Combinatorics of Fibonacci and golden mean number representations

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Short Abstract:
How many ways are there to represent a number as a sum of powers of the golden mean?
Among these, what is the best way to do this?
What is the relation with representing a number as a sum of Fibonacci numbers?
I will give some answers to these questions in my talk.

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"We'll have fun hearing this again!"

Short Abstract:
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Quick answers

How many ways are there to represent a number as a sum of powers of the golden mean?

Infinitely many.

Among these, what is the best way to do this?

Greedy algorithm $\Rightarrow$ Bergman expansion, which is NOT the best!

What is the relation with representing a number as a sum of Fibonacci numbers?

Golden mean shift: $100 \mapsto 011$.

Base phi representations

A base phi representation of a natural number N has the form

$$N = \sum_{i=-\infty}^{\infty} a_i \varphi^i,$$

a_i are arbitrary non-negative numbers,
$\varphi := (1+\sqrt{5})/2$: the golden mean.

Similarly to base 10 numbers, we write these representations as

$$\alpha(N) = a_L a_{L-1} \dots a_1 a_0 \cdot a_{-1} a_{-2} \dots a_{R+1} a_R.$$

Here L is the largest positive, and R is the smallest negative power of φ that occurs (when these exist).

Infinitely many Base phi representations

Even if we only use digits a_i from $\{0, 1\}$ there are infinitely many finite length representations.

Example One has $\varphi^3 + \varphi^{-1} + \varphi^{-4} = 5$.

So $\alpha(5) = 1000{\cdot}1001$.

But $\varphi^{-4} = \varphi^{-5} + \varphi^{-6}$, so also

$\alpha'(5) = 1000{\cdot}100011$, and

$\alpha''(5) = 1000{\cdot}10001011$, and $\ldots$

Golden mean shift: $100 \mapsto 011$.

Base phi: Bergman representation

A natural number N is written in Bergman base phi if

$$N = \sum_{i=-\infty}^{\infty} d_i \varphi^i,$$

with digits $d_i = 0$ or 1, and where $d_i d_{i+1} = 11$ is not allowed. Again we write

$$\beta(N) = d_L d_{L-1} \ldots d_1 d_0 \cdot d_{-1} d_{-2} \ldots d_{R+1} d_R.$$

Theorem *The Bergman representation of N is unique.*

Base phi: canonical representation

A natural number N is written in the canonical base phi representation if N has the form

$$N = \sum_{i=-\infty}^{\infty} c_i \varphi^i,$$

with digits $c_i = 0$ or 1, and where $c_{i+1}c_i = 11$ is not allowed, *except* that $c_1 c_0 = 11$, as soon as this is possible. We write

$$\gamma(N) = c_L c_{L-1} \dots c_1 c_0 \cdot c_{-1} c_{-2} \dots c_{R+1} c_R.$$

To obtain this representation one first looks if there exists a representation of N with $c_1 c_0 = 11$, and no other $c_{i+1}c_i = 11$, and if this is not the case, then $\gamma(N) = \beta(N)$.

Theorem *The canonical representation of N is unique.*

Examples

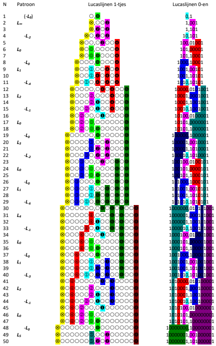
	Bergman	Canonical
N	$\beta(N)$	$\gamma(N)$
1	1·0	1·0
2	10·01	10·01
3	100·01	11·01
4	101·01	101·01
5	1000·1001	1000·1001
6	1010·0001	1010·0001
7	10000·0001	1011·0001
8	10001·0001	10001·0001
9	10010·0101	10010·0101
10	10100·0101	10011·0101
11	10101·0101	10101·0101
12	100000·101001	100000·101001

Theorem $\gamma(N) \neq \beta(N) \Leftrightarrow \exists n$, such that $N = \lfloor(\varphi + 2)n\rfloor$.

Why canonical??

ir A.W.W.J.M. van Loon:

“The Golden Ratio: the origin of nature?”



Why canonical? Part 1

Look at the length of the representations.

Base b: length n in the intervals $[b^{n-1}, b^n - 1]$.

Lucas numbers: (L_n) given by
$L_0 := 2$, $L_1 := 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

What are the intervals of constant expansion length for the Bergman representation?
Length $4n+1$ in the intervals $\Lambda_{2n} := [L_{2n}, L_{2n+1}]$,
length $4n+4$ in the intervals $\Lambda_{2n+1} := [L_{2n+1}+1, L_{2n+2}-1]$.

