Thue, 100 years later or 7/3

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Axel Thue was born in Tönsberg (Norway), a town on the Oslofjord, on February 19, 1863.

He was the son of Niels Thue, a master mariner and later a ship owner, and of Nicoline Cathinka Eger.

Axel Thue died on March 7, 1922 in Oslo (Norway).
7/3: anniversary of Thue's death.

The way Thue became interested in mathematics is rather strange:

After seeing an advertisement for a book by Elling Holst entitled "On pendulum's meaning of geometry", he ordered it, believing that it was a physics book about the pendulum.

But there had been a misprint in the advertisement: the title of the book was "On Poncelet's meaning of geometry".

Nevertheless Thue decided to read it.

(More on Thue's biography at
https://mathshistory.st-andrews.ac.uk/Biographies/Thue/.)

First part

Axel Thue is most known for his work in Number Theory and more precisely on Diophantine equations.

If $f(x, y)$ is a homogeneous polynomial with integer coefficients, irreducible in the rationals and of degree > 2 and c is a non-zero integer then $f(x, y) = c$ has only a finite number of integer solutions.

Application: *There are only finitely many pairs of integers satisfying $y^3 - 2x^2 = 1$.*

History of Thue's result between 1844 and 1955

Let α be algebraic of degree $n > 2$, and K be a positive constant, If, for infinitely many rationals p/q, $|\alpha - \frac{p}{q}| \leq \frac{1}{q^K}$, then K cannot be too large.

* Liouville (1844): $K \leq n$

* Thue (1908): $K \leq \frac{n}{2} + 1$

* Siegel (1921): $K \leq \frac{n}{s+1} + s$, $\forall s \in [1, n-1]$ (#);

(hence $K \preceq 2\sqrt{n}$)

* Dyson (1947): $K \leq \sqrt{2n}$

* Roth (1955): $K \leq 2$

(#) Siegel:

"When I tried then to read [Thue's article] I soon ended in confusion because of the numerous letters $c, k, \theta, \omega, m, n, a, s$, the deeper meaning of which seemed enigmatic to me. In order to be able to understand a bit more, though, I changed the ordering of the lemmas, introduced new symbols too, and among them, more by chance than by any deliberate thought, was a parameter which does not occur in Thue's paper, and which, to my amazement, provided a sharpening of the approximation theorem."

see:

– G. Poitou, Le théorème de Thue-Siegel-Roth, *Enseign. Math.* **7** (1961), 281–285 (1962)

– C. Goldstein(*), Axel Thue in context, *J. Théor. Nombres Bordeaux* **27** (2015), 309–337.

– https://mathshistory.st-andrews.ac.uk/Biographies/Thue/

– ...

Interestingly enough, Thue published 48 papers: 20 were written in Norwegian, 26 were written in German, (but published in Scandinavian journals), only 2 were published in journals ouside Scandinavia, in German. These last two appeared in *Crelle*: *Journal für die reine und angewandte Mathematik*.

Thus Thue has the reputation of having been isolated and mathematically independent, which is rather false as explained by C. Goldstein (ref. above).

Before going further, let us give quote two sentences of Thue:

For the development of the logical sciences it will be important to find wide fields for the speculative treatment of difficult problems, without regard to eventual applications.

The further removed from usefulness or practical application, the more important.

Second part

Axel Thue is also
one of the fathers
of combinatorics on words.

The famous Thue-Morse sequence (Prouhet-Thue-Morse sequence)

Definition through the morphism μ:

$$0 \to 01$$

$$1 \to 10$$

– – – – – – – – – – ––

0

01

0110

01101001

...

