Some decidable properties of linear dynamical systems

24.1.2022, One World Combinatorics on Words Seminar

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Joint work with C. Baier, F. Funke, S. Jantsch, T. Karimov, E. Lefaucheux, F. Luca, J. Ouaknine, D. Purser, A. Varonka, J. Worrell

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MAX PLANCK INSTITUTE FOR SOFTWARE SYSTEMS

UNIVERSITY OF OXFORD

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```
do {
    p = r = b + (2 * PTHRESH);
    if (r >= t) p = r = t;      /* too short to care about */
    else {
        while (((cmp(aTHX_ *(p-1), *p) > 0) == sense) &&
               ((p -= 2) > q)) {}
        if (p <= q) {
            /* b through r is a (long) run.
            ** Extend it as far as possible. */
            p = q = r;
            while (((p += 2) < t) &&
                   ((cmp(aTHX_ *(p-1), *p) > 0) == sense)) q = p;
            r = p = q + 2;      /* no simple pairs, no after-run */
        }
    }
    if (q > b) {                /* run of greater than 2 at b */
        gptr *savep = p;
        p = q += 2;
        /* pick up singleton, if possible */
        if ((p == t) &&
            ((t + 1) == last) &&
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            savep = r = p = q = last;
        p2 = NEXT(p2) = p2 + (p - b); ++runs;
        if (sense)
            while (b < --p) {
                const gptr c = *b;
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            }
        p = savep;
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    while (q < p) {             /* simple pairs */
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} while (b < t);
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}
return runs;
```

Modeling software systems

Does this program work correctly?

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- Decomposing into simple blocks
- Modeling blocks separately
- Correctness of each part ensures correctness of the program

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Does this while loop terminate/halt?

Loops as Dynamical Systems

Algorithm

$x := 1$
$y := 2$
$z := 3$
while $(x, y, z) \neq (2, 1, 0)$ **do**
  $x' := 2x + y$
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$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} := \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

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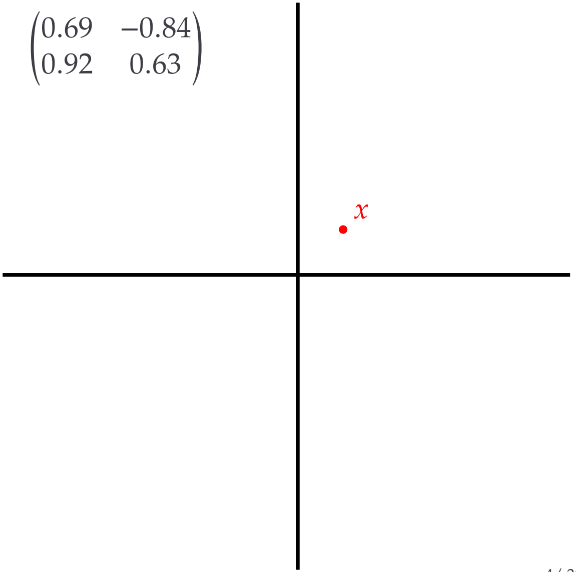
$$\text{Loop halts} \iff \exists n : M^n \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

Linear Dynamical Systems (LDS)

- Given a starting point $x \in \mathbb{R}^d$.
- A linear update map $M \in \mathbb{R}^{d\times d}$
- Defines an orbit $\langle x, Mx, M^2x, M^3x, \ldots \rangle$

$$\begin{pmatrix} 0.69 & -0.84 \\ 0.92 & 0.63 \end{pmatrix}$$



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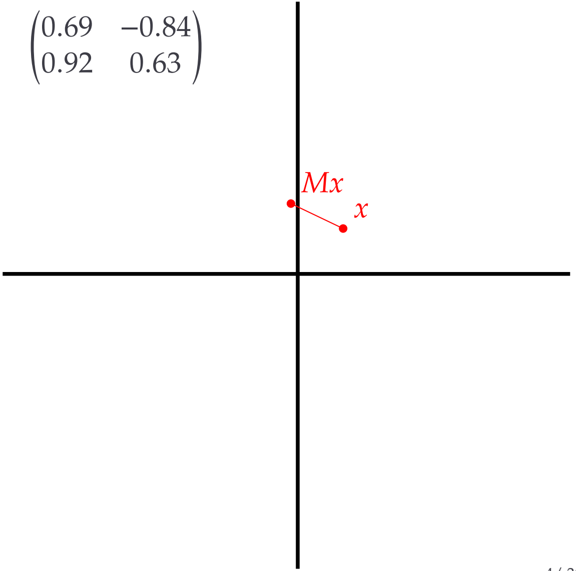

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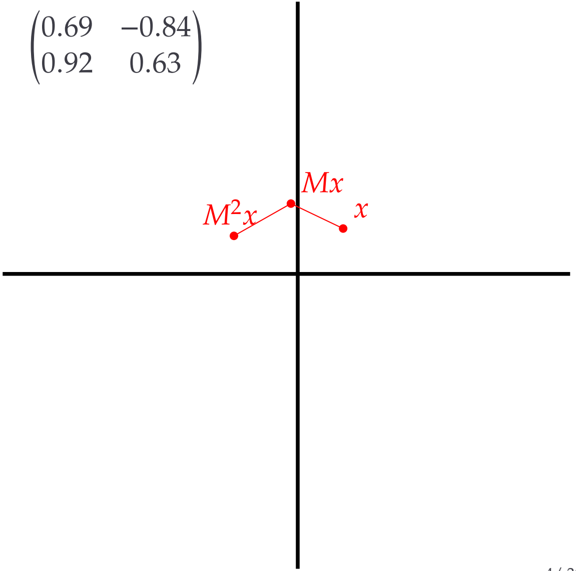

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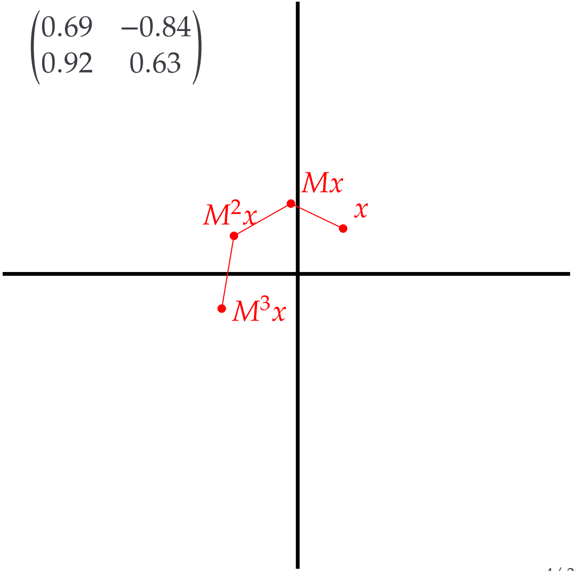

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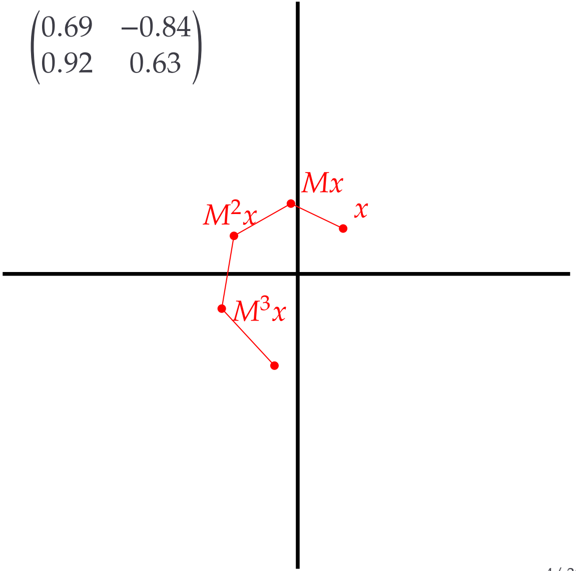

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Orbit Problem

Does the orbit contain the target?
$\exists n : M^n x = y$?

