 \leq d \leq 12$
- **D., Opočenská, Pelantová, Shur**, 2021:
  - new bound $\frac{d-1}{d-2}$ for $d \geq 12, d$ even
  - new conjecture: $\frac{d-1}{d-2}$ for $d \geq 11$

$$\frac{d}{d-1} < \frac{d-1}{d-2} < \frac{d-2}{d-3}$$

Conjecture for balanced sequences

- **Rampersad, Shallit, Vandomme**, 2019:
  the least critical exponent of balanced sequences over an alphabet of size d equals $\frac{d-2}{d-3}$ for $d \geq 5$
  - proven for $5 \leq d \leq 8$
- **Dolce, D., Pelantová**, 2021:
  - proven for $9 \leq d \leq 10$
  - disproven: new bound $\frac{d-1}{d-2}$ for $11 \leq d \leq 12$
- **D., Opočenská, Pelantová, Shur**, 2021:
  - new bound $\frac{d-1}{d-2}$ for $d \geq 12, d$ even
  - new conjecture: $\frac{d-1}{d-2}$ for $d \geq 11$

$$\frac{d}{d-1} < \frac{d-1}{d-2} < \frac{d-2}{d-3}$$

Program

Program

Definitions CoW

- *bispecial factor* of $\mathbf{u}$
- *Parikh vector* $\vec{V}(u)$ of a factor u of $\mathbf{u}$

Example

$\mathbf{u}_F = \mathtt{abaababaabaababaa}\ldots$

$\mathbf{u}_F = \varphi(\mathbf{u}_F)$, *where* $\varphi : \mathtt{a} \to \mathtt{ab}, \quad \mathtt{b} \to \mathtt{a}$

aba *is a bispecial factor since* aaba, baba *and* abab, abaa *are factors of* $\mathbf{u}_F$

$\vec{V}(\mathtt{aba}) = \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)$

Definitions CoW

- *return word* to a factor u of $\mathbf{u}$
- *derived sequence* $\mathbf{d}_{\mathbf{u}}(u)$ to a factor u of $\mathbf{u}$

Example

$\mathbf{u}_F = \mathtt{aba}\underline{\mathtt{aba}}\mathtt{baaba}\mathtt{ab}\underline{\mathtt{aba}}\mathtt{a}\ldots$

$r = \mathtt{aba}$ *and* $s = \mathtt{ab}$ *are return words to the factor* $u = \mathtt{aba}$

$\mathbf{d}_{\mathbf{u}_F}(u) = \mathtt{abaababaabaababa}\cdots = rsrrsr\ldots$

(Asymptotic) critical exponent

- *critical exponent* of $\mathbf{u}$
  $E(\mathbf{u}) = \sup\{e \in \mathbb{Q} : u^e \text{ is a non-empty factor of } \mathbf{u}\}$
- *asymptotic critical exponent* of $\mathbf{u}$
  $E^*(\mathbf{u}) = \lim_{n\to\infty} \sup\{e \in \mathbb{Q} : u^e \text{ is a factor of } \mathbf{u} \text{ and } |u| \geq n\}$

Evidently, $E^*(\mathbf{u}) \leq E(\mathbf{u})$.

Proposition (D., Medková, Pelantová, 2020)

Let $\mathbf{u}$ be a uniformly recurrent aperiodic sequence. Let w_n be the n-th bispecial of $\mathbf{u}$ and v_n a shortest return word to w_n. Then $E(\mathbf{u}) = 1 + \sup\{\frac{|w_n|}{|v_n|} : n \in \mathbb{N}\}$ *and* $E^*(\mathbf{u}) = 1 + \limsup_{n\to\infty} \frac{|w_n|}{|v_n|}$.

(Asymptotic) critical exponent

- *critical exponent* of $\mathbf{u}$
  $E(\mathbf{u}) = \sup\{e \in \mathbb{Q} : u^e \text{ is a non-empty factor of } \mathbf{u}\}$
- *asymptotic critical exponent* of $\mathbf{u}$
  $E^*(\mathbf{u}) = \lim_{n\to\infty} \sup\{e \in \mathbb{Q} : u^e \text{ is a factor of } \mathbf{u} \text{ and } |u| \geq n\}$

Evidently, $E^*(\mathbf{u}) \leq E(\mathbf{u})$.