(cf. cubefree / overlapfree)

A digression

Go to a browser and look for: Thue mu

Thue mu

Google

thue mu

Tous | Actualités | Vidéos | Images | Maps | Plus | Outils

Environ 50 100 000 résultats (0,75 secondes)

Résultats, y compris pour thue ***mue***
Essayez avec l'orthographe thue mu uniquement.

https://www.thueetmue.fr

Bienvenue sur le site de la commune nouvelle Thue et Mue

Thue et **Mue** est, depuis le 1er janvier 2017, une commune nouvelle française située dans le département du Calvados en région Normandie, peuplée de 5 449 ...

https://fr.wikipedia.org › wiki › Thue_et_Mue

Thue et Mue - Wikipédia

Thue et **Mue** est, depuis le 1er janvier 2017 , une commune nouvelle française située dans le ... 34 **m**. Max. 119 **m**. Superficie, 36,82 km2. Unité urbaine, Commune rurale.

Aire d'attraction : Caen; (commune de la cour… | Arrondissement : Caen
Gentilé : Thuemusien | Pays : France

Termes manquants : ~~mu~~ | Doit inclure : mu

Thue et Mue

Commune en France

Thue et Mue est, depuis le 1er janvier 201 française située dans le département du C Normandie, peuplée de 6 120 habitants. W

Superficie : 36,82 km²

Code postal : 14740, 14210 et 14250

Thue et Mue

From Wikipedia, the free encyclopedia

Coordinates: 49°12′40″N 0°30′50″W

Thue et Mue is a commune in the department of Calvados, northwestern France. The municipality was established on 1 January 2017 by merger of the former communes of Bretteville-l'Orgueilleuse (the seat), Brouay, Cheux, Le Mesnil-Patry, Putot-en-Bessin and Sainte-Croix-Grand-Tonne.[2]

See also [edit]

- Communes of the Calvados department

References [edit]

1. ^ "Populations légales 2018". The National Institute of Statistics and Economic Studies. 28 December 2020.
2. ^ Arrêté préfectoral 8 September 2016 (in French)

Thue et Mue

Commune

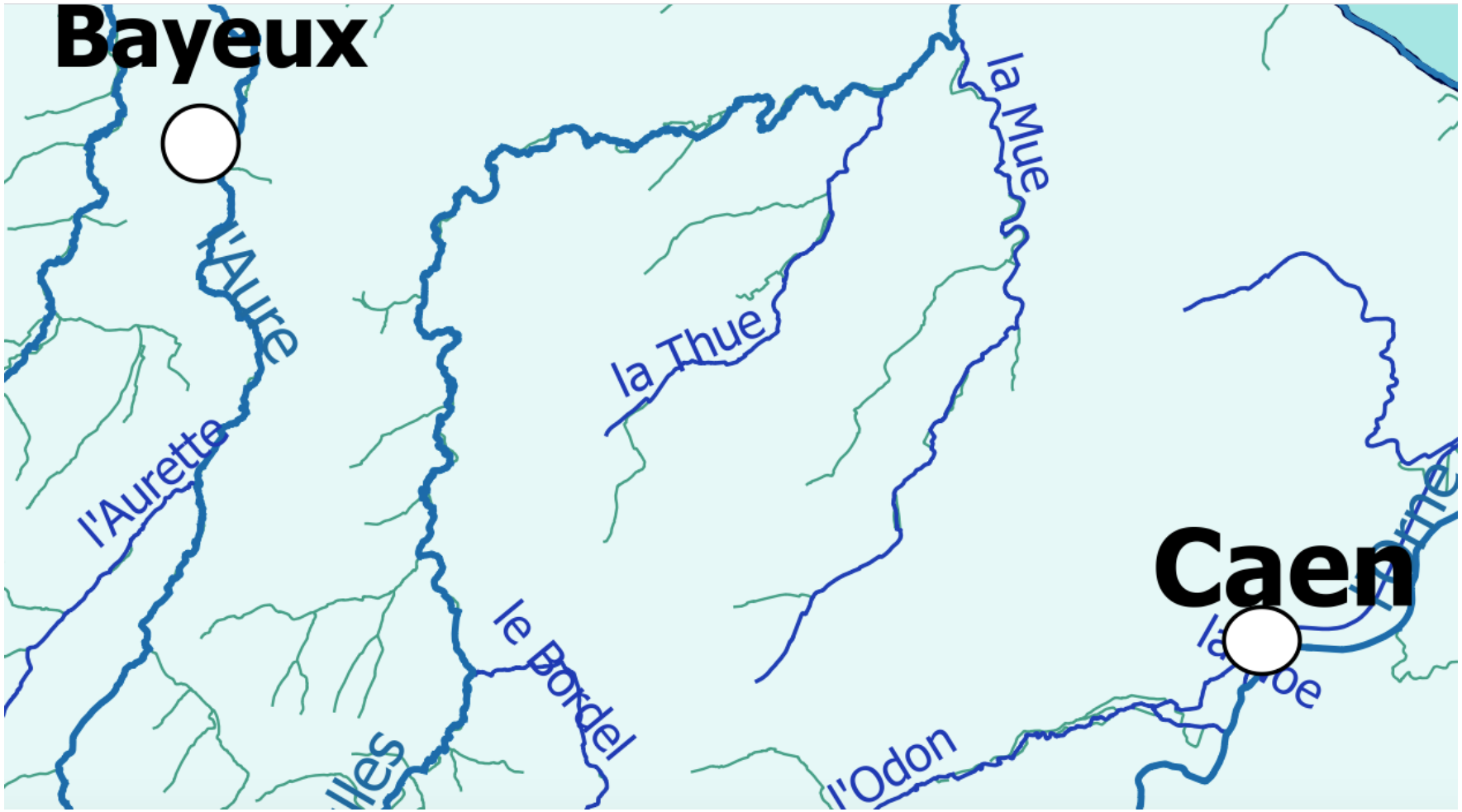
The church in Bretteville-l'Orgueilleuse