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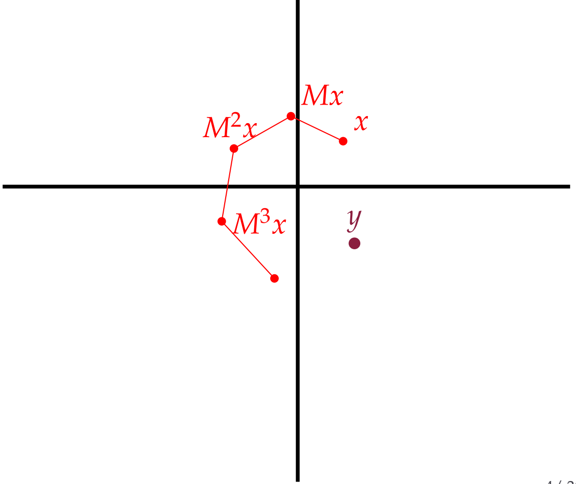

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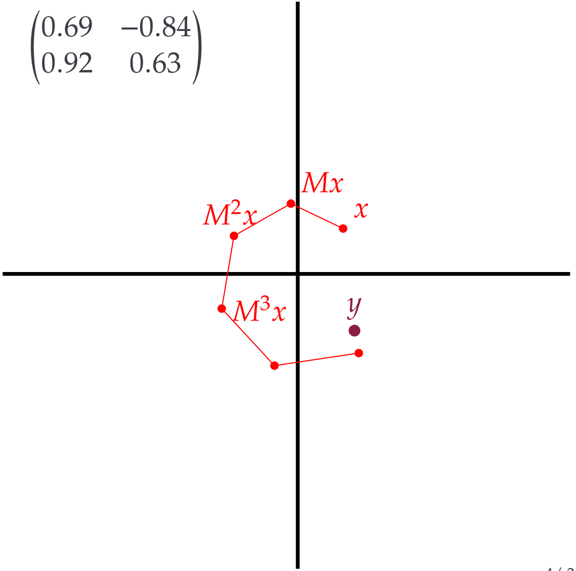

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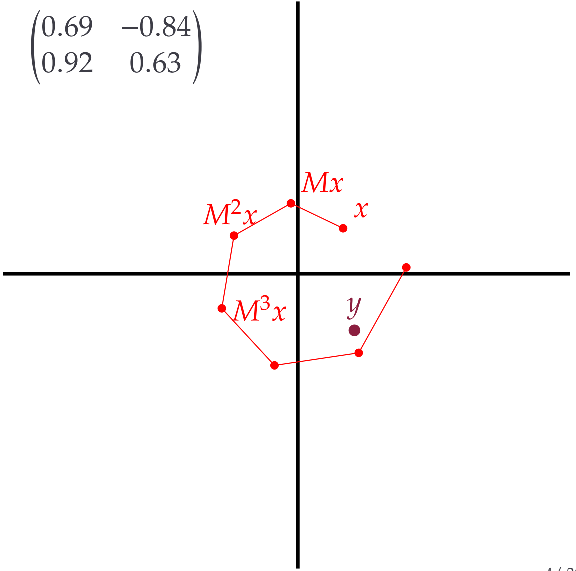

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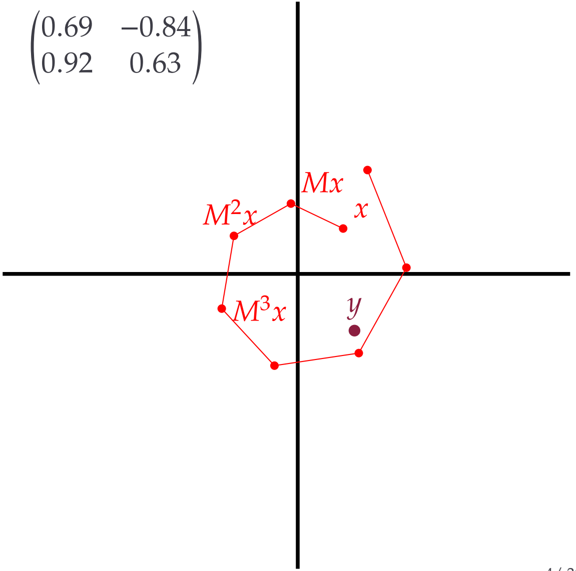

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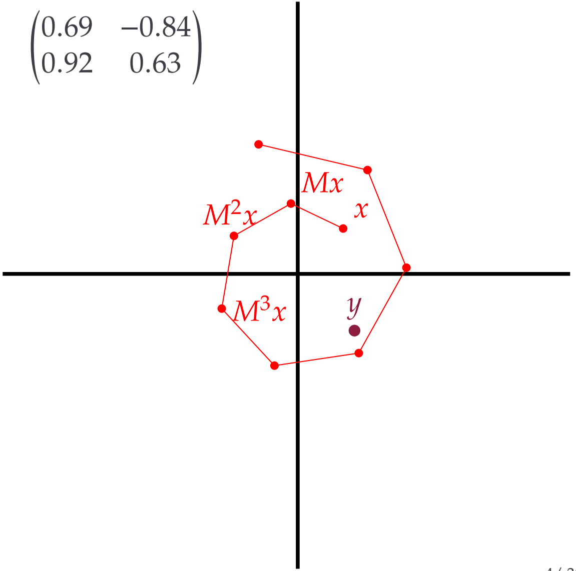

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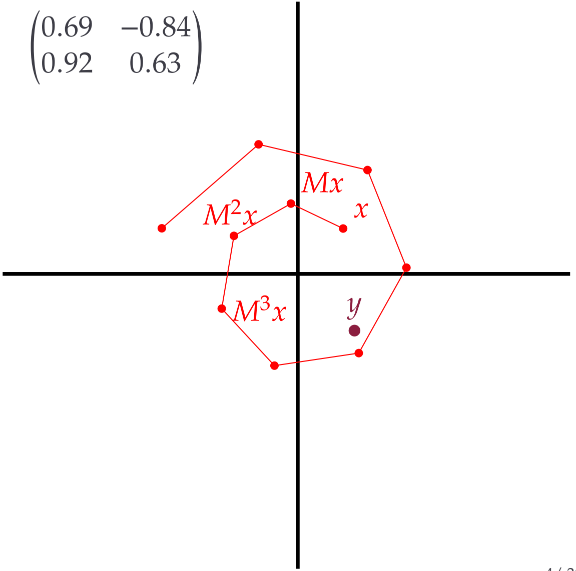

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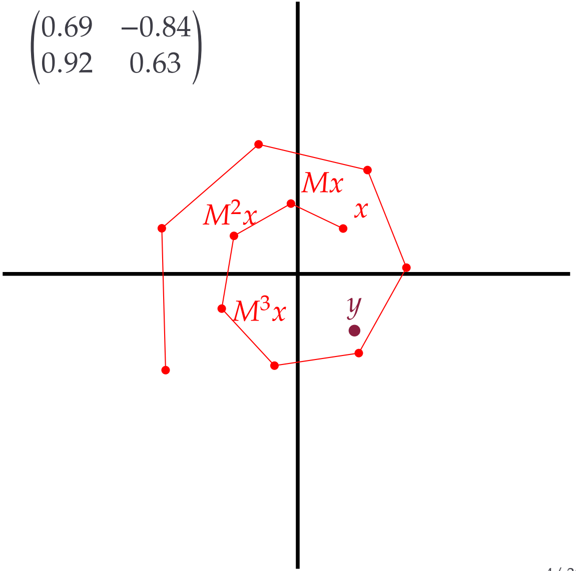

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Theorem (Kannan–Lipton'86)

Orbit Problem is decidable for x, y, M with ***algebraic entries****.*
In $\mathtt{PTIME}$ for x, y, M over $\mathbb{Q}$.

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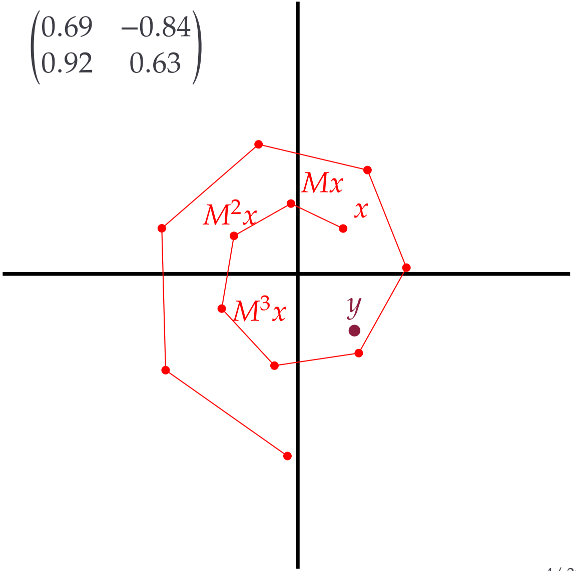

Reachability in LDS

Halting of Linear Dynamical Systems

Skolem's problem

Given x, u, M, decide

$$\exists n \in \mathbb{N} \text{ such that } u^\top M^n x = 0?$$

"Reaching a hyperplane."