Proposition (D., Medková, Pelantová, 2020)

Let $\mathbf{u}$ be a uniformly recurrent aperiodic sequence. Let w_n be the n-th bispecial of $\mathbf{u}$ and v_n a shortest return word to w_n. Then $E(\mathbf{u}) = 1 + \sup\{\frac{|w_n|}{|v_n|} : n \in \mathbb{N}\}$ *and* $E^*(\mathbf{u}) = 1 + \limsup_{n\to\infty} \frac{|w_n|}{|v_n|}$.

Example

$|w_n| = F_{n+2} + F_{n+1} - 2$ and $|v_n| = F_{n+1}$ with $F_0 = 0, F_1 = 1$
$E(\mathbf{u}_F) = 2 + \tau = 2 + \frac{1+\sqrt{5}}{2} = E^*(\mathbf{u}_F)$ – minimal for Sturmian

(Asymptotic) critical exponent

- *critical exponent* of $\mathbf{u}$
  $E(\mathbf{u}) = \sup\{e \in \mathbb{Q} : u^e \text{ is a non-empty factor of } \mathbf{u}\}$
- *asymptotic critical exponent* of $\mathbf{u}$
  $E^*(\mathbf{u}) = \lim_{n\to\infty} \sup\{e \in \mathbb{Q} : u^e \text{ is a factor of } \mathbf{u} \text{ and } |u| \geq n\}$

Evidently, $E^*(\mathbf{u}) \leq E(\mathbf{u})$.

Proposition (D., Medková, Pelantová, 2020)

Let $\mathbf{u}$ be a uniformly recurrent aperiodic sequence. Let w_n be the n-th bispecial of $\mathbf{u}$ and v_n a shortest return word to w_n. Then
$E(\mathbf{u}) = 1 + \sup\{\frac{|w_n|}{|v_n|} : n \in \mathbb{N}\}$ *and* $E^*(\mathbf{u}) = 1 + \limsup_{n\to\infty} \frac{|w_n|}{|v_n|}$.

Example

$|w_n| = F_{n+2} + F_{n+1} - 2$ and $|v_n| = F_{n+1}$ with $F_0 = 0, F_1 = 1$
$E(\mathbf{u}_F) = 2 + \tau = 2 + \frac{1+\sqrt{5}}{2} = E^*(\mathbf{u}_F)$ – minimal for Sturmian

Balanced sequences

Definition

$\mathbf{u}$ *over* $\mathcal{A}$ *balanced if* $|u| = |v| \Rightarrow |u|_a - |v|_a \leq 1$ *for all* $a \in \mathcal{A}$

Theorem (Graham 1973, Hubert 2000)

$\mathbf{v}$ *recurrent aperiodic is balanced iff* $\mathbf{v}$ *obtained from a Sturmian sequence* $\mathbf{u}$ *over* $\{\mathtt{a}, \mathtt{b}\}$ *by replacing*

- $\mathtt{a}$ *with a constant gap sequence* $\mathbf{y}$ *over* $\mathcal{A}$,
- $\mathtt{b}$ *with a constant gap sequence* $\mathbf{y}'$ *over* $\mathcal{B}$,

where $\mathcal{A}$ *and* $\mathcal{B}$ *disjoint. We write* $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$.

Balanced sequences

Definition

$\mathbf{u}$ *over* $\mathcal{A}$ *balanced if* $|u| = |v| \Rightarrow |u|_a - |v|_a \leq 1$ *for all* $a \in \mathcal{A}$

Theorem (Graham 1973, Hubert 2000)

$\mathbf{v}$ *recurrent aperiodic is balanced iff* $\mathbf{v}$ *obtained from a Sturmian sequence* $\mathbf{u}$ *over* $\{\mathtt{a}, \mathtt{b}\}$ *by replacing*

- $\mathtt{a}$ *with a constant gap sequence* $\mathbf{y}$ *over* $\mathcal{A}$,
- $\mathtt{b}$ *with a constant gap sequence* $\mathbf{y}'$ *over* $\mathcal{B}$,

where $\mathcal{A}$ *and* $\mathcal{B}$ *disjoint. We write* $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$.