Location of Thue et Mue [show]

Thue et Mue

From Wikipedia, the free encyclopedia

Thue et Mue is a commune in the department of Calvados, northwestern France. The municipality was established on 1 January 2017 by merger of the former communes of Bretteville-l'Orgueilleuse (the seat), Brouay, Cheux, Le Mesnil-Patry, Putot-en-Bessin and Sainte-Croix-Grand-Tonne.[2]


Bayeux
l'Aure
l'Aurette
la Thue
la Mue
le Bordel
l'Odon
Caen

Rivers entering the picture? Was this unexpected?

Well,
Prouhet-Thue-Morse $\longrightarrow$ PTM $\longrightarrow$ PoTaMos

ποταμός

= **river** in Greek
(cf. hippopotamus, Mesopotamia, etc.)

A "natural" occurrence of the morphism μ

It is well known that (for $x \neq k\pi$)

$$\cos(x)\cos(2x)\cos(4x)\ldots\cos(2^n x) = \frac{\sin(2^{n+1}x)}{2^{n+1}\sin(x)}.$$

What about the product

$$\sin(x)\sin(2x)\sin(4x)\ldots\sin(2^n x)?$$

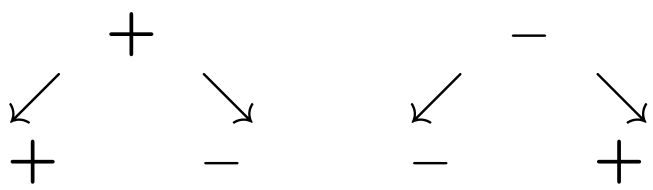
No nice formula.

At least we can look at its sign or at the sign of the following products on $[0, \pi]$:
$\sin(x)$
$\sin(x)\sin(2x)$
$\sin(x)\sin(2x)\sin(4x)$
...