- "Open" since the 1930s
- Decidable for instances with dimension ≤ 4.
  - Real algebraic entries.
- Open for systems with dimension ≥ 5
- Equivalent to zeros of linear recurrence sequences.

Halting of Linear Dynamical Systems

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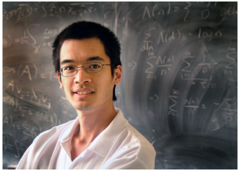
"A mathematical embarrassment ..."

Richard Lipton

"Reaching a hyperplane."



"It is faintly outrageous that this problem is still open; it is saying that we do not know how to decide the Halting Problem even for 'linear' automata!"

Terence Tao

So what can we do?

Reachability problems

Given $x \in \mathbb{Q}^d$, $M \in \mathbb{Q}^{d \times d}$, and a set $T \subseteq \mathbb{R}^d$, decide: $\exists n \in \mathbb{N}$ such that $M^n x \in T$?

Linear loops: T encodes the **halting condition**.

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Example (Skolem's problem; Not known to be decidable)

T is a hyperplane defined by $u \in \mathbb{Q}^d$: $\{z \in \mathbb{R}^d : (z, u) = 0\}$

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Example (Positivity problem; Provably mathematically hard![1])

T is a halfspace defined by $u \in \mathbb{Q}^d$: $\{z \in \mathbb{R}^d : (z, u) \geq 0\}$.

[1] Decidability entails major breakthroughs in field of Diophantine approximation.
Ouaknine, Worrell: Positivity problems for low-order linear recurrence sequences. (SODA2014)

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Have to restrict T!

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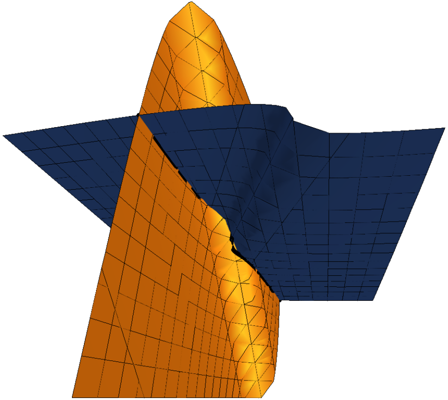
Restricting the targets: dimension

Intrinsic dimension

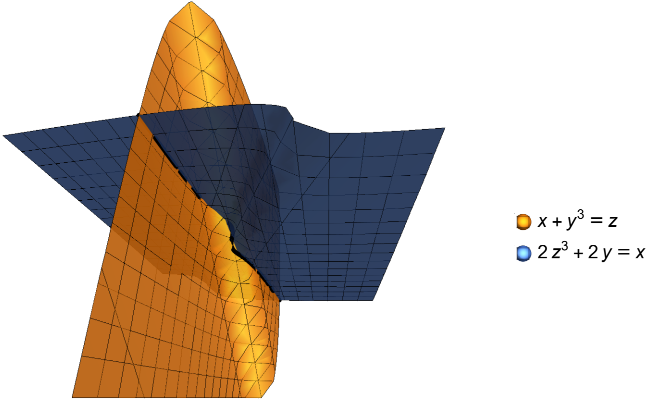


- $x + y^3 = z$
- $2z^3 + 2y = x$

In 3D ambient space:

- Surfaces: intrinsic dimension 2

Intrinsic dimension



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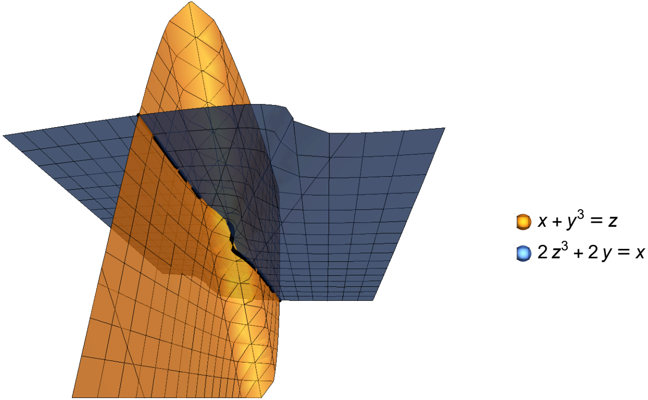

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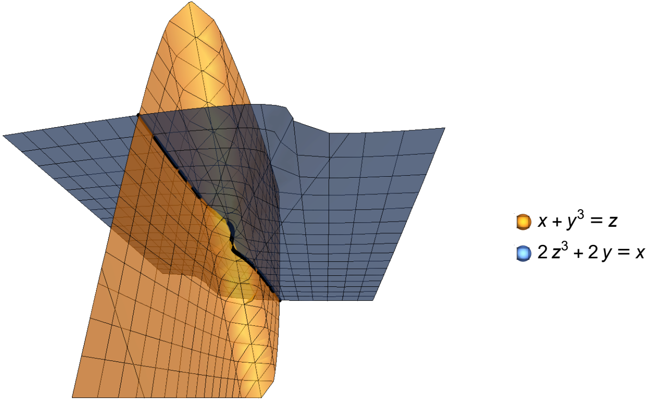

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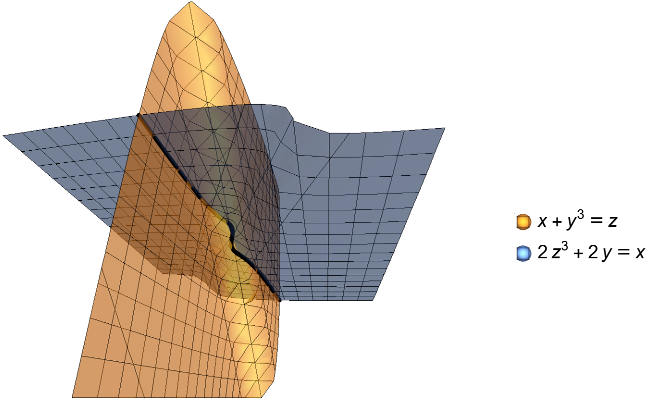

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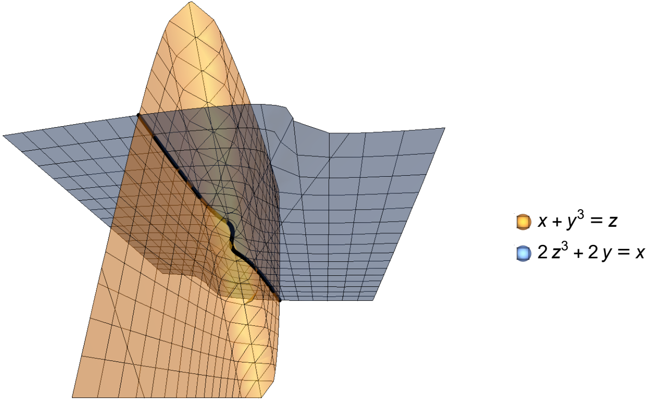

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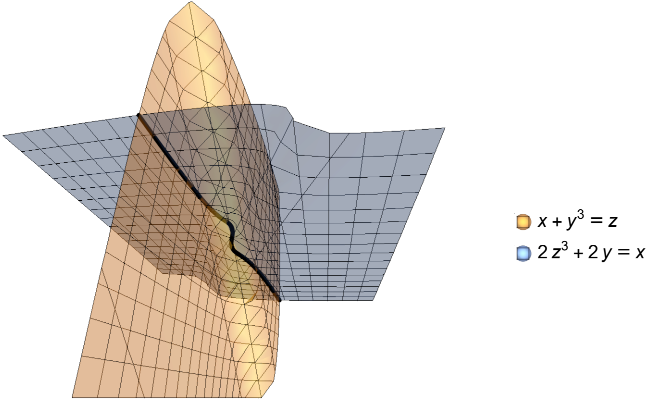

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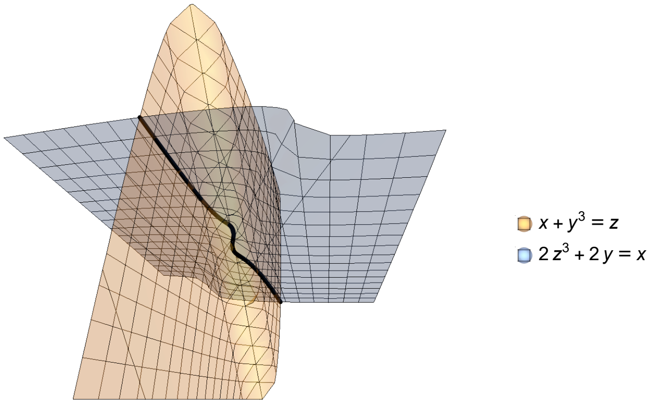

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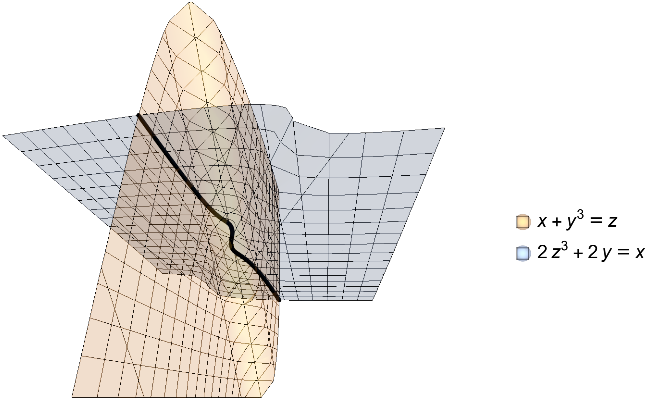
- Curves: intrinsic dimension 1

Restricting the targets: dimension

Intrinsic dimension



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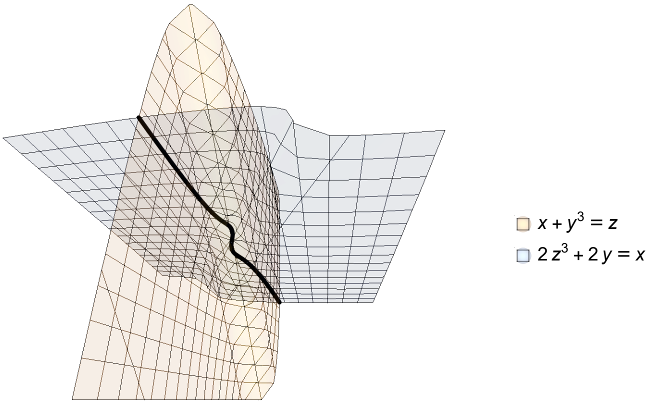

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Examples: Our targets

Example

Consider the semialgebraic sets in ambient dimension 4:

$$T_1 = \{(t, u, v, w)\colon t + u + v - w = 0 \wedge (t^3 = u^2 \vee w \geq 3t^2 + u)\}$$