What are the intervals of constant expansion length for the canonical representation?
Length $2n+1$ for n even, and $2n+2$ for n odd in the intervals $\Gamma_n := [L_n + 1, L_{n+1}]$.

Why canonical? Part 1 b

Base b: length n in the intervals $B_n = [b^{n-1}, b^n - 1]$.

Constant expansion length for the Bergman representation:
In the intervals $\Lambda_{2n} = [L_{2n}, L_{2n+1}]$, $\Lambda_{2n+1} = [L_{2n+1} + 1, L_{2n+2} - 1]$.

Constant expansion length for the canonical representation:
In the intervals $\Gamma_n := [L_n + 1, L_{n+1}]$.

What is 'wrong' with the Λ_n compared to the B_n?

Answer: the Λ_{2n+1} are too small compared to the Λ_{2n}:
$|\Lambda_{2n}| = L_{2n-1} + 1$, and $|\Lambda_{2n+1}| = L_{2n} - 1$.

The Γ_n are more like the B_n.

Why canonical? Part 2 a

Look at the length of the vertical runs in the table of representations.

Classical base b representations. The case $b = 2$:

N	expansion
0	0
1	1
2	10
3	11
4	100
5	101
6	110
7	111

N	expansion
8	1000
9	1001
10	1010
11	1011
12	1100
13	1101
14	1110
15	1111

N	expansion
16	10000
17	10001
18	10010
19	10011
20	10100
21	10101
22	10110
23	10111

In digit position i, for $i \geq 0$, only runs of 2^i 1's occur—separated by runs of 2^i 0's.

Why canonical? Part 2 b

For the Bergman expansion there is no such regularity: vertical runs of 1's of length 1,2,3,4,5,6 and 7 do occur.

This is completely different for the canonical expansion:

Theorem *In the canonical base phi expansion of the natural numbers only vertical runs of 1's with length a Lucas number occur, and all Lucas numbers occur as a run length. More precisely: in digit position i only runs of length L_{i-1} occur when $i \geq 1$, and only runs of length L_{-i} occur when $i \leq 0$.*

Next question:

How many ways are there to represent a number as a sum of powers of the golden mean?

Not much is known about this question.
A lot (dozens of papers) is known about a related question:

How many ways are there to represent a number as a sum of different Fibonacci numbers?

(Also known as Fibonacci partitions.)

D. A. Klarner (1966), L. Carlitz, (1968),.....,
S.Chow and T. Slattery, arXiv: 17 Sep 2020.

Minimal Fibonacci representations

These are also known as Zeckendorf representations.

Let $F_0 = 0,\ F_1 = 1,\ F_2 = 1, \ldots$ be the Fibonacci numbers.

Ignoring leading zeros, any natural number N can be written uniquely as

$$N = \sum_{i=2}^{\infty} d_i F_i,$$

with digits $d_i = 0$ or 1, and where $d_i d_{i+1} = 11$ is not allowed.

We write $Z(N) = d_L \ldots d_2$.

Example $Z(6) = 1001$, since $F_5 = 5, F_2 = 1$.

Zeckendorf and Bergman

N	$Z(N)$	$\beta(N)$
1	1	1·
2	10	10·01
3	100	100·01
4	101	101·01
5	1000	1000·1001
6	1001	1010·0001
7	1010	10000·0001
8	10000	10001·0001
9	10001	10010·0101
10	10010	10100·0101
11	10100	10101·0101
12	10101	100000·101001
13	100000	100010·001001
14	100001	100100·001001
15	100010	100101·001001

How many Fibonacci representations?

A000119 Number of representations of n as a sum of distinct Fibonacci numbers.

$T^{\mathrm{FIB}} = 1, 1, 1, 2, 1, 2, 2, 1, 3, 2, 2, 3, 1, 3, 3, 2, 4, 2, 3, 3, 1, 4, 3, 3, 5, \ldots$

Recursions given by Neville Robbins (1966). See also Chow and Slattery (2020).

$Z(8) = 10000$.
Other representations: 1100, 1011. So $T^{\mathrm{FIB}}(8) = 3$.

Golden mean shift

$Z(8) = 10000$.
Other representations: 1100, 1011. So $T^{\mathrm{FIB}}(8) = 3$.

Golden mean shift: a map $G = G_m$ on 0-1-words:

$$G: \; w_1 \dots w_m 100 w_{m+4} \dots w_k \mapsto w_1 \dots w_m 011 w_{m+4} \dots w_k.$$

So $G(Z(8)) = G(10000) = 1100$, and $G(1100) = 1001$.

This is the "word"-version of $F_n = F_{n-1} + F_{n-2}$.

But it is also the "word"-version of $\varphi^n = \varphi^{n-1} + \varphi^{n-2}$. !!