Write $+$ when a function is positive on some interval and $-$ if it is negative. Then

	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$\sin(x)$	$+$	$+$	$+$	$+$	
$\sin(2x)$	$+$	$-$	$+$	$-$	
$\sin(x)\sin(2x)$	$+$	$-$	$-$	$+$	
$\ldots$					

At each step the "rule of signs" implies



P. Séébold $\longrightarrow$ N. Rampersad

Séébold (1985): The only overlap-free fixed points of non-trivial binary morphisms are the Thue-Morse sequence beginning in 0 and the Thue-Morse sequence beginning in 1, i.e., the two fixed points of the morphism $0 \to 01$, $1 \to 10$.
(Séébold 1985; Berstel and Séébold 1993)

Rampersad (2004-2005): One can replace "overlap-free" in the above statement with "7/3-power-free" (and 7/3 is optimal).

(7/3 once again)

Actually 7/3 occurs again in a paper by Rampersad, Shallit and Shur.

Abstract: We prove a Fife-like characterization of the infinite binary 7/3-power-free words, by giving a finite automaton of 15 *states that encodes all such words. As a consequence, we characterize all such words that are 2-automatic.*

Remark: a result in the RSS paper states that the lexicographically least binary 7/3-power-free sequence is the sequence $001001\overline{\text{PTM}}$.

将棋 Shogi (Japanese chess)

千日手 (Sennichite) is a specific rule in Shogi about repetitions.

The game ends in a **draw** if the same position is repeated four times without involving checks. Note that in professional shogi, draws essentially do not count.

The sennichite rule was previously defined by a sequence of moves (and not a position) that had three repetitions.
The rule was changed in May 1983 (three repetitions replaced with four repetitions), after a play between Kunio Yonenaga et Kōji Tanigawa on March 8, 1983.

Again 7/3... Oops 8/3.

Chess

The so-called German rule states that a draw occurs if the same sequence of moves occurs three times in succession.

Euwe (1929) proved, using the cube-free property of a sequence (which happens to be the PTM sequence), that under such a rule infinite games of chess are possible.

Fairness

I. Palacios-Huerta, Tournaments, fairness and the Prouhet-Thue-Morse sequence, *Economic Inquiry* **50** (2012), 848–849.

$A \qquad B \qquad \longrightarrow \quad A\ B \quad A\ B \quad A\ B \ \ldots$

$A\ B \qquad B\ A \qquad \longrightarrow \quad A\ B\ B\ A \quad A\ B\ B\ A \quad A\ B\ B\ A \ \ldots$

In real life: cf. tennis (tie-break)

...

J. Cooper, A. Dutle, Greedy Galois Games, *Amer. Math. Monthly* **120** (2013), 441–451.

Transcendence(S) of the PTM sequence 0 1 1 0 1 0 0 1 ...

* The formal power series $\sum_{n\geq 0} a_n X^n$ is algebraic over $\mathbb{Z}/2\mathbb{Z}(X)$.

* The formal power series $\sum_{n\geq 0} a_n X^n$ is transcendental over $\mathbb{Z}/3\mathbb{Z}(X)$.

* The real number $\sum_{n\geq 0} \frac{a_n}{b^n}$ is transcendental over $\mathbb{Q}$, $(b \in \mathbb{N}_{\geq 2})$.

* The continued fraction $[a'_0, a'_1, \ldots]$ is transcendental over $\mathbb{Q}$.

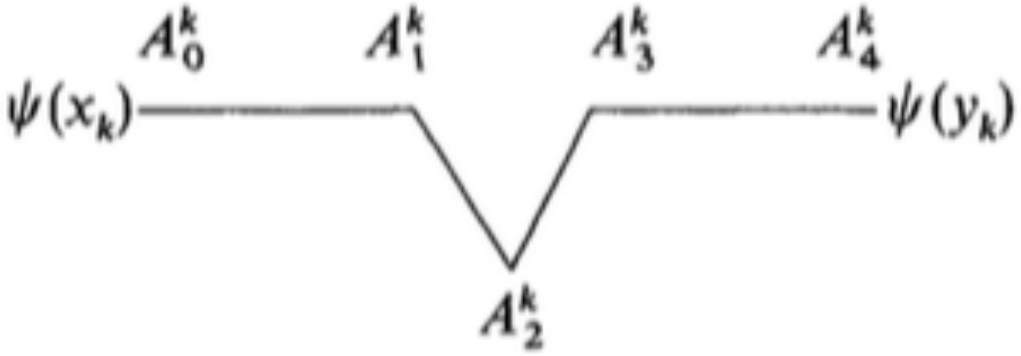
"Conversely" ($[a, b, b, a, b, a, a, b, \dots]$, a, b polynomials; in char. 2)