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Example

2D polytopes are semialgebraic sets contained in 3D subspaces.

Our results

Theorem

The reachability problem is decidable when T is a semialgebraic set

- *of* ***intrinsic dimension at most*** *1; or*
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Theorem (Skolem-hardness)

Skolem's problem at dimension 5 reduces to reachability of T a dimension-2 set.

Theorem (Chonev et al.[2])

- *Skolem's problem at dimension 5 reduces to reachability of a 3D polytope in $\mathbb{R}^4$.*
- *Reachability of a 4D polytope in $\mathbb{R}^4$ is* ***mathematically hard****.*

[2] Chonev, Ouaknine, Worrell: The Polyhedron-Hitting Problem. (SODA2015)

Semialgebraic targets of dimension 1

Proposition

Given x, M, and semialgebraic T of dimension 1, ***it is decidable whether*** *$\exists n \in \mathbb{N}$ such that $M^n x \in T$.*

Proof sketch.

We show that $\exists$ computable $N \in \mathbb{N}$ such that **if** such n exists, **then** $\exists n \leq N$.

1. Transform the system into Jordan normal form ($M = S^{-1}JS$).
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2. (Most cases) Solving instances of the Skolem problem.
3. (Corner cases) The system can be seen as a product of *arc hitting models / codings of rotations*.

$\square$

1. Transformation to Jordan normal form

- $M = S^{-1}JS$, where S is an invertible matrix and

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_s \end{pmatrix} \qquad \text{with} \quad J_i = \begin{pmatrix} \lambda_i & 1 & & \cdots \\ & \lambda_i & 1 & \cdots \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}$$

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- λ_i an eigenvalue of M.

$M^n x \in T \iff \underline{J^n y \in T'}$.

Interesting cases: Assume λ_i and λ_j are **multiplicatively independent**;
$\lambda_i^a \lambda_j^b = 1 \implies a = b = 0$.

Project $J^n y$ to the coordinates $(\lambda_i^n y_{i,s_i}, \lambda_j^n y_{j,s_j})$.

T' is complex semialgebraic of dimension 1;
There exists a bivariate polynomial P such that: $J^n y \in T'$ implies

$$P(\lambda_i^n y_{i,s_i}, \lambda_j^n y_{j,s_j}) = 0$$

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We get lucky: use known technology à la Shorey et al.[3] and Vereshchagin[4].

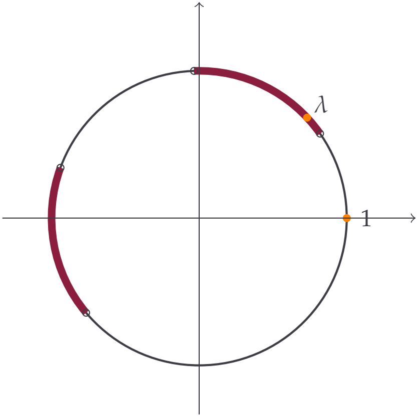
- (p-adic) Baker's theorem on linear forms in logarithms.

[3] Shorey, Tijdeman, Mignotte: *The distance between terms of an algebraic recurrence sequence*, (1984)
[4] Vereshchagin: *Occurrence of zero in a linear recursive sequence* (1985)

4. Corner cases; codings of rotations

Given $M = \lambda$ an algebraic number of modulus 1, $x = 1$, and open arcs $I_1, \ldots, I_k$ of unit circle in the complex plain (with algebraic endpoints)

Decide if there exists $n \geq 0$ such that $\lambda^n \in \bigcup_{j=1}^{k} I_j$. (Almost trivial!)