Example

$\mathbf{v} = \mathrm{colour}(\mathbf{u}_F, \mathbf{y}, \mathbf{y}')$, where $\mathbf{y} = (0102)^\omega$ and $\mathbf{y}' = (34)^\omega$

$$\begin{aligned} \mathbf{u}_F &= \mathtt{abaababaabaabab}\ldots \\ \mathbf{v} &= 031042301402304\ldots \qquad \pi(423) = \mathtt{bab} \end{aligned}$$

Balanced sequences

Definition

$\mathbf{u}$ *over* $\mathcal{A}$ *balanced if* $|u| = |v| \Rightarrow |u|_a - |v|_a \leq 1$ *for all* $a \in \mathcal{A}$

Theorem (Graham 1973, Hubert 2000)

$\mathbf{v}$ *recurrent aperiodic is balanced iff* $\mathbf{v}$ *obtained from a Sturmian sequence* $\mathbf{u}$ *over* $\{\mathtt{a}, \mathtt{b}\}$ *by replacing*

- $\mathtt{a}$ *with a constant gap sequence* $\mathbf{y}$ *over* $\mathcal{A}$,
- $\mathtt{b}$ *with a constant gap sequence* $\mathbf{y}'$ *over* $\mathcal{B}$,

where $\mathcal{A}$ *and* $\mathcal{B}$ *disjoint. We write* $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$.

Example

$\mathbf{v} = \mathrm{colour}(\mathbf{u}_F, \mathbf{y}, \mathbf{y}')$, where $\mathbf{y} = (0102)^\omega$ and $\mathbf{y}' = (34)^\omega$

$$\mathbf{u}_F = \mathtt{abaababaabaabab}\ldots$$

$$\mathbf{v} = 031042301402304\ldots \qquad \pi(423) = \mathtt{bab}$$

Program

Motivation

- **Rampersad**, May 2020, One World Numeration Seminar: *Ostrowski numeration and repetitions in words*
  - question by Cassaigne: “What about the asymptotic version?”
- **D., Medková, Pelantová**, 2020: *Complementary symmetric Rote sequences: the critical exponent and the recurrence function*

Motivation

- **Rampersad**, May 2020, One World Numeration Seminar: *Ostrowski numeration and repetitions in words*
  - question by Cassaigne: “What about the asymptotic version?”
- **D., Medková, Pelantová**, 2020: *Complementary symmetric Rote sequences: the critical exponent and the recurrence function*

Complementary symmetric Rote sequences

- **Rote sequence**: binary sequence with complexity $2n$
- **complementary symmetric sequence**: language closed under exchange of 0 and 1

$S(\mathbf{v}) = S(v_0 v_1 v_2 v_3 v_4 \ldots) = (v_0 + v_1 \mod 2)(v_1 + v_2 \mod 2)(v_2 + v_3 \mod 2)\ldots$

$S(\mathbf{v}_F) = S(00111001110001\ldots) = 0100101001001\ldots$

Complementary symmetric Rote sequences

- **Rote sequence**: binary sequence with complexity $2n$
- **complementary symmetric sequence**: language closed under exchange of 0 and 1

  $S(\mathbf{v})=S(v_0v_1v_2v_3v_4\dots)=(v_0+v_1 \mod 2)(v_1+v_2 \mod 2)(v_2+v_3 \mod 2)\dots$

  $S(\mathbf{v}_F)=S(00111001110001\dots)=0100101001001\dots$

Theorem (Rote 1994)

Let $\mathbf{u}$ and $\mathbf{v}$ be two binary sequences such that $\mathbf{u}=S(\mathbf{v})$. Then $\mathbf{v}$ is a CS Rote sequence iff $\mathbf{u}$ is a Sturmian sequence.