Let a, b be distinct, non-constant polynomials in $\mathbb{Z}/2\mathbb{Z}(X)$. Let $\xi_{a,b}$ be the power series in $\mathbb{Z}/2\mathbb{Z}((X^{-1}))$ whose sequence of partial quotients is the Thue-Morse sequence over $\{a, b\}$.
Then $\xi_{a,b}$ is algebraic of degree 4 over $\mathbb{Z}/2\mathbb{Z}(X)$.

If $\max(\deg a, \deg b) \leq 7$, see
Y. Hu, G.-N. Han, On the algebraicity of Thue-Morse and period-doubling continued fractions, *Acta Arith.* (2022),To appear.

General case: Y. Bugeaud, G.-N. Han, preprint 2022.

Thue-Morse and von Koch



J. Coquet, A summation formula related to the binary digits, *Invent. Math.* (1983) **73**, 107–116.

Also see Dekking (1982–1983), Ma and Holdener (2005), Allouche and Skordev (2007), Y.-Q. Li (2021).

Some more pictures

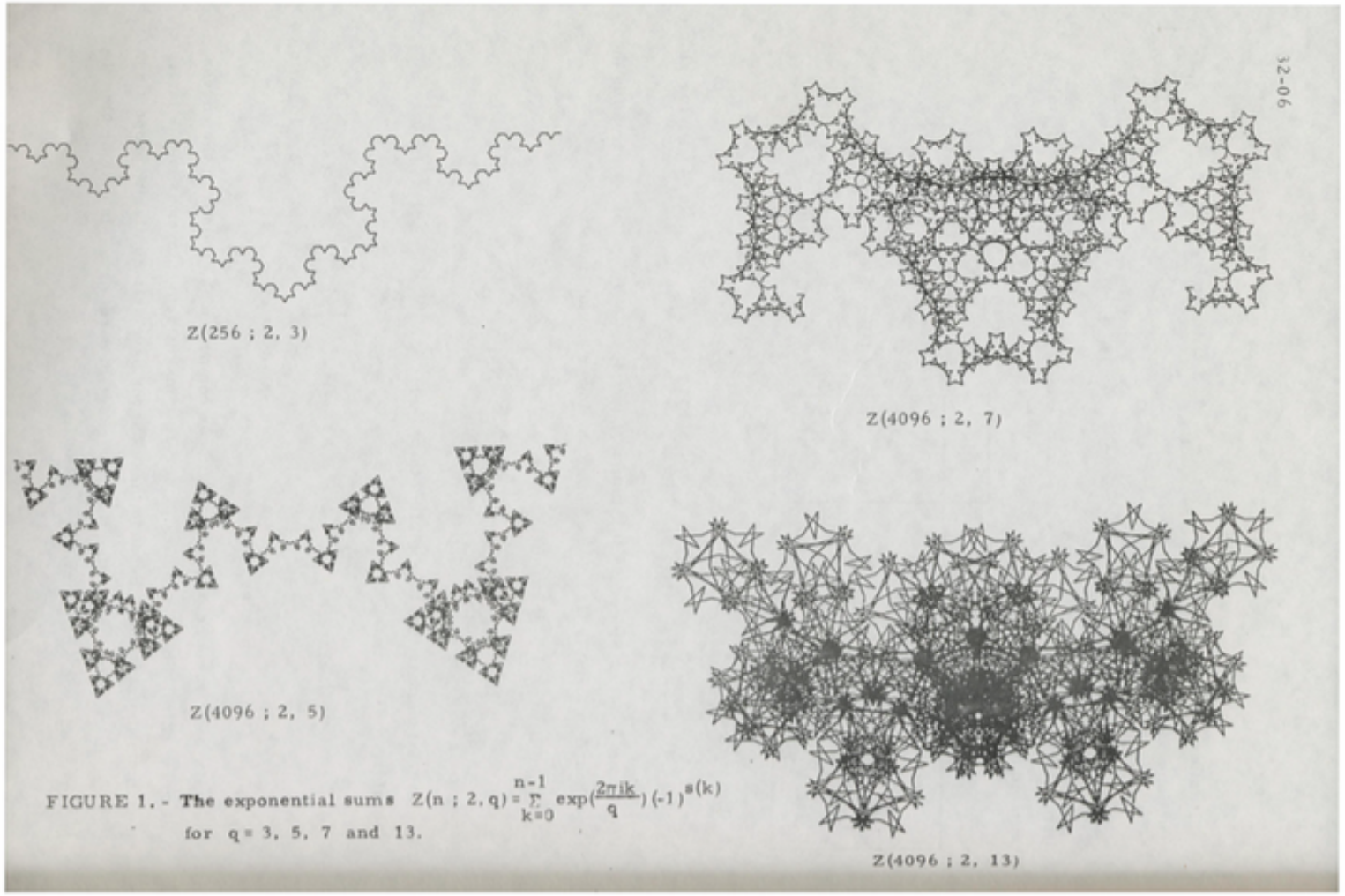


FIGURE 1. - **The exponential sums** $Z(n\,;2,q)=\sum_{k=0}^{n-1}\exp(\frac{2\pi ik}{q})(-1)^{s(k)}$ for $q=3,\ 5,\ 7$ and 13.

in Dekking (1982–1983)

More pictures (and more theory) in a similar vein, see, e.g.,

F. M. Dekking, M. Mendès France, Uniform distribution modulo one: a geometrical viewpoint. *J. Reine Angew. Math.* **329** (1981), 143–153.

J.-M. Deshouillers, Geometric aspect of Weyl sums, in *Elementary and analytic theory of numbers (Warsaw, 1982)*, Banach Center Publ., 17, PWN, Warsaw, 1985, pp. 75–82.

A quick allusion to semi-Thue systems

(A semi-Thue system is a restricted kind of a term rewriting system.) These systems were introduced in 1914 paper of Axel Thue: Probleme über Veränderungen von Zeichenreihen nach gegebenen Regeln, Kristiania: J. Dybvad. 34 S. Lex. 8 (1914).

