Construction and parametrization of conjugates of Christoffel words

One World CoW Seminar

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joint work with Yann Bugeaud

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Université du Québec à Montréal
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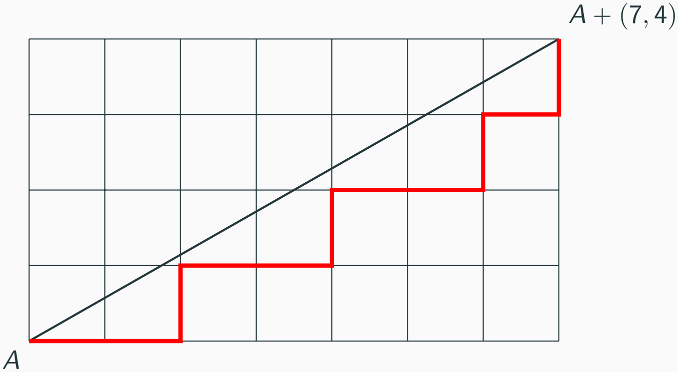
Christoffel words and their conjugates

Geometric definition Christoffel words



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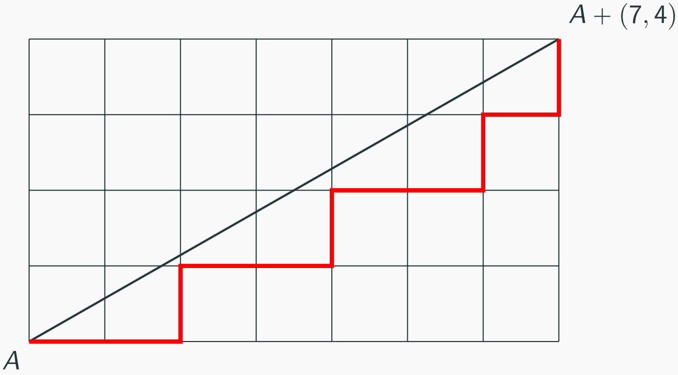

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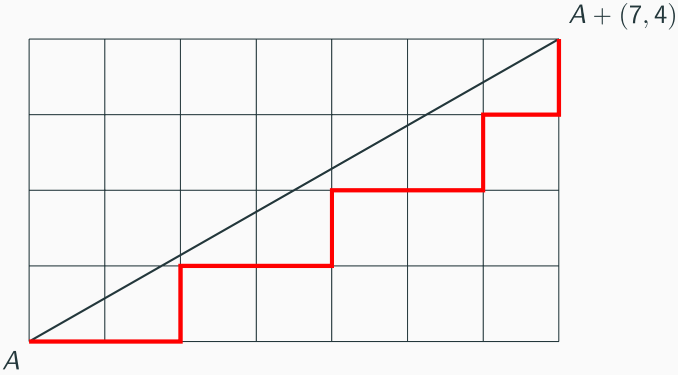

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The slope of w is the ratio $\frac{|w|_b}{|w|_a}$, equal to the slope of the diagonal

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Define a representation of the free monoid into $SL_2(\mathbb{Z})$ by

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The *Markoff number injectivity conjecture* is that this word is unique (open since 1913).

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- Conjugates coincide with the elements of the free group with two generators subject to the following conditions : they are positive (that is, with no inverted letter), they are cyclically reduced, and they are primitive (part of a basis of this free group) (Osborne-Zieschang 1981).

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- Besides the Christoffel words, which encode the Markoff binary forms and their minima, their conjugates correspond to the "small values" of these quadratic forms.

Parametrization of the conjugates of Christoffel words

Construction

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In order to have words in A^*, we add the condition $0 \leqslant d_i \leqslant b_i$.

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Example : *aaabaab* (lower), *aabaaba* (standard), *abaabaa*, *baabaaa* (upper), *aabaaab* (standard), *abaaaba*, *baaabaa*

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This representation for N is not unique. In the theorem, V_m depends only on N, not on the digits d_i of the Ostrowski representation.

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Corollary *Let N be an integer with legal Ostrowski representation $N = \sum_{1 \leqslant i \leqslant m} d_i q_{i-1}$. Then the prefix of length N of the central palindrome p is*

$$M_{m-1}^{d_m} \cdots M_0^{d_1}.$$

In particular this product depends only on N and not on the chosen legal Ostrowski representation of N.

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There may be several borders and periods. The least (nontrivial) period + the length of the longest border = the length of the word.

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In 2021, Gabric, Rampersad, and Shallit determine the set of all periods of a given finite prefix of a given infinite Sturmian word.

As a consequence they obtain the number of periods : it is equal to the sum of the digits of the lazy Ostrowski representation of $N =$ the length of the prefix ('lazy' will be defined further in the talk).

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Moreover, it is known that Christoffel words (as Lyndon words) have no border.

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Consider a conjugate $V_m = C^N(M_m) = V_{m-1}^{b_m - d_m} V_{m-2} V_{m-1}^{d_m}$, not a Christoffel word, with $N = \sum_{1 \leqslant i \leqslant m} d_i q_{i-1}$ (greedy Ostrowski representation).