Complementary symmetric Rote sequences

- **Rote sequence**: binary sequence with complexity $2n$
- **complementary symmetric sequence**: language closed under exchange of 0 and 1

  $S(\mathbf{v}) = S(v_0 v_1 v_2 v_3 v_4 \dots) = (v_0 + v_1 \mod 2)(v_1 + v_2 \mod 2)(v_2 + v_3 \mod 2)\dots$

  $S(\mathbf{v}_F) = S(00111001110001\dots) = 0100101001001\dots$

Theorem (Rote 1994)

Let $\mathbf{u}$ and $\mathbf{v}$ be two binary sequences such that $\mathbf{u} = S(\mathbf{v})$. Then $\mathbf{v}$ is a CS Rote sequence iff $\mathbf{u}$ is a Sturmian sequence.

- $E^*(\mathbf{v}) = E^*(\hat{\mathbf{v}})$, where $\mathbf{v}$ is a CS Rote sequence associated with $\mathbf{u}$ and $\hat{\mathbf{v}} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ by $\mathbf{y} = 0^\omega$ and $\mathbf{y}' = (12)^\omega$.
- The minimal critical exponent of ternary balanced sequences is the same as the minimal critical exponent of CS Rote sequences, and it equals $2 + \frac{1}{\sqrt{2}}$.

Complementary symmetric Rote sequences

- **Rote sequence**: binary sequence with complexity $2n$
- **complementary symmetric sequence**: language closed under exchange of 0 and 1

  $S(\mathbf{v}) = S(v_0 v_1 v_2 v_3 v_4 \ldots) = (v_0 + v_1 \mod 2)(v_1 + v_2 \mod 2)(v_2 + v_3 \mod 2)\ldots$

  $S(\mathbf{v}_F) = S(00111001110001\ldots) = 0100101001001\ldots$

Theorem (Rote 1994)

Let $\mathbf{u}$ and $\mathbf{v}$ be two binary sequences such that $\mathbf{u} = S(\mathbf{v})$. Then $\mathbf{v}$ is a CS Rote sequence iff $\mathbf{u}$ is a Sturmian sequence.

- $E^*(\mathbf{v}) = E^*(\hat{\mathbf{v}})$, where $\mathbf{v}$ is a CS Rote sequence associated with $\mathbf{u}$ and $\hat{\mathbf{v}} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ by $\mathbf{y} = 0^\omega$ and $\mathbf{y}' = (12)^\omega$.
- The minimal critical exponent of ternary balanced sequences is the same as the minimal critical exponent of CS Rote sequences, and it equals $2 + \frac{1}{\sqrt{2}}$.

Computation of asymptotic critical exponent

Recall $E^*(\mathbf{v}) = 1 + \limsup_{n\to\infty} \frac{|w_n|}{|v_n|}$

Proposition (Dolce, D., Pelantová, 2020)

Let $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$*. For a sufficiently long bispecial* w *in* $\mathbf{v}$ *its projection* $u = \pi(w)$ *is a bispecial in* $\mathbf{u}$*. The shortest return word to* w *is of length* $\min\{k|r| + \ell|s|\}$*, where*

1. $k\vec{V}(r) + \ell\vec{V}(s) = \begin{pmatrix} 0 \bmod \mathrm{Per}(\mathbf{y}) \\ 0 \bmod \mathrm{Per}(\mathbf{y}') \end{pmatrix}$;
2. $\binom{\ell}{k}$ *is the Parikh vector of a factor in* $\mathbf{d_u}(u)$.

Program implemented by Daniela Opočenská:
Input: slope α quadratic irrational, $\mathrm{Per}(\mathbf{y})$, $\mathrm{Per}(\mathbf{y}')$
Output: $E^*(\mathbf{v})$, where $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$