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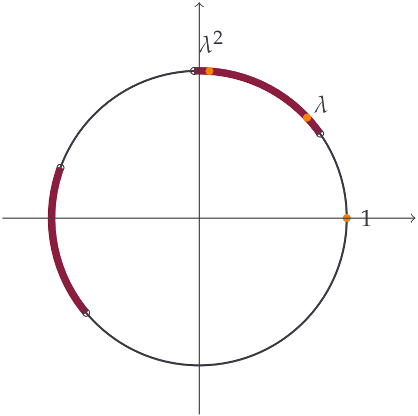

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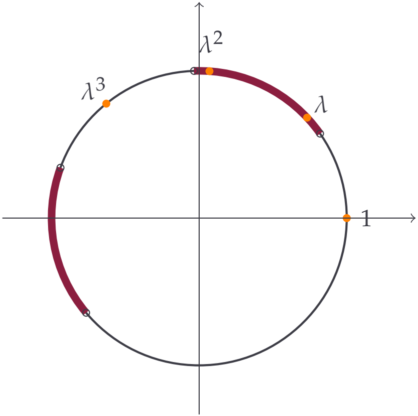

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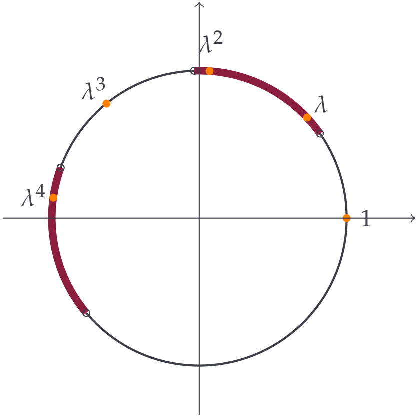

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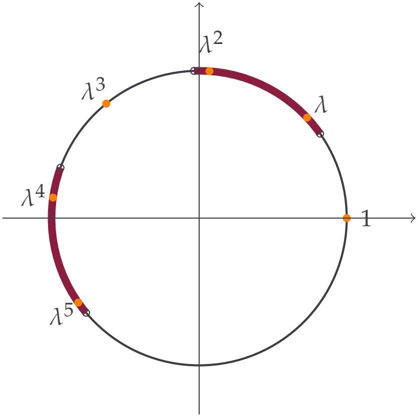

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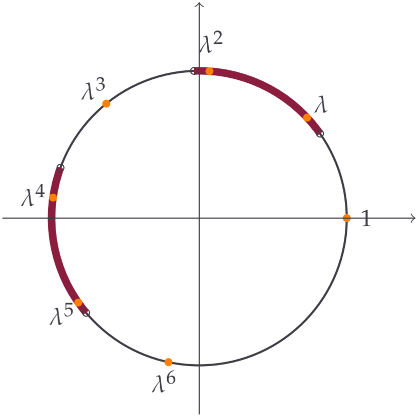

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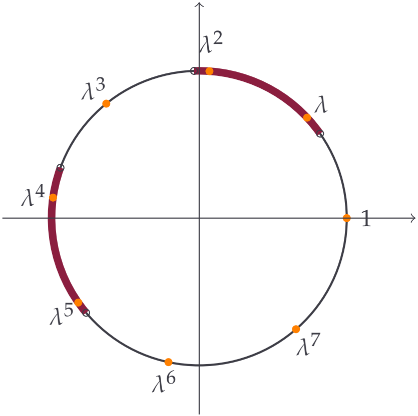

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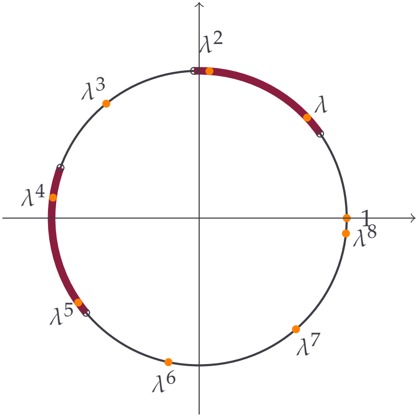

Beyond reachability

Beyond reachability

```
real t,te,time;
assume(te=14 and 16<=t and t<=17);
while true {
  time := 0; - timer measuring duration in each mode

  while (t<=22) { - heating mode
    t := 15/16*t-1/16*te+1; time++;
  }
  time := 0;
  while (t>=18){ - cooling mode
    t := 15/16*t-1/16*te; time++;
  }
}
```

Figure 2. A thermostat system, composed of two simple loops inside a outer loop.

from: Jeannet, Schrammel, Sankaranarayanan: *Abstract Acceleration of General Linear Loops*, POPL2014

Beyond reachability

Algorithm

$t := 1$
$u := -1$
$v := 2$
$w := 0$
while `true` **do**
  $t := 3t + 2u - 5w$
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Temporal analysis:
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Does the system satisfy the following LTL formula?

$$\mathbf{G}(P_1 \Rightarrow \mathbf{F}\neg P_2) \wedge \mathbf{F}(P_3 \vee \neg P_1)\,.$$

"whenever P_1 holds, then P_2 must eventually subsequently fail, and eventually either P_3 will hold or P_1 will fail"

Temporal properties of LDS

Let (M, x) be a LDS, let $T_1, \ldots, T_k$ be semialgebraic sets.

Definition (The characteristic word of the LDS (M, x) with respect to T_i)

$$\pi(M, x, T_1, \ldots, T_k) = a_0 a_1 a_2 \cdots \in \mathcal{P}(\{1, \ldots, k\})^{\mathbb{N}}$$

defined by

$$a_n = S \subseteq \{1, \ldots, k\}$$

if and only if $\forall i \in S \colon M^n x \in T_i$ and $\forall j \in \{1, \ldots, k\} \setminus S \colon M^n x \notin T_j$.

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Properties given by formulas from Monadic Second-Order Logic (capture ω-regular properties).

MSO: Monadic Second-Order Logic

MSO over the structure $(\mathbb{N}, <)$ and a finite collection of predicates $P_1, \ldots, P_k : \mathbb{N} \to \{\mathtt{true}, \mathtt{false}\}$; MSO on infinite words: Predicates P_i describes the indices with letter i.

The grammar of formulas:

$\psi := P(i)$	(where $P(i)$ is a predicate on position i of the word)
$\psi := \exists i \in \mathbb{N} : \psi \mid \forall i \in \mathbb{N} : \psi$	(first-order quantification)
$\psi := \exists X \subseteq \mathbb{N} : \psi \mid \forall X \subseteq \mathbb{N} : \psi$	(monadic second-order quantification)
$\psi := i \in X \mid i \notin X$	(subset membership testing)
$\psi := i < j \mid i = j$	(index comparison)
$\psi := \neg\psi \mid \psi \vee \psi \mid \psi \wedge \psi \mid \psi \Rightarrow \psi$	(standard logical operations)
$\psi := i = 0 \mid i = 1 \mid i = 2 \mid \ldots$	(fixed values)

MSO

For us, the predicate P_i corresponds to those n: $M^n x \in T_i$.

Example

Examples of MSO formulas for model checking LDS:

- Reachability of target T_i: $\exists n : P_i(n)$.
- Eventually trapped inside T_i: $\exists n \forall m : m > n \implies P_i(m)$.
- In target T_i at every odd position (O = the set of odd natural numbers): $\exists O \subseteq \mathbb{N} : 1 \in O \wedge \forall x \in O, \exists y, z : (y \notin O \wedge z \in O \wedge x < y < z \wedge \nexists t : x < t < y \vee y < t < z) \wedge \forall x : x \in O \implies P_i(x)$.
- Whenever T_i is visited T_j is visited some point later: $\forall n : P_i(n) \implies \exists m > n : P_j(m)$.
- Any linear temporal logic (LTL) formula over predicates $P_1, \ldots, P_k$.

The MSO model-checking problem for LDS

Theorem

Let (M, x) be a LDS, $T_1, \ldots, T_k \subseteq \mathbb{R}^d$ each T_i of which is a semialgebraic set

- *of* ***(semialgebraic) dimension at most*** *1; or*
- *which is* ***contained in a subspace of (linear) dimension*** *3.*

Then it is decidable whether $\pi(M, x, T_1, \ldots, T_k)$ satisfies a given MSO formula ψ.

Key notion: Effective almost periodicity

Definition (Almost periodic words)

An infinite word $w \in \Sigma^\omega$ is **almost periodic** if for every factor $u \in \Sigma^*$, there exists $p \in \mathbb{N}$ such that either:

- u does not occur in w after the position p,
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Theorem (Semënov'84)

Let w be an effectively almost periodic infinite word over alphabet Σ. The MSO theory over $(\mathbb{N}, <)$ extended with the unary predicates $\{P_a\}_{a \in \Sigma}$ remains decidable.

