Anti-powers in Aperiodic Recurrent Words

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One World Combinatorics on Words Seminar

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Powers and Anti-powers

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Definition (Fici–Restivo–Silva–Zamboni, 2016)

A k**-anti-power** is a word of the form $w^{(1)} \cdots w^{(k)}$ where $w^{(1)}, \ldots, w^{(k)}$ are distinct words of the same length.

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Example

The word $\underline{0100}\,\underline{1111}\,\underline{1100}$ is a 3-anti-power.
The word $\underline{0100}\,\underline{1111}\,\underline{0100}$ is *not* a 3-anti-power.

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Let $N(\ell, k)$ be the smallest positive integer such that every word of length $N(\ell, k)$ contains an ℓ-power or a k-anti-power.

Theorem (Fici–Restivo–Silva–Zamboni, 2016)

We have $k^2 - 1 \leq N(k, k) \leq k^3 \binom{k}{2}$.

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Consider a sequence $(\alpha_i)_{i\geq 1}$ of positive integers satisfying $\alpha_{i+1} \geq 5\alpha_i$ for all i. Let $x = x_1x_2\cdots$, where $x_n = 0$ if $n \notin \{\alpha_i : i \in \mathbb{Z}\}$ and $x_n = 1$ if $n \in \{\alpha_i : i \in \mathbb{Z}\}$.

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Theorem (Fici–Restivo–Silva–Zamboni, 2016)

The word x is aperiodic and avoids 4-anti-powers.

Uniformly Recurrent Words

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An infinite word is **uniformly recurrent** if every finite factor that appears in the word actually appears infinitely often and with bounded gaps.

Theorem (Fici–Restivo–Silva–Zamboni, 2016)

Every aperiodic uniformly recurrent word contains anti-powers of all orders starting at every position.

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$$\mathbf{t} =$$

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Theorem (D., 2017)

We have $\frac{1}{4} \leq \liminf_{k\to\infty} \frac{\gamma(k)}{k} \leq \frac{9}{10}$ *and* $\frac{1}{2} \leq \limsup_{k\to\infty} \frac{\gamma(k)}{k} \leq \frac{3}{2}$.

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Definition

Let $\mathrm{AP}_j(\mathbf{t}, k)$ be the set of positive integers m such that the factor of $\mathbf{t}$ of length km starting at position j is a k-anti-power. Let $\gamma_j(k) = \min \mathrm{AP}_j(\mathbf{t}, k)$.

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Theorem (Gaetz, 2021)

For all $j \geq 1$, $\dfrac{1}{10} \leq \liminf_{k\to\infty} \dfrac{\gamma_j(k)}{k} \leq \dfrac{9}{10}$ *and*
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For each fixed $j \geq 1$ and $k \geq 3$, we have

$$m \in \mathrm{AP}_j(\mathbf{t}, k) \iff 2m \in \mathrm{AP}_j(\mathbf{t}, k)$$

for all sufficiently large m.

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Proof Sketch.

Now suppose $m \in (2\mathbb{Z}^+ - 1) \setminus \mathrm{AP}_j(\mathbf{t}, k)$. Then there exists $0 \leq n_1 < n_2 \leq k-1$ such that $y := \mathbf{t}_{n_1 m + j} \cdots \mathbf{t}_{(n_1+1)m+j-1} = \mathbf{t}_{n_2 m + j} \cdots \mathbf{t}_{(n_2+1)m+j-1}$. Let $v = \mathbf{t}_{(n_1+1)m+j} \cdots \mathbf{t}_{n_2 m + j - 1}$.

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Theorem (D., 2017)

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How big is $(2\mathbb{Z}^+ - 1) \setminus \mathrm{AP}_j(\mathbf{t}, k)$?

What can be said about $\mathrm{AP}_j(\mathbf{x}, k)$ for other specific interesting words $\mathbf{x}$?

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Let $w_0 = 0$. For $n \geq 1$, let $w_n = w_{n-1} 1^{3|w_{n-1}|} w_{n-1}$. Let $w = \lim_{n\to\infty} w_n$.

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The word w is aperiodic and recurrent, and it avoids 6*-anti-powers.*

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Theorem (Berger–D., 2019)

Every aperiodic recurrent word contains 5-anti-powers.

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Let $\mathcal{A}^{\leq\omega}$ be the set of words over an alphabet $\mathcal{A}$. A **morphism** is a map $\mu : \mathcal{A}^{\leq\omega} \to \mathcal{A}^{\leq\omega}$ such that $\mu(uv) = \mu(u)\mu(v)$ for all $u, v \in \mathcal{A}^{\leq\omega}$. A morphism is r-**uniform** if $|\mu(a)| = r$ for all $a \in \mathcal{A}$.

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The Thue-Morse word $\mathbf{t} = 0110100110010110\cdots$ is a fixed point of the 2-uniform morphism μ given by $\mu(0) = 01$ and $\mu(1) = 10$.

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"Conjecture" (Berger–D., 2019)

If W is a "sufficiently well-behaved" aperiodic word that is fixed by a morphism μ, then there exists a constant $C = C(W)$ such that for all $j, k \geq 1$, W contains a k-anti-power j-fix with blocks of length at most Ck.

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Theorem (Garg and Postic (independently), 2019)

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Uniform Morphic Words

Garg also proved the conjecture for the Fibonacci word, which is fixed by the **non-uniform** morphism μ given by $\mu(0) = 01$ and $\mu(1) = 0$.

Theorem (Garg, 2019)

For every $j, k \geq 1$, the Fibonacci word contains a k-anti-power j-fix with blocks of length at most $2.89k$.

Thank You!