Completion of table

d	α	$\mathbf{y}$	$\mathbf{y}'$	$E(\mathbf{v})$	$E^*(\mathbf{v})$
3	$[0,\overline{2}]$	$(01)^\omega$	2^ω	$2+\frac{1}{\sqrt{2}}$	$2+\frac{1}{\sqrt{2}}$
4	$[0,2,\overline{1}]$	$(01)^\omega$	$(23)^\omega$	$1+\frac{1+\sqrt{5}}{4}$	$1+\frac{1+\sqrt{5}}{4}$
5	$[0,\overline{2}]$	$(0102)^\omega$	$(34)^\omega$	$\frac{3}{2}$	$\frac{3}{2}$
6	$[0,1,2,1,1,\overline{1,1,1,2}]$	0^ω	$(123415321435)^\omega$	$\frac{4}{3}$	$\frac{4}{3}$
7	$[0,1,1,3,\overline{1,2,1}]$	$(01)^\omega$	$(234526432546)^\omega$	$\frac{5}{4}$	$\frac{5}{4}$
8	$[0,1,3,1,\overline{2}]$	$(01)^\omega$	$(234526732546237526432576)^\omega$	$\frac{6}{5}=1.2$	$\frac{12+3\sqrt{2}}{14}\doteq 1.16$
9	$[0,1,2,3,\overline{2}]$	$(01)^\omega$	$(234567284365274863254768)^\omega$	$\frac{7}{6}$?	$1+\frac{2\sqrt{2}-1}{14}\doteq 1.13$
10	$[0,1,4,2,\overline{3}]$	$(01)^\omega$	$(234567284963254768294365274869)^\omega$	$\frac{8}{7}$?	$1+\frac{\sqrt{13}}{26}\doteq 1.139$

Table: Baranwal, Rampersad, Shallit, Vandomme: d-ary balanced sequences with the least critical exponent.

Computation of critical exponent

Recall $E(\mathbf{v}) = 1 + \sup\{\frac{|w_n|}{|v_n|} : n \in \mathbb{N}\}$

Our result: $E(\mathbf{v}) = \max\left\{E^*(\mathbf{v}), 1 + \frac{|w_i|}{|v_i|}\right\}$ for finitely many i

Computation of critical exponent

Recall $E(\mathbf{v}) = 1 + \sup\{\frac{|w_n|}{|v_n|} : n \in \mathbb{N}\}$

Our result: $E(\mathbf{v}) = \max\left\{E^*(\mathbf{v}), 1 + \frac{|w_i|}{|v_i|}\right\}$ for finitely many i

Proposition (Dolce, D., Pelantová, 2020)

Let $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$. *Let* w *be a bispecial factor of* $\mathbf{v}$ *with projection* $u = \pi(w)$ *in* $\mathbf{u}$. *The shortest return word to* w *is of length* $\min\{k|r| + \ell|s|\}$, *where*

1. $k\vec{V}(r) + \ell\vec{V}(s) = \binom{0 \bmod n}{0 \bmod n'}$, *where* $n \in \mathrm{gap}(\mathbf{y}, |u|_{\mathtt{a}})$ *and* $n' \in \mathrm{gap}(\mathbf{y}', |u|_{\mathtt{b}})$;
2. $\binom{\ell}{k}$ *is the Parikh vector of a factor in* $\mathbf{d_u}(u)$.

Example

For $\mathbf{y} = (0102)^\omega$, we have $\mathrm{gap}(\mathbf{y}, 1) = \{2, 4\}$ and $\mathrm{gap}(\mathbf{y}, m) = \{4\}$ for $m \geq 2$.

Computation of critical exponent

Recall $E(\mathbf{v}) = 1 + \sup\{\frac{|w_n|}{|v_n|} : n \in \mathbb{N}\}$

Our result: $E(\mathbf{v}) = \max\left\{E^*(\mathbf{v}), 1 + \frac{|w_i|}{|v_i|}\right\}$ for finitely many i

Proposition (Dolce, D., Pelantová, 2020)

Let $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$. *Let* w *be a bispecial factor of* $\mathbf{v}$ *with projection* $u = \pi(w)$ *in* $\mathbf{u}$. *The shortest return word to* w *is of length* $\min\{k|r| + \ell|s|\}$, *where*

1. $k\vec{V}(r) + \ell\vec{V}(s) = \begin{pmatrix} 0 \bmod n \\ 0 \bmod n' \end{pmatrix}$, *where* $n \in \mathrm{gap}(\mathbf{y}, |u|_{\mathtt{a}})$ *and* $n' \in \mathrm{gap}(\mathbf{y}', |u|_{\mathtt{b}})$;
2. $\begin{pmatrix} \ell \\ k \end{pmatrix}$ *is the Parikh vector of a factor in* $\mathbf{d}_{\mathbf{u}}(u)$.