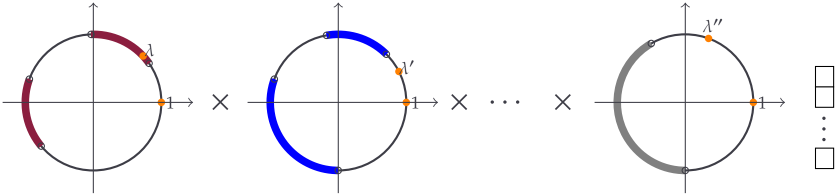
Proof sketch

Analysis from decidability of reachability shows:

Observation

Let (M, x) be a LDS and T be a semialgebraic set of dimension 1 or a semialgebraic set contained in a subspace of linear dimension 3. There exists a computable $\ell \in \mathbb{N}$ such that for all $0 \leq r < \ell$ the word $\pi(M^\ell, M^r x, T)$ is either

- *eventually constant; or*
- *a coding of a rotation up to a finite computable prefix.*



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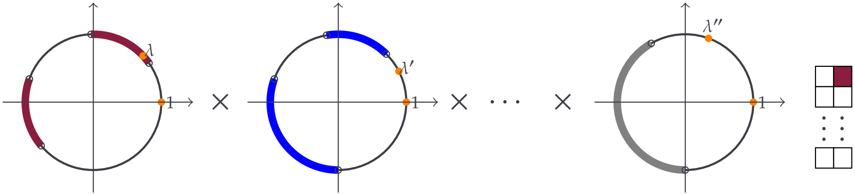

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Let (M, x) be a LDS and T be a semialgebraic set of dimension 1 or a semialgebraic set contained in a subspace of linear dimension 3. There exists a computable $\ell \in \mathbb{N}$ such that for all $0 \le r < \ell$ the word $\pi(M^\ell, M^r x, T)$ is either

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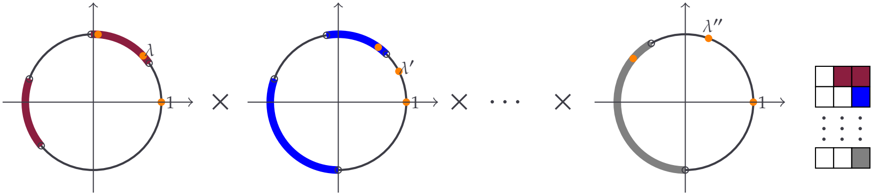

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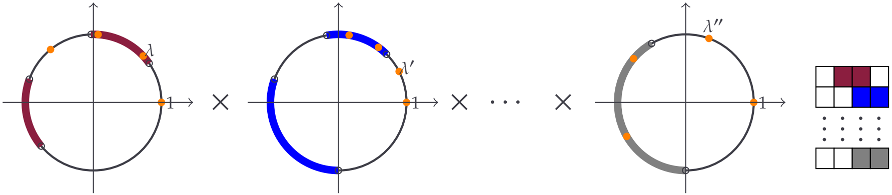

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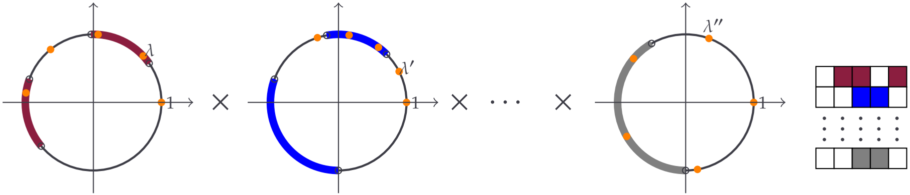

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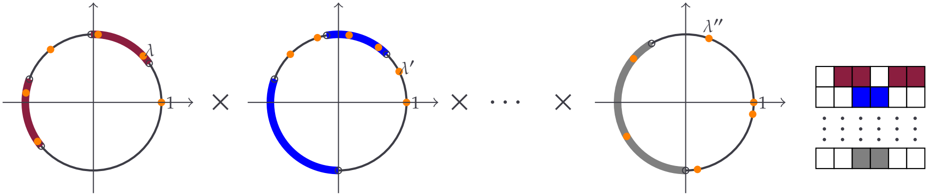

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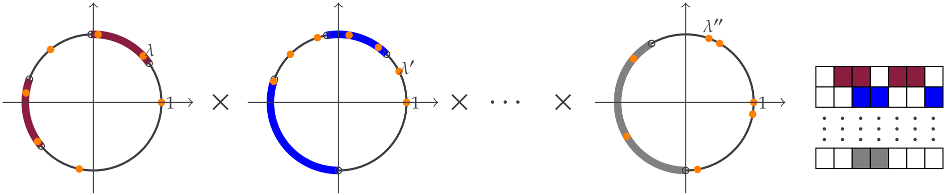

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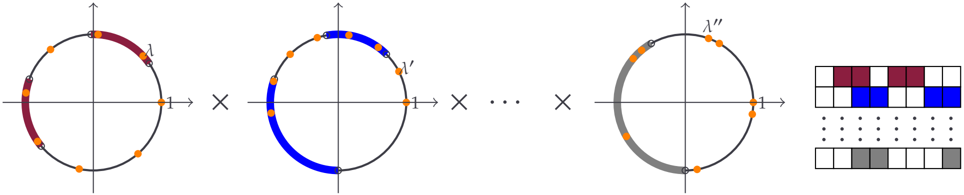

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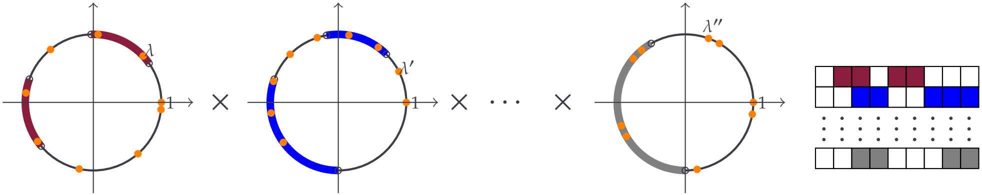

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Corollary

$\pi(M, x, T_0, \ldots, T_k)$ is an interleaving of codings of rotations on $\mathbb{T}^k$ (up to a finite prefix).

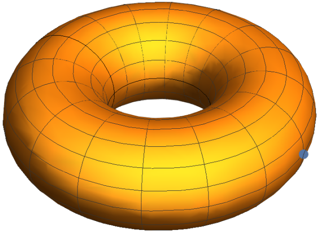
Coding of a rotation

Definition (Coding of a rotation on $\mathbb{T}^k$.)

Given a semialgebraic set $A_1 \subseteq \mathbb{T}^k$ and algebraic point $\vec{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \mathbb{T}^k$. Let $A_0 = \mathbb{T}^k \setminus A_1$.
Coding: $\mathbf{w} = a_0 a_1 \cdots \in \{0,1\}^{\mathbb{N}}$ where $a_n = 1 \Leftrightarrow \vec{\lambda}^n = (\lambda_1^n, \ldots, \lambda_k^n) \in A_1$.