How many base phi representations?

We have seen: there are infinitely many ways!

Proposal by Ron Knott: only count those not ending in 011.

A289749 Number of ways not ending in 011 to write n in base phi.

$T^{\kappa} = 1, 1, 2, 3, 3, 5, 5, 5, 8, 8, 8, 5, 10, 13, 12, 12, 13, 10, 7, 15, 18, 21, 16, \ldots$

1 all forms: 1

2 all forms: 10·01, 1·11

3 all forms: 100·01, 11·01, 10·1111

4 all forms: 101·01, 100·1111, 11·1111

5 all forms: 1000·1001, 110·1001, 110·0111, 101·1111, 1000·0111

6 all forms: 1010·0001, 1001·1001, 111·1001, 111·0111, 1001·0111

Trimming Knott

3 all forms: 100·01, 11·01, 10·1111

4 all forms: 101·01 , 100·1111, 11·1111

The representations of $N = 3, N = 4$ are obtained in a special way.

$101{\cdot}01 \to 101{\cdot}0011 \to 100{\cdot}1111.$

We remove these all the time, obtaining the total number of base phi representations

$T^{\varphi} = 1, 1, 2, 2, 1, 5, 5, 4, 5, 4, 3, 1, 10, 13, 12, 12, 13, 10, 6, 11, 12, \dots$

instead of

$T^{\kappa} = 1, 1, 2, 3, 3, 5, 5, 5, 8, 8, 8, 5, 10, 13, 12, 12, 13, 10, 7, 15, 18, \dots$

Theorem COUNT: $T^{\varphi}(N) = T^{\mathrm{FIB}}(F_{-R(N)+2}N)$.

Proof: Suppose that $\beta(N) = d_L \dots d_R$, so $N = \sum_R^L d_i \varphi^i$.
Multiply by φ^{-R+2}:

$$\varphi^{-R+2}N = \sum_{i=R}^{L} d_i \varphi^{i-R+2} = \sum_{j=2}^{L-R+2} d_{j+R-2}\varphi^j = \sum_{j=2}^{L-R+2} e_j \varphi^j$$

where we substituted $j = i - R + 2$, and defined $e_j := j + R - 2$.

Next we use the well known equation $\varphi^j = F_j\varphi + F_{j-1}$:

$$[F_{-R+2}\varphi + F_{-R+1}]N = \sum_{j=2}^{L-R+2} e_j[F_j\varphi + F_{j-1}].$$

This implies that

$$F_{-R+2}N = \sum_{j=2}^{L-R+2} e_j F_j.$$

So left side = Zeckendorf expansion of the number $F_{-R+2}N$.
But the manipulations above can be made for any 0-1-word of length $L - R + 1 \Rightarrow$ golden mean shifts of $e_2 \dots e_{L-R+2}$ are in 1-to-1 correspondence with golden mean shifts of $d_L \dots d_R$. $\square$

Base phi and Lucas numbers

The Lucas numbers $(L_n) = (2, 1, 3, 4, 7, 11, 18, 29, \dots) :$

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad \text{for } n \geq 2.$$

From $L_{2n} = \varphi^{2n} + \varphi^{-2n}$, and $L_{2n+1} = L_{2n} + L_{2n-1}$:

$$\beta(L_{2n}) = 10^{2n} \cdot 0^{2n-1} 1, \quad \beta(L_{2n+1}) = 1(01)^n \cdot (01)^n.$$

We read off: $R(L_{2n}) = -2n, R(L_{2n+1}) = -2n.$

Also clear that $T^{\varphi}(L_{2n}) = 2n$, and $T^{\varphi}(L_{2n+1}) = 1.$

So **Theorem COUNT** gives new (?!) information on the Fibonacci representations:

$$T^{\mathrm{FIB}}(F_{2n+2} L_{2n}) = 2n, \; T^{\mathrm{FIB}}(F_{2n+2} L_{2n+1}) = 1 \text{ for all } n \geq 1.$$

Fib and Luc

From Miklos Kristof, Mar 19 2007: (Start)

Let $L(n)$ = A000032(n) = Lucas numbers. Then

For $a >= b$ and odd b, $F(a+b) + F(a-b) = L(a) * F(b)$.

For $a >= b$ and even b, $F(a+b) + F(a-b) = F(a) * L(b)$.

For $a >= b$ and odd b, $F(a+b) - F(a-b) = F(a) * L(b)$.

.....(End)

So $F_{2n+2}L_{2n+1} = F_{4n+3} - F_1 = F_{4n+3} - 1$.

But $T^{\mathrm{FIB}}(F_n - 1) = 1$ is a well-known formula!

Stop

THE END