Let $\ell = \min\{b_m - d_m, d_m\}$ Let B be the longest border of V_m.

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If $d_m = b_m$, then $B = V_{m-1}$.

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+ several other cases

Statement and proof are technical.

Sturmian graph revisited

Lazy

A legal Ostrowski representation $N = \sum_i d_i q_{i-1}$ is called *lazy* if for any i, $d_i = 0 \Rightarrow d_{i-1} = b_{i-1}$ (Epifanio, Frougny, Gabriele, Mignosi and Shallit 2012).

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Then Frid's result (on the factorization of prefixes of p) implies : *Each suffix of p has a unique factorization*

$$L_0^{d_1} L_1^{d_2} \cdots L_{m-1}^{d_m}$$

where $\sum_{1 \leqslant i \leqslant m} d_i q_{i-1}$ *is the lazy Ostrowski representation of its length.*

Compact graph

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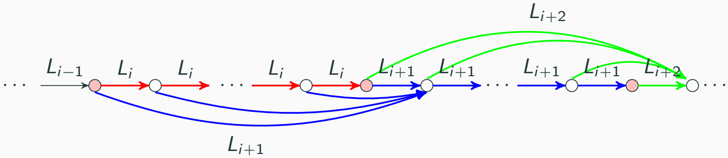

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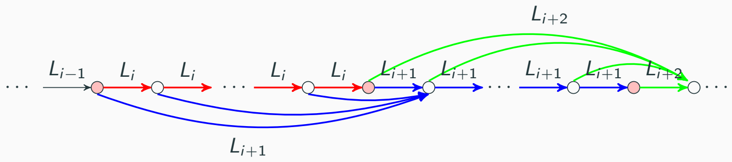

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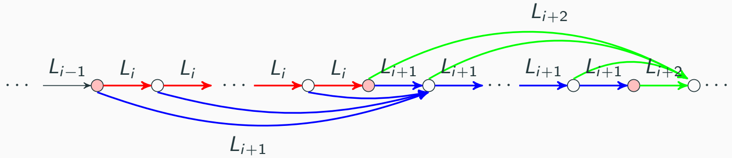

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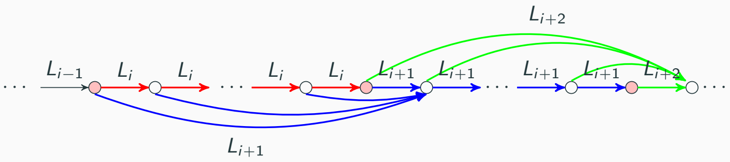

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For each suffix s of p, there is a unique path in this graph, starting from the origin, and with label s.

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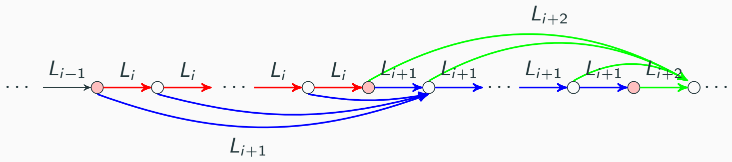

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This graph has been obtained by another method by Epifanio, Mignosi, Shallit and Venturini 2007, and called *compact graph*. They obtain it from the minimal automaton of the set of suffixes of the central palindrome p, after an operation called *compaction* (Blumer, Blumer, Haussler, McConnell, Ehrenfeucht 1987).

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For each natural number $N = 0, 1, \ldots, q_m - 2 = |p|$ *there is a unique path in the Sturmian graph, starting form the origin, with (additive) label* N.

Embeddings of these graphs

The Sturmian graph may be embedded in the Stern-Brocot tree ; and the compact graph may be embedded in the tree of central words (a vertex in this tree is a binary word u, and its label is $Pal(u)$, where Pal is the *iterated palindromization* of Aldo de Luca 1997).

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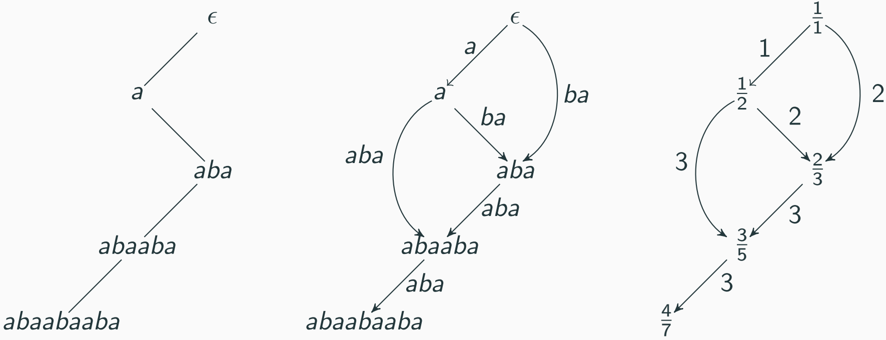

An open problem : determine all borders of a given conjugate. Count them. Equivalently count the number of its periods. Does the lazy representation plays a role, as it does for the determination and counting of the periods of a given prefix of a Sturmian infinite word ? (quoted result of Gabric, Rampersad and Shallit).

Merci

Thank you !