Proof sketch (of effective almost periodicity)

The closure C of $\{(\lambda_1^n, \ldots, \lambda_k^n)\}_{n\in\mathbb{N}}$ is semialgebraic. Representation can be effectively computed![5]



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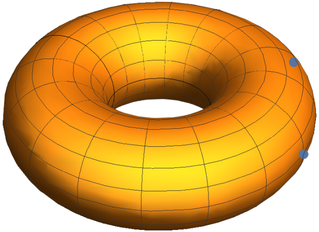

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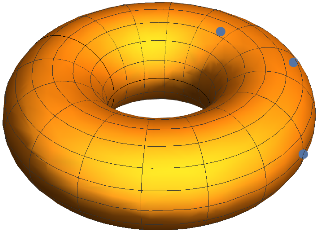

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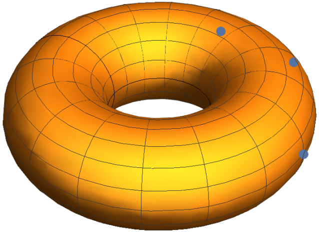

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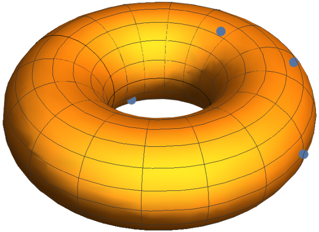

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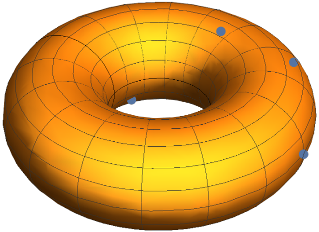

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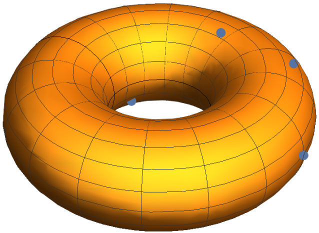

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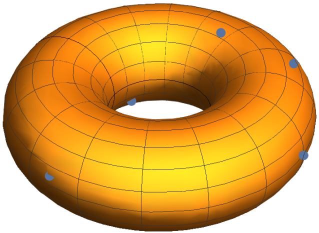

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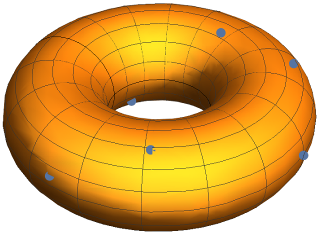

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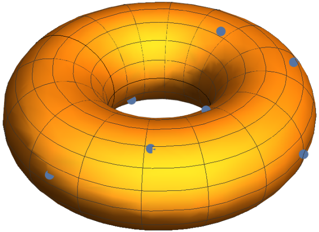

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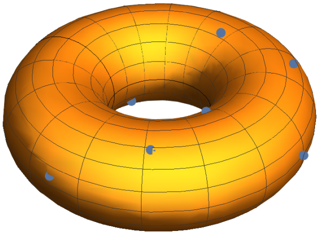

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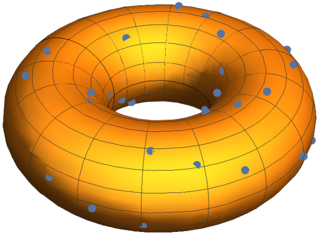

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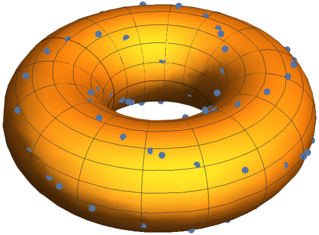

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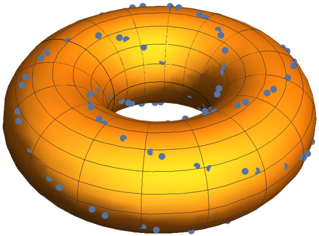

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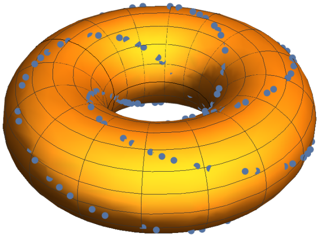

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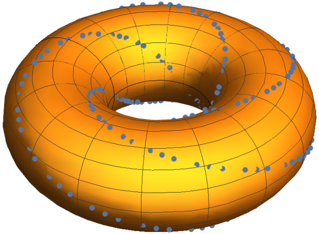

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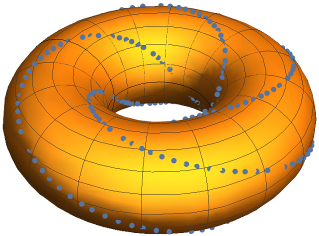

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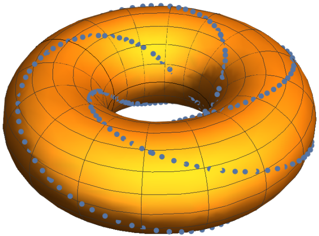

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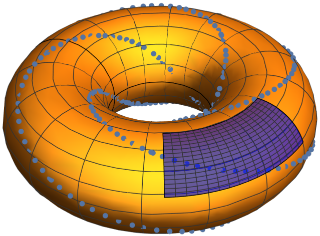

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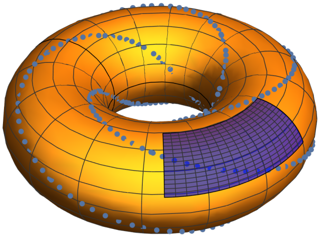

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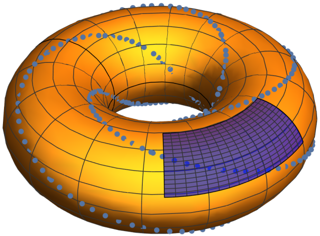

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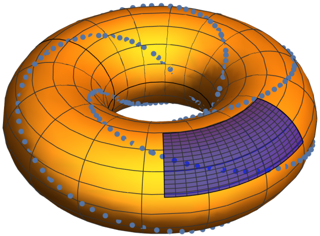

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Effectiveness by Tarski: First-order theory of the reals is decidable.

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Parametric reachability

Parametric Loops as Parametric Dynamical Systems

Algorithm

$\vec{x} := \vec{a}$

while $\vec{x} \neq \vec{y}$ **do**
  $\vec{x} := M\vec{x}$
end while

Theorem (Kannan–Lipton'86)

*Orbit Problem is decidable for x, y, M with **algebraic entries**.*
In $\mathtt{PTIME}$ for x, y, M over $\mathbb{Q}$.

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For fixed input $\vec{a} = (a_1, \ldots, a_k)$:
loop halts $\iff \exists n : M(\vec{a})^n \vec{x}(\vec{a}) = \vec{y}(\vec{a})$.

- Halting depends on input.
- Halting on different n for different inputs.

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Algorithm with **Input** $a_1, \ldots, a_k \in \mathbb{R}$

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Problem

Does there exist input $a_1, \ldots, a_k$ such that the loop halts?

$$\exists a_1, \ldots, a_k \in \mathbb{R},\ n \in \mathbb{N} : M(a_1, \ldots, a_k)^n \vec{x}(a_1, \ldots, a_k) = \vec{y}(a_1, \ldots, a_k)?$$

Problem statement

$\mathbb{Q}(z_1, \ldots, z_k)$: Rational functions in variables $z_1, \ldots, z_k$

Parametric Orbit Problem

Given

- initial vector $x \in \mathbb{Q}(z_1, \ldots, z_k)^d$,
- update matrix $M \in \mathbb{Q}(z_1, \ldots, z_k)^{d \times d}$, and
- target vector $y \in \mathbb{Q}(z_1, \ldots, z_k)^d$.

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- update matrix $M \in \mathbb{Q}(z_1, \ldots, z_k)^{d \times d}$, and
- target vector $y \in \mathbb{Q}(z_1, \ldots, z_k)^d$.