Example

For $\mathbf{y} = (0102)^\omega$, we have $\mathrm{gap}(\mathbf{y}, 1) = \{2, 4\}$ and $\mathrm{gap}(\mathbf{y}, m) = \{4\}$ for $m \geq 2$.

Computation of critical exponent

Program implemented by Opočenská:
Input: slope α quadratic irrational, $\mathbf{y}$, $\mathbf{y}'$
Output: $E(\mathbf{v})$, where $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$

Description of algorithms for computation of (asymptotic) critical exponent published:
Dolce, D., Pelantová: *On balanced sequences and their critical exponent*, arXiv 2021

Computation of critical exponent

Program implemented by Opočenská:
Input: slope α quadratic irrational, $\mathbf{y}$, $\mathbf{y}'$
Output: $E(\mathbf{v})$, where $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$

Description of algorithms for computation of (asymptotic) critical exponent published:
Dolce, D., Pelantová: *On balanced sequences and their critical exponent*, arXiv 2021

Completion of table – continued

d	α	$\mathbf{y}$	$\mathbf{y}'$	$E(\mathbf{v})$	$E^*(\mathbf{v})$
3	$[0, \overline{2}]$	$(01)^\omega$	2^ω	$2 + \frac{1}{\sqrt{2}}$	$2 + \frac{1}{\sqrt{2}}$
4	$[0, 2, \overline{1}]$	$(01)^\omega$	$(23)^\omega$	$1 + \frac{1+\sqrt{5}}{4}$	$1 + \frac{1+\sqrt{5}}{4}$
5	$[0, \overline{2}]$	$(0102)^\omega$	$(34)^\omega$	$\frac{3}{2}$	$\frac{3}{2}$
6	$[0, 1, 2, 1, 1, \overline{1, 1, 1, 2}]$	0^ω	$(123415321435)^\omega$	$\frac{4}{3}$	$\frac{4}{3}$
7	$[0, 1, 1, 3, \overline{1, 2, 1}]$	$(01)^\omega$	$(234526432546)^\omega$	$\frac{5}{4}$	$\frac{5}{4}$
8	$[0, 1, 3, 1, \overline{2}]$	$(01)^\omega$	$(234526732546237526432576)^\omega$	$\frac{6}{5} = 1.2$	$\frac{12+3\sqrt{2}}{14} \doteq 1.16$
9	$[0, 1, 2, 3, \overline{2}]$	$(01)^\omega$	$(234567284365274863254768)^\omega$	$\frac{7}{6} \doteq 1.167$	$1 + \frac{2\sqrt{2}-1}{14} \doteq 1.13$
10	$[0, 1, 4, 2, \overline{3}]$	$(01)^\omega$	$(2345672849632547682943652748 69)^\omega$	$\frac{8}{7} \doteq 1.14$	$1 + \frac{\sqrt{13}}{26} \doteq 1.139$
11	$[0, 1, 5, 1, \overline{1, 1, 1, 2}]$	$(01)^\omega$	$(\mathtt{234567892A436587294A638527496A832547698A})^\omega$	$\frac{10}{9} \doteq 1.11$	$\frac{415+5\sqrt{105}}{424} \doteq 1.0996$
12	$[0, 1, 1, 3, \overline{2}]$	$(012345)^\omega$	$(\mathtt{6789AB})^\omega$	$\frac{11}{10} = 1.1$	$\frac{8-\sqrt{2}}{6} \doteq 1.0976$

Table: Baranwal, Rampersad, Shallit, Vandomme: d-ary balanced sequences with the least critical exponent.