Also see `https://arxiv.org/abs/1308.5858`: J. F. Propper, Thue's 1914 paper: a translation (44 pages).

Among the authors who were interested in semi-Thue systems, one can cite Gérard Huet (e.g., G. Huet, Confluent Reductions: abstract properties and applications to term rewriting systems, *Journal of the ACM* **27** (1980), 797–821).

But:

Huet $\longrightarrow$ Hue t $\longrightarrow$ Thue

Playing with words has to do with Oulipo (OuLiPo is a gathering of (mainly) French-speaking writers and mathematicians who seek to create works using constrained writing techniques. It was founded in 1960 by Raymond Queneau and François Le Lionnais. One of their famous members is Georges Perec.

Georges Perec was born on... March 7, 1936

7/3

Recall that Thue wrote

The further removed from usefulness or practical application, the more important.

Really? We have seen the semi-Thue systems. One could also cite the programming language "Thue" that Wikipedia describes as follows:

Thue is an esoteric programming language invented by John Colagioia in early 2000. It is a meta-language that can be used to define or recognize Type-0 languages from the Chomsky hierarchy. Because it is able to define languages of such complexity, it is also Turing-complete itself. Thue is based on a nondeterministic string rewriting system called semi-Thue grammar, which itself is named after the Norwegian mathematician Axel Thue.

Thank you!

ありがとうございます

Merci !

谢谢

Vielen Dank !