do there exist $\vec{a} = (a_1, \ldots, a_k) \in \mathbb{R}^k$ and $n \in \mathbb{N}$ such that

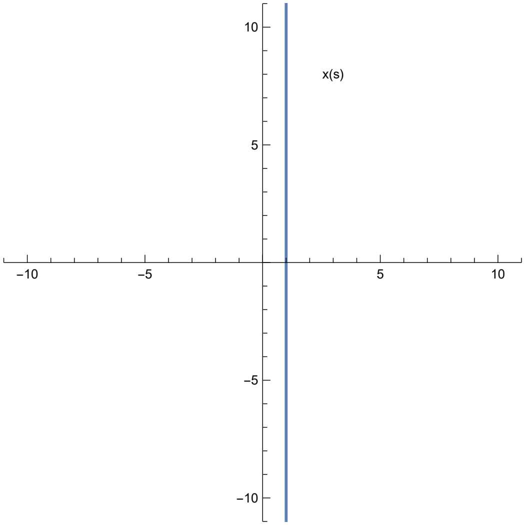
$$y(\vec{a}\,) = M(\vec{a}\,)^n x(\vec{a}\,)?$$

Parametric Linear Dynamical Systems

Input: $s \in \mathbb{R}$

$$x(s) = \begin{pmatrix} 1 \\ s \end{pmatrix}$$

$$M(s) = \begin{pmatrix} 1 & s-1 \\ 1 & -1 \end{pmatrix}$$



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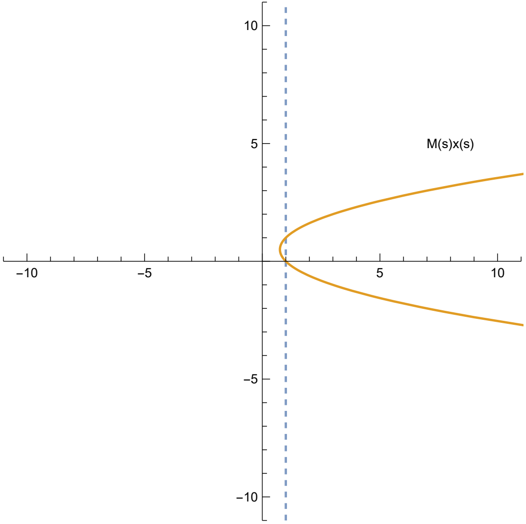

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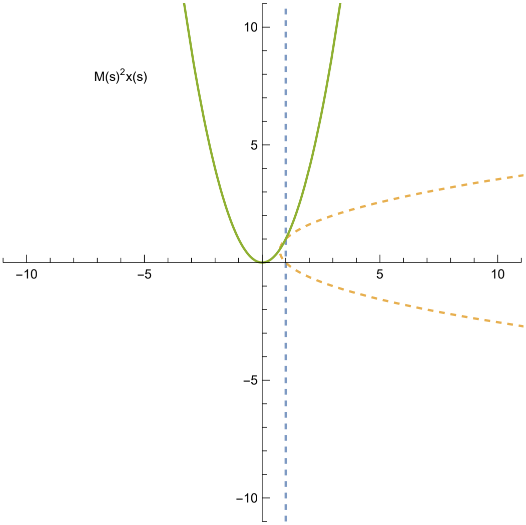

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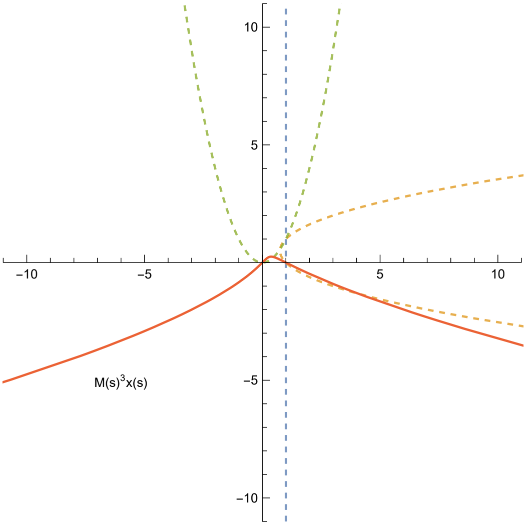

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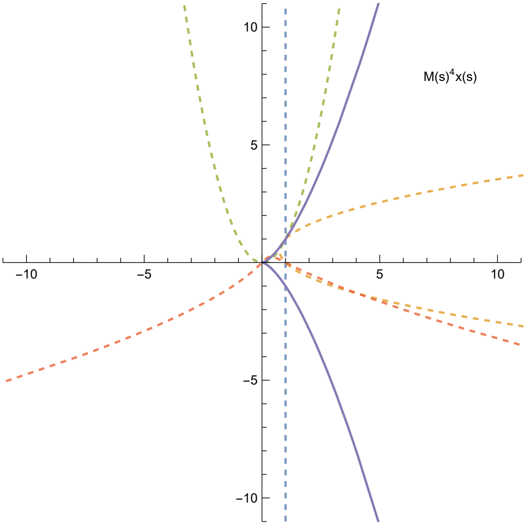

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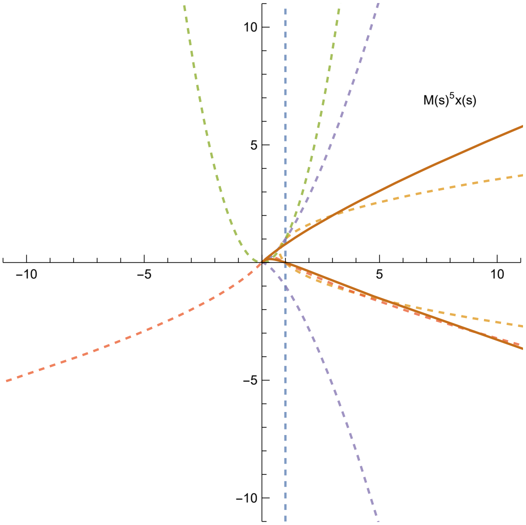

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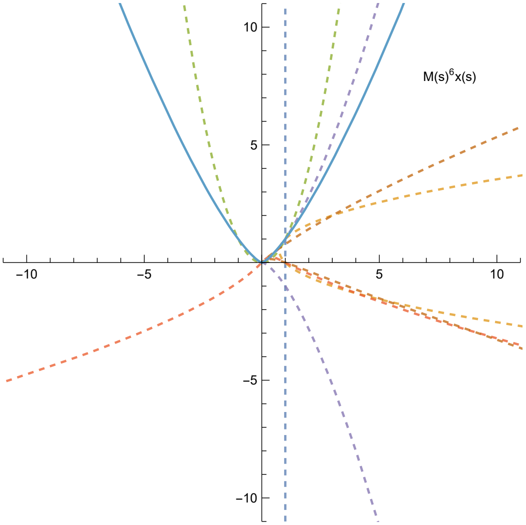

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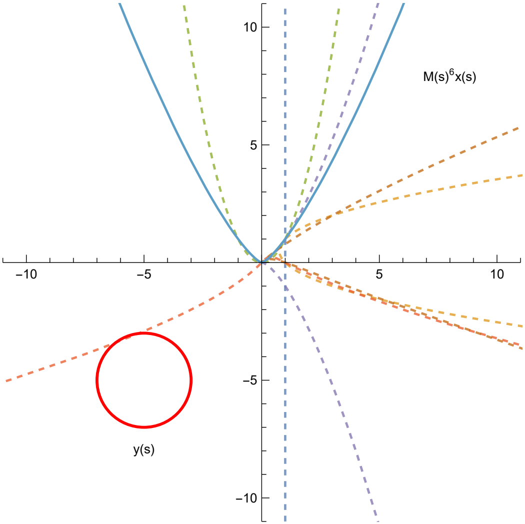
$$y(s) = \begin{pmatrix} -5 + 2\frac{1-s^2}{1+s^2} \\ -5 + 2\frac{2s}{1+s^2} \end{pmatrix}$$



Results

Parametric Orbit Problem

Given

- $x, y \in \mathbb{Q}(z_1, \ldots, z_k)^d$,
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Theorem

- *Univariate systems ($k = 1$): Problem is decidable.*

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Theorem

- *Univariate systems ($k = 1$): Problem is decidable.*
- *Multivariate systems ($k \geq 2$): Problem is* ***Skolem hard****.*

Thank you!

Talk based on

- C. Baier, F. Funke, S. Jantsch, T. Karimov, E. Lefaucheux, F. Luca, J. Ouaknine, D. Purser, M.A. Whiteland, and J. Worrell:
  The Orbit Problem for Parametric Linear Dynamical Systems
  CONCUR 2021, #28, doi:10.4230/LIPIcs.CONCUR.2021.28
- T. Karimov, E. Lefaucheux, J. Ouaknine, D. Purser, A. Varonka, M.A. Whiteland, J. Worrell:
  What's Decidable about Linear Loops?
  POPL 2022, #65, doi:10.1145/3498727

The Bivariate Orbit Problem is hard!

$$\exists n \in \mathbb{N},\ a_1, a_2 \in \mathbb{R} : M(a_1, a_2)^n\, x(a_1, a_2) = y(a_1, a_2)$$

Proposition

Skolem-5 reduces to the Parametric Orbit Problem (with two variables).

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$\vec{u} M^n \vec{x} = a\lambda^n + \overline{a\lambda^n} + b\gamma^n + \overline{b\gamma^n} + c$ with $1 \leq |\lambda| = |\gamma|$.

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$\vec{u}M^n\vec{x} = A\Re(\lambda^n) + B\Im(\lambda^n) + C\Re(\gamma^n) + D\Im(\gamma^n) + E$ with $1 \leq |\lambda| = |\gamma|$.

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- $2D$ algebraic target set in $\mathbb{C}^4$; T: $\begin{cases} A\mathtt{x} + B\mathtt{y} + C\mathtt{z} + D\mathtt{w} + E = 0 \\ \mathtt{x}^2 + \mathtt{y}^2 = \mathtt{z}^2 + \mathtt{w}^2 \end{cases}$

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    - the graph of a **target vector**: $T = \{y(a_1, a_2) : a_i \in \mathbb{R}\}$
- Then $\vec{u}M^n\vec{x} = 0$ if and only if $\exists a_1, a_2 \in \mathbb{R} : N^n\vec{x}' = y(a_1, a_2)$. $\square$