Towards a new conjecture

- **Dvořáková**, September 2021, WORDS 2021:
  Critical exponent of balanced sequences
  - conjecture $\frac{d-2}{d-3}$ refuted by examples over 11 and 12 letters
  - new conjecture: $\frac{d-1}{d-2}$ or $\frac{d}{d-1}$?
  - **Shur**: the lower bound $\frac{d-1}{d-2}$
- **D., Opočenská, Pelantová, Shur**, 2021:
  On minimal critical exponent of balanced sequences, arXiv 2021
  - new conjecture $\frac{d-1}{d-2}$ for $d \geq 11$
  - proven for even $d \geq 12$

Lower bounds

Observation 7 *If* $4 \in \text{gap}(\mathbf{y}, 1)$ *and* $a_1 = 1$ *and* $a_2 \geq 2$*, then* $E(\mathbf{v}) \geq \frac{10}{9}$.

Proof. Use Proposition 5 with $u = \mathtt{a}$ and $f = \mathtt{ababab^2ab}$.

Observation 8 *If* $6 \in \text{gap}(\mathbf{y}', 2)$ *and* $a_1 = 1, a_2 = 2$ *and* $a_3 \geq 2$*, then* $E(\mathbf{v}) \geq \frac{6}{5}$.

Proof. We use Proposition 5 with $u = \mathtt{b}^2$ and $f = \mathtt{b^2abab^2aba}$.

Observation 9 *If* $7 \in \text{gap}(\mathbf{y}', 2)$ *and* $a_1 \geq 2$*, then* $E(\mathbf{v}) \geq \frac{6}{5}$.

Proof. We apply Proposition 5 with $u = \mathtt{b}^2$ and the following f:

- If $a_1 \geq 4$, then $f = \mathtt{b^5ab^2}$.
- If $a_1 = 3$, then $f = \mathtt{b^3ab^4a}$.
- If $a_1 = 2$ and $a_2 \geq 2$, then $f = \mathtt{b^2ab^2ab^2ab}$.
- If $a_1 = 2$ and $a_2 = 1$, then $f = \mathtt{b^2ab^3ab^2a}$.

Observation 10 *If* $8 \in \text{gap}(\mathbf{y}', 2)$ *and* $a_1 \geq 2$*, then* $E(\mathbf{v}) \geq \frac{7}{6}$.

Proof. We apply Proposition 5 with $u = \mathtt{b}^2$ and the following f:

- If $a_1 \geq 5$, then $f = \mathtt{b^5ab^3}$.
- If $a_1 \in \{3, 4\}$, then $f = \mathtt{b^3ab^4ab}$.
- If $a_1 = 2$ and $a_2 \geq 3$, then $f = \mathtt{b^2ab^2ab^2ab^2a}$.
- If $a_1 = 2$ and $a_2 \in \{1, 2\}$, then $f = \mathtt{b^2ab^3ab^2ab}$.

Completion of table – continued

d	α	$\mathbf{y}$	$\mathbf{y}'$	$E(\mathbf{v})$	$E^*(\mathbf{v})$
3	$[0, \overline{2}]$	$(01)^\omega$	2^ω	$2 + \frac{1}{\sqrt{2}}$	$2 + \frac{1}{\sqrt{2}}$
4	$[0, 2, \overline{1}]$	$(01)^\omega$	$(23)^\omega$	$1 + \frac{1+\sqrt{5}}{4}$	$1 + \frac{1+\sqrt{5}}{4}$
5	$[0, \overline{2}]$	$(0102)^\omega$	$(34)^\omega$	$\frac{3}{2}$	$\frac{3}{2}$
6	$[0, 1, 2, 1, 1, \overline{1, 1, 1, 2}]$	0^ω	$(123415321435)^\omega$	$\frac{4}{3}$	$\frac{4}{3}$
7	$[0, 1, 1, 3, \overline{1, 2, 1}]$	$(01)^\omega$	$(234526432546)^\omega$	$\frac{5}{4}$	$\frac{5}{4}$
8	$[0, 1, 3, 1, \overline{2}]$	$(01)^\omega$	$(23452673254623752643257 6)^\omega$	$\frac{6}{5} = 1.2$	$\frac{12+3\sqrt{2}}{14} \doteq 1.16$
9	$[0, 1, 2, 3, \overline{2}]$	$(01)^\omega$	$(2345672843652748632547 68)^\omega$	$\frac{7}{6} \doteq 1.167$	$1 + \frac{2\sqrt{2}-1}{14} \doteq 1.13$
10	$[0, 1, 4, 2, \overline{3}]$	$(01)^\omega$	$(234567284963254768294365274869)^\omega$	$\frac{8}{7} \doteq 1.14$	$1 + \frac{\sqrt{13}}{26} \doteq 1.139$
11	$[0, 1, 5, 1, \overline{1, 1, 1, 2}]$	$(01)^\omega$	$(234567892\mathtt{A}436587294\mathtt{A}638527496\mathtt{A}832547698\mathtt{A})^\omega$	$\frac{10}{9} \doteq 1.11$	$\frac{415+5\sqrt{105}}{424} \doteq 1.0996$
12	$[0, 1, 1, 3, \overline{2}]$	$(012345)^\omega$	$(6789\mathtt{AB})^\omega$	$\frac{11}{10} = 1.1$	$\frac{8-\sqrt{2}}{6} \doteq 1.0976$
$d \geq 14$ even	$[0, 1, 1, \lfloor d/4 \rfloor, \overline{1}]$	$(12 \ldots \mathtt{d}/2)^\omega$	$(1'2' \ldots \mathtt{d}/2')^\omega$	$\frac{d-1}{d-2}$	$1 + \frac{2}{d\tau^{N-1}}$, where $\tau^{N+1} < d/2 < \tau^{N+2}$

Table: Baranwal, Rampersad, Shallit, Vandomme: d-ary balanced sequences with the least critical exponent.

Open problems

- Proof of conjecture $\frac{d-1}{d-2}$ for odd $d \geq 13$
  - using our computer program done for $13 \leq d \leq 33$
- Minimal asymptotic critical exponent of d-ary balanced sequences
  - Is there an analogy of Dejean's conjecture for E^*?

Open problems

- Proof of conjecture $\frac{d-1}{d-2}$ for odd $d \geq 13$
  - using our computer program done for $13 \leq d \leq 33$
- Minimal asymptotic critical exponent of d-ary balanced sequences
  - Is there an analogy of Dejean's conjecture for E^*?
  - Is there a better lower bound than $E^*(\mathbf{v}) \geq 1 + \frac{1}{\mathrm{Per}(\mathbf{y})\mathrm{Per}(\mathbf{y}')}$?

Open problems

- Proof of conjecture $\frac{d-1}{d-2}$ for odd $d \geq 13$
  - using our computer program done for $13 \leq d \leq 33$
- Minimal asymptotic critical exponent of d-ary balanced sequences
  - Is there an analogy of Dejean's conjecture for E^*?
  - Is there a better lower bound than $E^*(\mathbf{v}) \geq 1 + \frac{1}{\mathrm{Per}(\mathbf{y})\mathrm{Per}(\mathbf{y}')}$?
- What is the minimal critical exponent of a d-ary 2-balanced sequence?

Open problems

- Proof of conjecture $\frac{d-1}{d-2}$ for odd $d \geq 13$
  - using our computer program done for $13 \leq d \leq 33$
- Minimal asymptotic critical exponent of d-ary balanced sequences
  - Is there an analogy of Dejean's conjecture for E^*?
  - Is there a better lower bound than $E^*(\mathbf{v}) \geq 1 + \frac{1}{\mathrm{Per}(\mathbf{y})\mathrm{Per}(\mathbf{y}')}$?
- What is the minimal critical exponent of a d-ary 2-balanced sequence?
- What is the critical exponent of colourings of non-Sturmian sequences by constant gap sequences?

Open problems

- Proof of conjecture $\frac{d-1}{d-2}$ for odd $d \geq 13$
  - using our computer program done for $13 \leq d \leq 33$
- Minimal asymptotic critical exponent of d-ary balanced sequences
  - Is there an analogy of Dejean's conjecture for E^*?
  - Is there a better lower bound than $E^*(\mathbf{v}) \geq 1 + \frac{1}{\mathrm{Per}(\mathbf{y})\mathrm{Per}(\mathbf{y}')}$?
- What is the minimal critical exponent of a d-ary 2-balanced sequence?
- What is the critical exponent of colourings of non-Sturmian sequences by constant gap sequences?

Thank you for attention